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A Generalization of the Banach Contraction Principle in Noncomplete Metric Spaces

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Abstract. We give a sufficient condition on metric spaces possessing the Banach fixed point property (BFPP). Further we also give a sufficient condition on not possessing BFPP.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

The following famous theorem is referred to as the *Banach contraction principle*.

Theorem 1.1 ([1, 5]). *Let (X, d) be a complete metric space and let T be a contraction on X, that is, there exists $r \in (0, 1)$ such that*

$$d(Tx, Ty) \le r\, d(x, y) \tag{1}$$

for all $x, y \in X$. Then T has a unique fixed point z and $\{T^n x\}$ converges to z for any $x \in X$.

This theorem is very forceful and simple and it became a classical tool in nonlinear analysis. Moreover it has many generalizations; see [3, 6–8, 11, 13, 15–19, 21–23, 25] and others. On the other hand, Connell [9] gave an example of a metric space X such that X is not complete and every contraction on X has a fixed point. Thus, Theorem 1.1 cannot characterize the metric completeness of X. We have discussed the metric completeness about the fixed point property for other mappings; see [14, 20, 23, 26] and others. See also [2, 24].

Definition 1.2 ([12]). *A metric space (X, d) is said to possess the Banach fixed point property (BFPP, for short) if every contraction on X has a fixed point.*

Theorem 1.1 tells that every complete metric space possesses BFPP. Borwein in [4] gave another example of a metric space possessing BFPP.

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Theorem 1.3 (Borwein [4]). *Define a subset X of the 2-dimensional Euclidean space $\mathbb{R}^2$ by*

$$X = \{0\} \cup \bigcup_{k=1}^{\infty} L_k,$$

where

$$L_k = \{(t, t\,2^{-k}) : t \in (0, 1]\}$$

for $k \in \mathbb{N}$. Then X possesses BFPP.

In 2007, Xiang proved some splendid results on BFPP. The following is one of them, which includes Theorem 1.3, but does not include Theorem 1.1.

Theorem 1.4 (Xiang [27]). *A locally Lipschitz-connected metric space possesses BFPP iff it is Lipschitz-complete.*

Motivated by the above, in this paper, we prove a generalization of both Theorems 1.1 and 1.3. Our approach differs from that of Xiang [27]. We also give a sufficient condition on not possessing BFPP.

2. A General Case

Let (X, d) be a metric space. Throughout this paper we denote by $\mathrm{CauS}(X)$ the set of all Cauchy sequences in X. We also denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

In this section, we prove a fixed point theorem in a very general setting.

Definition 2.1. *Let (X, d) be a metric space and let ℓ be a function from $X \times \mathrm{CauS}(X)$ into $[0, \infty]$. Then X is said to satisfy Condition (ℓ) if for every $\{x_n\} \in \mathrm{CauS}(X)$, there exists $w \in X$ such that $\ell(w, \{x_n\}) < \infty$ and $(1/2)\,\ell(w, \{x_n\}) < \ell(x, \{x_n\})$ for any $x \in X \setminus \{w\}$.*

Theorem 2.2. *Let (X, d) be a metric space and ℓ be a function from $X \times \mathrm{CauS}(X)$ into $[0, \infty]$. Let T be a mapping on X. Assume the following:*

(i) *X satisfies Condition (ℓ).*
(ii) *There exists $v \in X$ such that $\{T^n v\} \in \mathrm{CauS}(X)$.*
(iii) *There exists $r \in (0, 1)$ such that*

$$\ell(Tx, \{T^n v\}) \le r\,\ell(x, \{T^n v\}) \tag{2}$$

for any $x \in X$.

Then the following hold:

(j) *T has a unique fixed point z.*

(jj) *$\ell(z, \{T^n v\}) = 0$ holds.*

(jjj) *$\lim_m \ell(T^m x, \{T^n v\}) = 0$ holds for any $x \in X$.*

Proof. Since X satisfies Condition (ℓ), there exists $z \in X$ such that $\ell(z, \{T^n v\}) < \infty$ and $(1/2)\,\ell(z, \{T^n v\}) < \ell(x, \{T^n v\})$ for any $x \in X \setminus \{z\}$. We consider the following two cases:

- $\ell(z, \{T^n v\}) = 0$
- $\ell(z, \{T^n v\}) > 0$

In the first case, we note $\ell(x, \{T^n v\}) > 0$ for any $x \in X \setminus \{z\}$. Since $\ell(Tz, \{T^n v\}) = 0$ holds from (2), z is a fixed point of T. In the second case, from (2) we have

$$\lim_{m\to\infty} \ell(T^m z, \{T^n v\}) \leq \lim_{m\to\infty} r^m\, \ell(z, \{T^n v\}) = 0.$$

Hence there exists $\mu \in \mathbb{N}$ satisfying

$$\begin{aligned}\ell(T^\mu z, \{T^n v\}) &< (1/2)\,\ell(z, \{T^n v\}) \\ &\leq \min\Big\{\inf\Big\{\ell(x, \{T^n v\}) : x \in X \setminus \{z\}\Big\}, \ell(z, \{T^n v\})\Big\} \\ &= \inf\Big\{\ell(x, \{T^n v\}) : x \in X\Big\},\end{aligned}$$

which is a contradiction. So the second case cannot be possible. As above,

$$\lim_{m\to\infty} \ell(T^m x, \{T^n v\}) = 0$$

holds for any $x \in X$. Therefore the fixed point z is unique. □

Using Theorem 2.2, we obtain the following fixed point theorem.

Theorem 2.3. *Let (X, d) be a metric space and ℓ be a function from $X \times \mathrm{CauS}(X)$ into $[0, \infty]$. Let T be a contraction on X. Assume the following:*

(i) *X satisfies Condition (ℓ).*

(ii) *There exists $v \in X$ such that*

$$\lim_{m\to\infty} d(x, T^m v) \leq \ell(x, \{T^n v\})$$

for any $x \in X$.

(iii) *There exists $r \in (0, 1)$ satisfying (2) for any $x \in X$.*

Then T has a unique fixed point z and $\{T^n x\}$ converges to z for any $x \in X$.

Remark 2.4. *Let $\{x_n\} \in \mathrm{CauS}(X)$. Then it is well known that a function ρ from X into $[0, \infty)$ defined by*

$$\rho(x) = \lim_{n\to\infty} d(x, x_n) \tag{3}$$

for $x \in X$ is well defined. Also it is well known that

$$|\rho(x) - \rho(y)| \leq d(x, y) \leq \rho(x) + \rho(y) \tag{4}$$

for any $x, y \in X$.

Proof. Since T is a contraction, there exists $s \in (0, 1)$ such that $d(Tx, Ty) \leq s\, d(x, y)$ for all $x, y \in X$. Fix $v \in X$. Then since

$$\sum_{n=1}^{\infty} d(T^n v, T^{n+1} v) \leq \sum_{n=1}^{\infty} s^n\, d(v, Tv) < \infty,$$

we have $\{T^n v\} \in \mathrm{CauS}(X)$. Thus, (ii) of Theorem 2.2 holds. So by Theorem 2.2, (j)–(jjj) of Theorem 2.2 hold. Thus, T has a unique fixed point z. By (4) and (ii), we have

$$d(x, y) \leq \lim_{m\to\infty} d(x, T^m v) + \lim_{m\to\infty} d(y, T^m v) \leq \ell(x, \{T^n v\}) + \ell(y, \{T^n v\}) \tag{5}$$

for any $x, y \in X$. Using (5) and Theorem 2.2 (jj) and (jjj), we have

$$\lim_{m\to\infty} d(T^m x, z) \leq \lim_{m\to\infty} \Big(\ell(T^m x, \{T^n v\}) + \ell(z, \{T^n v\})\Big) = 0$$

for any $x \in X$. Thus we obtain the desired result. □

In order to understand Condition (ℓ), Theorems 2.2 and 2.3 well, we prove the Banach contraction principle (Theorem 1.1 above) by using Theorem 2.3.

In the remainder of this section, let (X, d) be a metric space and let ℓ be a function from $X \times \mathrm{CauS}(X)$ into $[0, \infty)$ defined by

$$\ell(x, \{x_n\}) = \lim_{n\to\infty} d(x, x_n) \tag{6}$$

for $(x, \{x_n\}) \in X \times \mathrm{CauS}(X)$.

Proposition 2.5. *Let (X, d) be a metric space and define a function ℓ by (6). Then X is complete iff X satisfies Condition (ℓ).*

Proof. Obvious. □

Proof of Theorem 1.1. By Proposition 2.5, X satisfies Condition (ℓ). Fix $v \in X$. Then we have

$$d(Tx, T^{n+1}v) \le r\, d(x, T^n v)$$

and hence (2) holds for any $x \in X$. So by Theorem 2.3, we obtain the desired result. □

3. A Special Case

In this section, we prove a fixed point theorem in metric spaces satisfying Condition (ℓ) for some fixed ℓ.

Definition 3.1. *Let (X, d) be a metric space, let $x, y \in X$, $\{x_n\} \in \mathrm{CauS}(X)$ and $\varepsilon > 0$.*

- *A finite sequence $\{y_1, \cdots, y_m\}$ in X is said to be ε-chain linking x and y [10] if $y_1 = x$, $y_m = y$ and $d(y_j, y_{j+1}) < \varepsilon$ for any $j \in \{1, \cdots, m-1\}$.*
- *(x, y) is said to be ε-chainable if there exists ε-chain linking x and y.*
- *$(x, \{x_n\})$ is said to be ε-chainable if there exists $\nu \in \mathbb{N}$ such that (x, x_n) is ε-chainable for any $n \ge \nu$.*

In this section, we let (X, d) be a metric space. For $\varepsilon > 0$, we define a function ℓ_ε from $X \times \mathrm{CauS}(X)$ into $[0, \infty]$ by

$$\ell_\varepsilon(x, \{x_n\}) = \limsup_{n\to\infty} \inf\Big\{ \sum_{j=1}^{m-1} d(y_j, y_{j+1}) : \{y_1, \cdots, y_m\} \text{ is } \varepsilon\text{-chain linking } x \text{ and } x_n \Big\}, \tag{7}$$

where $\inf \varnothing = \infty$. We also define a function ℓ from $X \times \mathrm{CauS}(X)$ into $[0, \infty]$ by

$$\ell(x, \{x_n\}) = \sup\Big\{ \ell_\varepsilon(x, \{x_n\}) : \varepsilon > 0 \Big\} \tag{8}$$

for $(x, \{x_n\}) \in X \times \mathrm{CauS}(X)$.

Lemma 3.2. *Let (X, d) be a metric space and let ℓ be a function defined by (8). Let $\{x_n\} \in \mathrm{CauS}(X)$ and define a function ρ by (3). Then the following hold:*

(i) *For any $x \in X$ and $\varepsilon, \varepsilon' \in (0, \infty)$ with $\varepsilon < \varepsilon'$,*

$$\rho(x) \le \ell_{\varepsilon'}(x, \{x_n\}) \le \ell_\varepsilon(x, \{x_n\}) \le \ell(x, \{x_n\})$$

holds.

(ii) *For any $w \in X$, $\{x_n\}$ converges to w iff $\ell(w, \{x_n\}) = 0$.*

(iii) *If X satisfies Condition (ℓ), then $\{x_n\}$ does not converge iff $0 < \inf\{\ell(x, \{x_n\}) : x \in X\} < \infty$.*

Proof. (i) is obvious. In order to show (ii), we fix $w \in X$. It follows from (i) that $\ell(w, \{x_n\}) = 0$ implies that $\{x_n\}$ converges to w. In order to show the converse implication, we assume that $\{x_n\}$ converges to w. For any $\varepsilon > 0$, a two-length sequence $\{w, x_n\}$ is ε-chain linking w and x_n for sufficiently large $n \in \mathbb{N}$. So, $\ell_\varepsilon(w, \{x_n\}) = 0$ holds and hence $\ell(w, \{x_n\}) = 0$ holds. We shall show (iii). We put

$$t = \inf\{\ell(x, \{x_n\}) : x \in X\}.$$

Since X satisfies Condition (ℓ), there exists $w \in X$ such that $\ell(w, \{x_n\}) < \infty$ and $(1/2)\,\ell(w, \{x_n\}) < \ell(x, \{x_n\})$ for any $x \in X \setminus \{w\}$. So $t < \infty$ holds. We assume $t = 0$. Then $\ell(w, \{x_n\}) = 0$ holds. So, by (ii), $\{x_n\}$ converges to w. Conversely, we assume that $\{x_n\}$ converges to some $x \in X$. Then by (ii), we have $\ell(x, \{x_n\}) = 0$ and hence $t = 0$. We have shown (iii). □

Theorem 3.3. *Let (X, d) be a metric space and let ℓ be a function defined by* (8). *Assume that X satisfies Condition (ℓ). Let T be a contraction on X. Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for any $x \in X$.*

Proof. There exists $r \in (0, 1)$ satisfying (1) for any $x, y \in X$. Fix $v \in X$. By Lemma 3.2, (ii) of Theorem 2.3 holds. In order to show (iii) of Theorem 2.3, we fix $x \in X$. We consider the following two cases:

- $\ell(x, \{T^n v\}) = \infty$
- $\ell(x, \{T^n v\}) < \infty$

In the first case,

$$\ell(Tx, \{T^n v\}) \le \infty = r\,\ell(Tx, \{T^n v\})$$

holds. In the second case, we fix $\varepsilon > 0$. Then from the definition of ℓ, $(x, \{T^n v\})$ is ε-chainable. So, there exists $\nu \in \mathbb{N}$ such that $(x, T^n v)$ is ε-chainable for any $n \ge \nu$. Fix $n \ge \nu$ and let $\{y_1, y_2, \cdots, y_m\}$ be an arbitrary ε-chain linking x and $T^n v$. Then since T is a contraction, $\{Ty_1, Ty_2, \cdots, Ty_m\}$ is $(r\,\varepsilon)$-chain linking Tx and $T^{n+1} v$ and hence is ε-chain. Also

$$\sum_{j=1}^{m-1} d(Ty_j, Ty_{j+1}) \le r \sum_{j=1}^{m-1} d(y_j, y_{j+1})$$

is obvious. Since $\{y_1, y_2, \cdots, y_m\}$ is arbitrary, we obtain

$$\ell_\varepsilon(Tx, \{T^n v\}) \le r\,\ell_\varepsilon(x, \{T^n v\}).$$

Since $\varepsilon > 0$ is arbitrary, we obtain (2). We have shown (iii) of Theorem 2.3 holds. So by Theorem 2.3, we obtain the desired result. □

Using Theorem 3.3, we prove Theorem 1.3.

Proof of Theorem 1.3. Define a function ℓ by (8). Let $x \in X$ and $\{x_n\} \in \mathrm{CauS}(X)$. We consider the following three cases:

- $\{x_n\}$ converges to 0.
- $\{x_n\}$ converges to some $w \in X \setminus \{0\}$.
- $\{x_n\}$ does not converge.

In the first case, it is obvious that $\ell(x, \{x_n\}) = d(x, 0)$ holds. So,

$$\ell(0, \{x_n\}) = 0 < \infty$$

and

$$(1/2)\,\ell(0, \{x_n\}) = 0 < d(x, 0) = \ell(x, \{x_n\})$$

for any $x \in X \setminus \{0\}$. In the second case, we choose $k \in \mathbb{N}$ satisfying $w \in L_k$. It is obvious that

$$\ell(x, \{x_n\}) = \begin{cases} d(x, w) & \text{if } x \in L_k \\ d(x, 0) + d(0, w) & \text{if } x \notin L_k \end{cases}$$

holds. So,

$$\ell(w, \{x_n\}) = 0 < \infty$$

and

$$(1/2)\, \ell(w, \{x_n\}) = 0 < d(x, w) \leq \ell(x, \{x_n\})$$

for any $x \in X \setminus \{w\}$. In the third case, there exists a unique element w of the completion of X satisfying $\lim_n d(x_n, w) = 0$. It is not difficult to show

$$\ell(x, \{x_n\}) = d(x, 0) + d(0, w) = d(x, 0) + \lim_{n \to \infty} d(0, x_n)$$

for any $x \in X$. So, putting $t = \lim_n d(0, x_n)$, we have

$$\ell(0, \{x_n\}) = t < \infty$$

and

$$(1/2)\, \ell(0, \{x_n\}) = t/2 < t < d(x, 0) + t = \ell(x, \{x_n\})$$

for any $x \in X \setminus \{w\}$. Therefore X satisfies Condition (ℓ). So by Theorem 3.3, T has a unique fixed point. □

Remark 3.4. *Therefore we can tell that Theorem 2.3 is a generalization of both Theorems 1.1 and 1.3,*

4. Not Possessing BFPP

In this section, we give a sufficient condition on not possessing BFPP. While Theorem 1.4 is of continuous type, the following is of discrete type in some sense.

Theorem 4.1. *Let (X, d) be a metric space and let $\{x_n\} \in \mathrm{CauS}(X)$. Assume that for any $x \in X$, there exists $\varepsilon > 0$ such that $(x, \{x_n\})$ is not ε-chainable. Then X does not have BFPP.*

Proof. Define a function ρ from X into $[0, \infty)$ by (3). From the assumption, $\{x_n\}$ does not converge. So, $\rho(x) > 0$ for any $x \in X$. Taking a subsequence, without loss of generality, we may assume $\rho(x_{n+1}) < \rho(x_n)/3$ for any $n \in \mathbb{N}$. Define a function h from X into $[0, \infty)$ by

$$h(x) = \inf \left\{ \varepsilon \in (0, \infty) : (x, \{x_n\}) \text{ is } \varepsilon\text{-chainable} \right\}$$

for $x \in X$. From the assumption, $h(x) > 0$ holds. Also, since $(x, \{x_n\})$ is ε-chainable for any $\varepsilon > \rho(x)$, $h(x) \leq \rho(x)$ holds for any $x \in X$. Define a contraction T on X by

$$Tx = \begin{cases} x_2 & \text{if } \rho(x_1) \leq h(x) \\ x_j & \text{if } \rho(x_{j-1}) \leq h(x) < \rho(x_{j-2}) \text{ for some } j \in \mathbb{N} \text{ with } j \geq 3 \end{cases}$$

for $x \in X$. We shall show that T is a contraction. Fix $x, y \in X$ with $x \neq y$ and $h(x) \leq h(y)$. We consider the following two cases:

- $d(x, y) \leq h(x)$
- $d(x, y) > h(x)$

In the first case, for any $\varepsilon > h(x)$, from the definition of h, $(x, \{x_n\})$ is ε-chainable. Since $d(x, y) < \varepsilon$, $(y, \{x_n\})$ is also ε-chainable. We have $h(y) \leq \varepsilon$ and hence $h(x) = h(y)$. So

$$d(Tx, Ty) = 0 \leq (2/3)\, d(x, y).$$

In the second case, let $i, j \geq 2$ satisfy $Tx = x_i$ and $Ty = x_j$. For any $\varepsilon > d(x, y)$, since $h(x) < \varepsilon$, $(x, \{x_n\})$ is ε-chainable. Since $d(x, y) < \varepsilon$, $(y, \{x_n\})$ is also ε-chainable. Hence we obtain $h(y) \leq d(x, y)$. We have

$$\begin{aligned} d(Tx, Ty) &= d(x_i, x_j) \\ &\leq \rho(x_i) + \rho(x_j) \\ &< (1/3)\big(\rho(x_{i-1}) + \rho(x_{j-1})\big) \\ &\leq (1/3)\big(h(x) + h(y)\big) \\ &\leq (2/3)\, d(x, y). \end{aligned}$$

We have shown that T is a contraction. It is obvious that T does not have a fixed point. Thus, X does not have BFPP. □

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