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Coefficient Estimates for Certain Subclass of Analytic Functions Defined by Subordination

Nirupam Ghosh[a], **A. Vasudevarao**[a]

[a] *Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur-721 302, West Bengal, India.*

Abstract. In this article we determine the coefficient bounds for functions in certain subclasses of analytic functions defined by subordination which are related to the well-known classes of starlike and convex functions. The main results deal with some open problems proposed by Q.H. Xu *et al.*([20], [21]). An application of Jack lemma for certain subclass of starlike functions has been discussed.

To the memory of Professor Lj. Ćirić (1935–2016)

1. introduction

Let $\mathcal{A}$ denote the family of analytic functions f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f'(0) - 1$. If $f \in \mathcal{A}$ then f has the following representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

A function f is said to be univalent in a domain $\Omega \subsetneq \mathbb{C}$ if it is injective in Ω. Let $\mathcal{S}$ denote the class of univalent functions in $\mathcal{A}$. A function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*(\alpha)$, called starlike functions of order α, if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad \text{for} \quad z \in \mathbb{D}$$

and in the class $\mathcal{C}(\alpha)$, called convex functions of order α, if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad \text{for} \quad z \in \mathbb{D}.$$

Clearly the classes $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} := \mathcal{C}(0)$ are the well- known classes of starlike and convex functions respectively. It is well -known that $\mathcal{C} \subsetneq \mathcal{S}^* \subsetneq \mathcal{S}$. A function $f \in \mathcal{A}$ is in the class $\mathcal{SP}(\alpha)$, called α-Spiral functions, if

$$\mathrm{Re}\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) > 0 \quad \text{for} \quad z \in \mathbb{D}.$$

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Email addresses: nirupamghoshmath@gmail.com (Nirupam Ghosh), alluvasu@maths.iitkgp.ernet.in (A. Vasudevarao)

The class $\mathcal{SP}(\alpha)$ has been introduced by Špaček [17] in 1933.

Let f and g be analytic functions in the unit disk $\mathbb{D}$. A function f is said to be subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$ such that $f(z) = g(\omega(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

For $A, B \in \mathbb{C}$ with $|B| \leq 1$, let $\mathcal{S}^*[A,B]$ denote the class of functions $f \in \mathcal{A}$ which satisfy the following subordination relation

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad \text{for} \quad z \in \mathbb{D}.$$

Without loss of generality we may assume that B is a real. In view of $S^*[A,B] = S^*[-A,-B]$, we can consider $-1 \leq B \leq 0$. For particular choice of parameters A and B, we can obtain $\mathcal{S}^* := \mathcal{S}^*[1,-1]$ and $\mathcal{S}^*(\alpha) := \mathcal{S}^*[1-2\alpha,-1]$. If we choose $A = e^{-2i\alpha}$ and $B = -1$ then $\mathcal{SP}(\alpha) := S^*[e^{-2i\alpha},-1]$.

Nasr and Aouf [10–12] and Wiatrowski [22] extended the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ by introducing $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$, the class of starlike functions of complex order γ and the class of convex functions of complex order γ respectively.

More preciously, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\gamma)$, if it satisfies the following condition

$$\operatorname{Re}\left(1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right) > 0 \quad \text{for} \quad z \in \mathbb{D} \quad \text{and} \quad \gamma \in \mathbb{C} \setminus \{0\}.$$

Similarly, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}(\gamma)$, if it satisfies the following condition

$$\operatorname{Re}\left(1 + \frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)}\right)\right) > 0 \quad \text{for} \quad z \in \mathbb{D} \quad \text{and} \quad \gamma \in \mathbb{C} \setminus \{0\}.$$

The function classes $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$ have been extensively studied by many authors (for example, see [2–6]). For fixed $\beta > 1$, the classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ are defined by

$$\begin{aligned}
\mathcal{M}(\beta) &:= \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta \quad \text{for } z \in \mathbb{D} \right\} \quad \text{and} \\
\mathcal{N}(\beta) &:= \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \beta \quad \text{for } z \in \mathbb{D} \right\}.
\end{aligned}$$

If we choose $\gamma = (1-\beta)$ then the class $\mathcal{M}(\beta) := \mathcal{S}^*(1-\beta)$ and $\mathcal{N}(\beta) := \mathcal{C}(1-\beta)$. The classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ have been extensively discussed by Obradovic *et al.* [13] and Firoz Ali and Vasudevarao [1].

In 2007, Altintas *et al.* [7] introduced the classes $\mathcal{S}c(\gamma,\lambda,\beta)$ and $\mathcal{B}(\gamma,\lambda,\beta,\mu)$. A function $f \in \mathcal{A}$ is in the class $\mathcal{S}c(\gamma,\lambda,\beta)$ for $\gamma \in \mathbb{C} \setminus \{0\}, 0 \leq \lambda \leq 1$ and $0 \leq \beta < 1$ if it satisfies the following condition

$$\operatorname{Re}\left(1 + \frac{1}{\gamma}\left(\frac{z(\lambda zf'(z) + (1-\lambda)f(z))'}{(\lambda zf'(z) + (1-\lambda)f(z))} - 1\right)\right) > \beta \quad \text{for} \quad z \in \mathbb{D}.$$

Clearly $\mathcal{S}^*(\gamma) := \mathcal{S}c(\gamma,0,0)$ and $\mathcal{C}(\gamma) := \mathcal{S}c(\gamma,1,0)$. A function $w = f(z)$ belongs to $\mathcal{A}$ is said to be in the class $\mathcal{B}(\gamma,\lambda,\beta,\mu)$ if it satisfies the following non-homogeneous Cauchy-Euler differential equation

$$z^2\frac{d^2w}{dz^2} + \mu z\frac{dw}{dz} + \mu(\mu+1)w = (\mu+1)(\mu+2)g(z),$$

where $g \in \mathcal{S}c(\gamma,\lambda,\beta)$ and $\mu \in \mathbb{R} \setminus (-\infty,-1]$. In [7], the authors obtained the coefficient bounds for functions in the classes $\mathcal{S}c(\gamma,\lambda,\beta)$ and $\mathcal{B}(\gamma,\lambda,\beta,\mu)$ but the results were not sharp.

In 2011, Srivastava *et al.* [18] introduced the classes $\mathcal{S}(\lambda,\gamma,A,B)$ and $\mathcal{K}(\lambda,\gamma,A,B,m,\mu)$. A function $f \in \mathcal{A}$ is in the class $\mathcal{S}(\lambda,\gamma,A,B)$ if it satisfies the following subordination condition

$$1 + \frac{1}{\gamma}\left(\frac{z(\lambda zf'(z) + (1-\lambda)f(z))'}{(\lambda zf'(z) + (1-\lambda)f(z))} - 1\right) \prec \frac{1+Az}{1+Bz} \quad \text{for} \quad z \in \mathbb{D},$$

where $\gamma \in \mathbb{C} \setminus \{0\}, 0 \le \lambda \le 1$ and $-1 \le B < A \le 1$. Similarly, a function $w = f(z)$ belongs to $\mathcal{A}$ is said to be in the class $\mathcal{K}(\lambda, \gamma, A, B, m, \mu)$ if it satisfies the following non-homogeneous Cauchy-Euler type differential equation of order m

$$z^m \frac{d^m w}{dz^m} + \binom{m}{1}(\mu + m - 1) z^{m-1} \frac{d^{m-1} w}{dz^{m-1}} + \cdots + \binom{m}{m} \prod_{j=0}^{m-1} (\mu + j) w = g(z) \prod_{j=0}^{m-1} (\mu + j + 1), \tag{2}$$

where $g \in \mathcal{S}(\lambda, \gamma, A, B)$, $\mu \in \mathbb{R} \setminus (-\infty, -1]$ and $m \in \mathbb{N} \setminus \{1\}$. For particular choice of the parameters A and B, we obtain $\mathcal{S}c(\gamma, \lambda, \beta) := \mathcal{S}(\lambda, \gamma, 1 - 2\beta, -1)$, $\mathcal{S}^*(\gamma) := \mathcal{S}(0, \gamma, 1, -1)$ and $C(\gamma) := \mathcal{S}(1, \gamma, 1, -1)$. The coefficient bounds for functions in the classes $\mathcal{S}(\lambda, \gamma, A, B)$ and $\mathcal{K}(\lambda, \gamma, A, B, m, \mu)$ have been investigated by Srivastava *et al.* [18] but the results are not sharp. Recently, Q-H Xu *et al.* [20] obtained the following sharp coefficient bounds for functions in classes $\mathcal{S}(\lambda, \gamma, A, B)$ and $\mathcal{K}(\lambda, \gamma, A, B, m, \mu)$ with some restriction on the parameters.

Theorem A. *[20] Let $f \in \mathcal{S}(\lambda, \gamma, A, B)$ be given by (1), where $\gamma \in \mathbb{C} \setminus \{0\}, 0 \le \lambda \le 1$ and $-1 \le B < A \le 1$. If*

$$|\gamma(A - B) - B(n - 2)| \ge (n - 2),$$

then

$$|a_n| \le \frac{\prod_{j=0}^{n-2} |(A - B)\gamma - jB|}{(1 + \lambda(n - 1))(n - 1)!}, \quad n \in \mathbb{N} \setminus \{1\} \tag{3}$$

and the estimates in (3) are sharp.

Theorem B. *[20] Let $f \in \mathcal{K}(\lambda, \gamma, A, B, m, \mu)$ be given by (1), where $\gamma \in \mathbb{C} \setminus \{0\}, 0 \le \lambda \le 1, -1 \le B < A \le 1$, $\mu \in \mathbb{R} \setminus (-\infty, -1]$ and $m \in \mathbb{N} \setminus \{1\}$. If*

$$|\gamma(A - B) - B(n - 2)| \ge (n - 2),$$

then

$$|a_n| \le \frac{\prod_{j=0}^{n-2} |(A - B)\gamma - jB| \prod_{j=0}^{m-1} (\mu + j + 1)}{(1 + \lambda(n - 1))(n - 1)! \prod_{j=0}^{m-1} (\mu + j + n)}, \quad n, m \in \mathbb{N} \setminus \{1\} \tag{4}$$

and the estimates in (4) are sharp.

In 2013, Xu *et al.* [20] proposed the following two problems concerning the coefficient bounds for functions in the class $\mathcal{S}(\lambda, \gamma, A, B)$.

Problem 1.1. *If the function $f \in \mathcal{S}(\lambda, \gamma, A, B)$ is given by (1) with $\gamma \in \mathbb{C} \setminus \{0\}, 0 \le \lambda \le 1$ and $-1 \le B < A \le 1$ then prove or disprove that*

$$|a_n| \le \frac{\prod_{j=0}^{n-2} |(A - B)\gamma - jB|}{(1 + \lambda(n - 1))(n - 1)!}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{5}$$

Problem 1.2. *If the coefficient estimates in (5) do hold true then prove or disprove that these estimates are sharp.*

In 2013, Xu *et al.* [21] considered the class $\mathcal{S}^\beta(A, B)$ by the condition that a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^\beta(A, B)$ if it satisfies

$$(1 + i \tan \beta) \frac{z f'(z)}{f(z)} - i \tan \beta \prec \frac{1 + Az}{1 + Bz} \quad \text{for} \quad z \in \mathbb{D},$$

where $-\pi/2 < \beta < \pi/2$ and $-1 \le B < A \le 1$ and obtained the following coefficient bounds for functions in this class.

Theorem C. *[21] Let $f \in \mathcal{S}^{\beta}(A,B)$ be given by (1) with $-\pi/2 < \beta < \pi/2$, $-1 \le B < A \le 1$ and $n \in \mathbb{N} \setminus \{1\}$. Suppose also that*

$$(A-(n-1)B)^2\cos^2\beta + (n-2)^2(B^2\sin^2\beta - 1) \ge 0. \tag{6}$$

Then

$$|a_n| \le \prod_{j=0}^{n-2}\left(\frac{|(A-B)e^{-i\beta}\cos\beta - jB|}{j+1}\right), \quad n \in \mathbb{N}\setminus\{1\} \tag{7}$$

and the estimates in (7) are sharp .

We note that Theorem C is proved under the additional assumption (6). In the same paper the authors proposed the following two problems concerning the coefficient bounds for functions in class $\mathcal{S}^{\beta}(A,B)$ without assuming the additional condition (6).

Problem 1.3. *If the function $f \in \mathcal{S}^{\beta}(A,B)$ is given by (1) with $-\pi/2 < \beta < \pi/2$ and $-1 \le B < A \le 1$, then prove or disprove that*

$$|a_n| \le \prod_{j=0}^{n-2}\left(\frac{|(A-B)e^{-i\beta}\cos\beta - jB|}{j+1}\right), \quad n \in \mathbb{N}\setminus\{1\}. \tag{8}$$

Problem 1.4. *If the coefficient estimates in (8) do hold true then prove or disprove that these estimates are sharp.*

It is interesting to note that if we choose $\lambda = 0$ and $\gamma = 1/(1 + i\tan\beta)$ then the class $\mathcal{S}(\lambda,\gamma,A,B)$ reduces to $\mathcal{S}^{\beta}(A,B)$. Hence it is sufficient to study Problem 1.1 and Problem 1.2 for functions in the class $\mathcal{S}(\lambda,\gamma,A,B)$.

The problem of coefficient estimates is one of the most exciting problem in the theory of univalent functions. For $f \in \mathcal{S}$ of the form (1), it was proved that $|a_2| \le 2$ and proposed a conjecture $|a_n| \le n$ for $n \ge 3$ by Bieberbach in 1916. This celebrated conjecture was proved affirmatively by Branges in 1984. This motivates us to determine the coefficient bounds for functions in some subclasses of analytic functions which are defined by the subordination and these classes are related to the well-known classes of starlike and convex functions.

The main aim of this paper is to attempt the aforementioned problems in much detailed. In fact, the main results of this paper deal with some open problems proposed by Q.H. Xu *et al.*([20], [21]).

Before proving our main results, we recall the following lemma due to Xu *et al.* [20].

Lemma 1.5. *[20] Let the parameters A,B λ,γ and m satisfy $\gamma \in \mathbb{C}\setminus\{0\}, 0 \le \lambda \le 1, -1 \le B < A \le 1$ and $m \in \mathbb{N}\setminus\{1\}$. If $|\gamma(A-B) - B(m-2)| \ge (m-2)$, then*

$$|\gamma|^2(A-B)^2 + \sum_{k=2}^{m-1}\left[\frac{\left||\gamma(A-B)-B(k-1)|^2-(k-1)^2\right|}{((k-1)!)^2}\right]\prod_{j=0}^{k-2}|\gamma(A-B)-jB|^2$$
$$= \frac{\prod_{j=0}^{m-2}|\gamma(A-B)-Bj|^2}{((m-2)!)^2}.$$

2. Coefficient estimates

In this section, we will estimate the modulus of the coefficients of function of the form (1), which belong to the class of $\mathcal{S}(\lambda,\gamma,A,B)$ and $\mathcal{K}(\lambda,\gamma,A,B,m,\mu)$. Moreover, the inequalities obtained will be examined in terms of sharpness.

Theorem 2.1. *Let $f \in \mathcal{S}(\lambda, \gamma, A, B)$ be of the form (1), where $\gamma \in \mathbb{C} \setminus \{0\}, 0 \leq \lambda \leq 1, -1 \leq B < A \leq 1$ and $n \in \mathbb{N} \setminus \{1\}$ be fixed and define $A_k = |\gamma(A - B) - B(k - 1)| - (k - 1)$.*

(i) If $A_2 \leq 0$, then

$$|a_n| \leq \frac{|\gamma|(A - B)}{(n - 1)(1 + \lambda(n - 1))}. \tag{9}$$

(ii) If $A_{n-1} \geq 0$, then

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} |\gamma(A - B) - jB|}{(n - 1)!(1 + \lambda(n - 1))}. \tag{10}$$

(iii) If $A_k \geq 0$ and $A_{k+1} \leq 0$ for $k = 2, 3, \ldots, n - 2$, then

$$|a_n| \leq \frac{\prod_{j=0}^{k-1} |\gamma(A - B) - jB|}{(k - 1)!(n - 1)(1 + \lambda(n - 1))}. \tag{11}$$

The estimates in (9) and (10) are sharp.

Proof. The proof of part (ii) can be found in [20]. But for the sake of completeness of the result, we include it here. Let $f \in \mathcal{S}(\lambda, \gamma, A, B)$. Then there exists an analytic function $\omega(z)$ in $\mathbb{D}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that

$$1 + \frac{1}{\gamma}\left(\frac{z(\lambda z f'(z) + (1 - \lambda)f(z))'}{(\lambda z f'(z) + (1 - \lambda)f(z))} - 1\right) = \frac{1 + A\omega(z)}{1 + B\omega(z)}. \tag{12}$$

Using the series expansion (1) of $f(z)$ in (12) and then after simplification we obtain

$$\sum_{k=2}^{\infty}(k - 1)(1 + \lambda(k - 1))a_k z^k = \left(\gamma(A - B)z + \sum_{k=2}^{\infty}(\gamma(A - B) - B(k - 1))(1 + \lambda(k - 1))a_k z^k\right)\omega(z)$$

which can be written as

$$\sum_{k=2}^{n}(k - 1)(1 + \lambda(k - 1))a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k = \left(\gamma(A - B)z + \sum_{k=2}^{n-1}(\gamma(A - B) - B(k - 1))(1 + \lambda(k - 1))a_k z^k\right)\omega(z)$$

for certain coefficients b_k. Since $|\omega(z)| < 1$, an application of Parseval's theorem gives

$$\sum_{k=2}^{n}(k - 1)^2(1 + \lambda(k - 1))^2|a_k|^2 + \sum_{k=n+1}^{\infty} |b_k|^2 \leq |\gamma|^2(A - B)^2 + \sum_{k=2}^{n-1}\left(|\gamma(A - B) - B(k - 1)|^2\right)(1 + \lambda(k - 1))^2|a_k|^2$$

and therefore

$$(n - 1)^2(1 + \lambda(n - 1)^2)|a_n|^2 \leq |\gamma|^2(A - B)^2 + \sum_{k=2}^{n-1}\left(|\gamma(A - B) - B(k - 1)|^2 - (k - 1)^2\right)(1 + \lambda(k - 1))^2|a_k|^2. \tag{13}$$

For $n = 2$, it follows from (13) that

$$|a_2| \leq \frac{|\gamma|(A - B)}{1 + \lambda}. \tag{14}$$

Note that if $A_k \geq 0$ then $A_{k-1} \geq 0$ for $k = 2, 3, \ldots$, because

$$|\gamma(A - B) - (k - 2)B| \geq |\gamma(A - B) - (k - 1)B| - |B| \geq (k - 1) - 1 = k - 2.$$

Again, if $A_k \le 0$ then $A_{k+1} \le 0$ for $k = 2, 3, \ldots$, because

$$|\gamma(A-B) - kB| \le |\gamma(A-B) - (k-1)B| + |B| \le (k-1) + 1 = k.$$

If $A_2 \le 0$ then from the above discussion we can conclude that $A_k \le 0$ for all $k > 2$. It follows from (13) that

$$(n-1)^2(1+\lambda(n-1)^2)|a_n|^2 \le |\gamma|^2(A-B)^2$$

and consequently

$$|a_n| \le \frac{|\gamma|(A-B)}{(n-1)(1+\lambda(n-1))}. \tag{15}$$

Equality in (15) is attained for the functions $f_n(z)$ where $f_n(z)$ satisfies the following differential equation

$$\lambda z f_n'(z) + (1-\lambda) f_n(z) = z(1 + Bz^{n-1})^{\frac{\gamma(A-B)}{B(n-1)}}.$$

Next, let $A_{n-1} \ge 0$. Then from the above discussion we have $A_2, A_3, A_4, \ldots, A_{n-2} \ge 0$. From (14) it is clear that (10) is true for $n = 2$. Suppose that (10) is true for $k = 2, 3, \ldots, n-1$. Then using the induction hypothesis, it follows from (13) that

$$\begin{aligned}
&(n-1)^2(1+\lambda(n-1))^2|a_n|^2 \\
&\le |\gamma|^2(A-B)^2 + \sum_{k=2}^{n-1} \left(\left||\gamma(A-B) - B(k-1)|^2 - (k-1)^2\right|\right)(1+\lambda(k-1))^2|a_k|^2 \\
&\le |\gamma|^2(A-B)^2 + \sum_{k=2}^{n-1} \left(\left||\gamma(A-B) - B(k-1)|^2 - (k-1)^2\right|\right)(1+\lambda(k-1))^2 \frac{\prod_{j=0}^{k-2} |\gamma(A-B) - jB|^2}{((k-1)!)^2(1+\lambda(k-1))^2}.
\end{aligned}$$

An application of Lemma 1.5 shows that

$$(n-1)^2(1+\lambda(n-1))^2|a_n|^2 \le \frac{\prod_{j=0}^{n-2} |\gamma(A-B) - jB|^2}{((n-2)!)^2}$$

and consequently,

$$|a_n| \le \frac{\prod_{j=0}^{n-2} |\gamma(A-B) - jB|}{((n-1)!)(1+\lambda(n-1))}.$$

By the mathematical induction, (10) is true for all $n \ge 2$. The equality in (10) is attained for the following function

$$f(z) = \begin{cases}
\frac{\lambda-1}{\lambda} \int_0^z \frac{t^{\frac{\lambda-1}{\lambda}}}{\lambda(1+Bt)^{\frac{B-A}{B}\gamma}}\, dt & \text{for} \quad B \ne 0, \lambda \ne 0 \\
\frac{z}{(1+Bz)^{\frac{B-A}{B}\gamma}} & \text{for} \quad B \ne 0, \lambda = 0 \\
\frac{1}{\lambda} \int_0^z t^{\frac{1-\lambda}{\lambda}} e^{A\gamma t}\, dt & \text{for} \quad B = 0, \lambda \ne 0 \\
z e^{A\gamma z} & \text{for} \quad B = 0, \lambda = 0.
\end{cases}$$

Now if we assume that $A_k \geq 0$ and $A_{k+1} \leq 0$ for $k = 2, 3, \ldots, n-2$. Then $A_2, A_3, A_4, \ldots, A_{k-1} \geq 0$ and $A_{k+2}, A_{k+3}, \ldots, A_{n-2} \leq 0$. Using (10) and Lemma 1.5 in (13), we obtain

$$\begin{aligned}
&(n-1)^2(1+\lambda(n-1)^2)|a_n|^2 \\
&\leq |\gamma|^2(A-B)^2 + \sum_{l=2}^{k}\Big(\big||\gamma(A-B)-B(l-1)|^2-(l-1)^2\big|\Big)(1+\lambda(l-1))^2|a_l|^2 \\
&\leq |\gamma|^2(A-B)^2 + \sum_{l=2}^{k}\Big(\big||\gamma(A-B)-B(l-1)|^2-(l-1)^2\big|\Big)\frac{\prod_{j=0}^{l-2}|\gamma(A-B)-jB|^2}{((l-1)!)^2} \\
&= \frac{\prod_{j=0}^{k-1}|\gamma(A-B)-jB|^2}{((k-1)!)^2},
\end{aligned}$$

from which (11) follows. □

Theorem 2.2. *Let $f \in \mathcal{K}(\lambda, \gamma, A, B, m, \mu)$ be of the form (1) and $\gamma \in \mathbb{C} \setminus \{0\}, 0 \leq \lambda \leq 1, -1 \leq B < A \leq 1, m \in \mathbb{N} \setminus \{1\}$ and $\mu \in \mathbb{R} \setminus (-\infty, -1]$. Define $A_k = |\gamma(A-B) - B(k-1)| - (k-1)$.*

(i) If $A_2 \leq 0$, then

$$|a_n| \leq \frac{|\gamma|(A-B)}{(n-1)(1+(n-1)\lambda)}\frac{\prod_{j=0}^{m-1}(\mu+j+1)}{\prod_{j=0}^{m-1}(\mu+j+n)}. \tag{16}$$

(ii) If $A_{n-1} \geq 0$, then

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}|\gamma(A-B)-jB|}{(n-1)!(1+\lambda(n-1))}\frac{\prod_{j=0}^{m-1}(\mu+j+1)}{\prod_{j=0}^{m-1}(\mu+j+n)}. \tag{17}$$

(iii) If $A_k \geq 0$ and $A_{k+1} \leq 0$ for $k = 2, 3, \ldots, n-2$, then

$$|a_n| \leq \frac{\prod_{j=0}^{k-1}|\gamma(A-B)-jB|}{(n-1)(k-1)!(1+\lambda(n-1))}\frac{\prod_{j=0}^{m-1}(\mu+j+1)}{\prod_{j=0}^{m-1}(\mu+j+n)}. \tag{18}$$

The estimates in (16) and (17) are sharp.

Proof. Let $f \in \mathcal{K}(\lambda, \gamma, A, B, m, \mu)$ be of the form (1). Then there exists $g \in \mathcal{S}(\lambda, \gamma, A, B)$ of the form $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ such that (2) holds. By comparing the coefficients on both sides of (2), we obtain

$$a_n = \left(\frac{\prod_{j=0}^{m-1}(\mu+j+1)}{\prod_{j=0}^{m-1}(\mu+j+n)}\right)b_n,$$

where $m, n \in \mathbb{N} \setminus \{1\}$ and $\mu \in \mathbb{R} \setminus (-\infty, -1]$. Then the desired results follow from Theorem 2.1. The sharpness of (16) and (17) easily follow from the sharpness of (9) and (10). □

Corollary 2.3. *Let $f \in \mathcal{S}c(\gamma, \lambda, \beta)$ be given by (1).*

(i) If $|2\gamma(1-\beta)+1| \leq 1$, then

$$|a_n| \leq \frac{2|\gamma|(1-\beta)}{(n-1)(1+(n-1)\lambda)}. \tag{19}$$

The equality in (19) occurs for the solution of equation

$$\lambda z f_n'(z) + (1-\lambda)f_n(z) = z(1-z^{n-1})^{\frac{-2\gamma(1-\beta)}{(n-1)}}.$$

(ii) *If* $|2\gamma(1-\beta)+(n-2)| \geq (n-2)$, *then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}|2\gamma(1-\beta)+j|}{(n-1)!(1+(n-1)\lambda)}. \tag{20}$$

The inequality (20) is sharp.

(iii) *If* $|2\gamma(1-\beta)+(k-1)| \geq (k-1)$, *then*

$$|a_n| \leq \frac{\prod_{j=0}^{k-1}|2\gamma(1-\beta)+j|}{(n-1)(k-1)!(1+(n-1)\lambda)}.$$

Corollary 2.4. *Let* $f \in \mathcal{B}(\gamma,\lambda,\beta,\mu)$ *be given by (1).*

(i) *If* $|2\gamma(1-\beta)+1| \leq 1$, *then*

$$|a_n| \leq \frac{2|\gamma|(1-\beta)}{(n-1)(1+(n-1)\lambda)}\frac{(\mu+1)(\mu+2)}{(\mu+n)(\mu+n+1)}. \tag{21}$$

The inequality (21) is sharp.

(ii) *If* $|2\gamma(1-\beta)+(n-2)| \geq (n-2)$, *then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}|2\gamma(1-\beta)+j|}{(n-1)!(1+(n-1)\lambda)}\frac{(\mu+1)(\mu+2)}{(\mu+n)(\mu+n+1)}. \tag{22}$$

The inequality (22) is sharp.

(iii) *If* $|2\gamma(1-\beta)+(k-1)| \geq (k-1)$, *then*

$$|a_n| \leq \frac{\prod_{j=0}^{k-1}|2\gamma(1-\beta)+j|}{(n-1)(k-1)!(1+(n-1)\lambda)}\frac{(\mu+1)(\mu+2)}{(\mu+n)(\mu+n+1)}.$$

The following two results give the sharp coefficient bounds for functions in the classes $\mathcal{S}^*(\gamma)$ and $C(\gamma)$ under some assumptions.

Corollary 2.5. *Let* $f \in \mathcal{S}^*(\gamma)$ *be given by (1).*

(i) *If* $|2\gamma+1| \leq 1$, *then*

$$|a_n| \leq \frac{2|\gamma|}{n-1}. \tag{23}$$

The equality in (23) occurs for the functions $f_n(z)$ *where* $f_n(z)$ *is defined by*

$$f_n(z) = z(1-z^{n-1})^{\frac{-2\gamma}{(n-1)}}.$$

(ii) *If* $|2\gamma+(n-2)| \geq (n-2)$, *then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}|2\gamma+j|}{(n-1)!}. \tag{24}$$

The inequality (24) is sharp for the function $f(z)$ *where* $f(z)$ *is defined by*

$$f(z) = \frac{z}{(1-z)^{2\gamma}}.$$

Corollary 2.6. *Let* $f \in C(\gamma)$ *be given by (1).*

(i) If $|2\gamma + 1| \le 1$, *then*

$$|a_n| \le \frac{2|\gamma|}{n(n-1)}. \tag{25}$$

The equality in (25) occurs for the functions $f_n(z)$ *where* $f_n(z)$ *is defined by*

$$f_n'(z) = (1 - z^{n-1})^{\frac{-2\gamma}{(n-1)}}.$$

(ii) If $|2\gamma + (n-2)| \ge (n-2)$, *then*

$$|a_n| \le \frac{\prod_{j=0}^{n-2} |2\gamma + j|}{n!}. \tag{26}$$

The inequality (26) is sharp for the function $f(z)$ *where* $f(z)$ *is defined by*

$$f(z) = \int_0^z \frac{dt}{(1-t)^{2\gamma}}.$$

It is interesting to note that if we choose $\gamma = 1 - \beta$ in Corollaries 2.5 and 2.6 then we can obtain the sharp coefficient bounds for functions in the classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$. In fact these results extend the results obtained by Firoz Ali and Vasudevarao [1].

3. Application of Jack Lemma

In 1999, Silverman [16] investigated the class $\mathcal{G}_b$ for $0 < b \le 1$ which involves the quotient of analytic representations of convexity and starlikeness of a function. More precisely, for $0 < b \le 1$, consider the following class

$$\mathcal{G}_b := \left\{ f \in \mathcal{A} : \left| \frac{1 + zf''(z)/f(z)}{zf'(z)/f(z)} - 1 \right| \le b \quad \text{for } z \in \mathbb{D} \right\}.$$

It was proved [16] that $\mathcal{G}_b \subset \mathcal{S}^*(2/(1+\sqrt{1+8b}))$. In 2000, Obradović and Tuneski [14] improved this result by showing $\mathcal{G}_b = S^*[0, -b] \subset S^*(2/(1+\sqrt{1+8b}))$. In 2003, Tuneski [19] found a nice relation among A, B and b so that functions f in the class $\mathcal{G}_b$ also belong to the class $S^*[A, B]$. In this paper, we prove a sufficient condition for function $f \in \mathcal{G}_b$ to be in the class $\mathcal{SP}(\alpha)$.

The following lemma, known as Jack lemma, is helpful in proving for our main results.

Lemma 3.1. *[8] Let* ω *be a non-constant analytic function in the unit disk* $\mathbb{D}$ *with* $\omega(0) = 0$. *If* $|\omega(z)|$ *attains its maximum value on the circle* $|z| = r$ *at the point* z_0 *then* $z_0\omega'(z_0) = k_0\omega(z_0)$ *and* $k_0 \ge 1$.

The recent applications of Jack lemma we refer to [9, 15]. Using the above Jack lemma we prove the following lemma.

Lemma 3.2. *Let* p *be an analytic function in the unit disk* $\mathbb{D}$ *with* $p(0) = 1$ *and* $A = e^{-2i\alpha}$ *be a complex constant with* $|\alpha| < \pi/2$. *If* p *satisfies the following condition*

$$\frac{zp'(z)}{p^2(z)} \prec \frac{(A+1)z}{(1+Az)^2} := h_1(z) \quad \textit{for} \quad z \in \mathbb{D} \tag{27}$$

then

$$p(z) \prec \frac{1+Az}{1-z} \quad \textit{for} \quad z \in \mathbb{D}, \tag{28}$$

that is, $p \in \mathcal{SP}(\alpha)$.

Proof. Let $p(z) = (1 + A\omega(z))/(1 - \omega(z))$. Then ω is analytic in $\mathbb{D}$ and $\omega(0) = 0$. A simple computation shows that

$$\frac{zp'(z)}{p^2(z)} \prec \frac{(A+1)z\omega'(z)}{(1+\omega(z))^2} \quad \text{for} \quad z \in \mathbb{D}.$$

Now the subordination relation (28) holds if and only if $|\omega(z)| < 1$ for z in $\mathbb{D}$. Assume that there exists a point $z_0 \in \mathbb{D}$ such that $|\omega(z_0)| = 1$. Then by Jack lemma, $z_0\omega'(z_0) = k_0\omega(z_0)$ and $k_0 \geq 1$. For such z_0 we have $z_0p'(z_0)/p^2(z_0) = k_0h_1(\omega(z_0))$ which does not contain in $h_1(\mathbb{D})$ because $|\omega(z_0)| = 1$ and $k_0 \geq 1$. This contradicts the subordination condition (27). Hence $|\omega(z)| < 1$ for all $z \in \mathbb{D}$ which yields the desired result. □

Using Lemma 3.2 we prove the following theorem.

Theorem 3.3. *Let $f \in \mathcal{A}$ and $A = e^{-2i\alpha}$ be a complex constant with $|\alpha| < \pi/2$. If*

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{(1+A)z}{(1+Az)^2} \quad \textit{for} \quad z \in \mathbb{D}$$

then $f \in \mathcal{SP}(\alpha)$.

Proof. Let $p(z) = \frac{zf'(z)}{f(z)}$. Then p is analytic in $\mathbb{D}$ and $p(0) = 1$. A simple computation shows that

$$\frac{zp'(z)}{p^2(z)} = \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \prec \frac{(1+A)z}{(1+Az)^2} \quad \text{for} \quad z \in \mathbb{D}.$$

In view of Lemma 3.2, it follows that $p(z) \prec (1 + Az)/(1 - z)$ and hence $f \in \mathcal{SP}(\alpha)$.

Using Theorem 3.3, we obtain the following result. □

Corollary 3.4. *Let $A = e^{-2i\alpha}$ be a complex constant with $|\alpha| < \pi/2$. Then $\mathcal{G}_b \subset \mathcal{S}^*[A, -1] := \mathcal{SP}(\alpha)$ when $b = |1 + A|/4$.*

Proof. For $f \in \mathcal{G}_b$, we have

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + bz \quad \text{for} \quad z \in \mathbb{D}.$$

Let $h_2(z) = 1 + \frac{(1+A)z}{(1+Az)^2}$. Then a simple computation shows that

$$\min\{|h_2(e^{i\theta}) - 1| : \theta \in [0, 2\pi)\} = \frac{|1+A|}{4}.$$

If $b = |1 + A|/4$ then by using the definition of subordination we obtain $1 + bz \prec h_2(z)$. Therefore from Theorem 3.3, it follows that $f \in \mathcal{S}^*[A, -1] := \mathcal{SP}(\alpha)$. □

3.1. Starlike univalent functions of order α

Let $B(z_0; r)$ denote the open ball centered at z_0 and radius r. We say that $f \in \mathcal{H}(\alpha)$, $0 < \alpha < 1$, if $f \in \mathcal{A}$ and $A_\alpha(z) = \frac{2\alpha f(z)}{zf'(z)}$ maps the unit disk $\mathbb{D}$ into $B(1; 1)$. Since the conformal mapping $B(w) = (1 + w)^{-1}$ maps $\mathbb{D}$ onto $\operatorname{Re} w > 1/2$, one can see that the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{H}(\alpha)$ coincide.

Let $f \in \mathcal{H}(\alpha)$ and consider the function $h(z) := h_\beta(z) = (\frac{z}{f(z)})^\beta - 1$, where $0 < \beta \leq 1$. If $f \in \mathcal{H}(\alpha)$, $1/2 \leq \alpha < 1$, using Jack's lemma, Örnek [15] showed that h satisfies the condition of the Schwarz lemma: h maps $\mathbb{D}$ onto itself and $h(0) = 0$, and he has proved

Lemma 3.5. *Let $f \in \mathcal{H}(\alpha)$, $1/2 \leq \alpha < 1$ and $1/\beta = 2(1 - \alpha)$. Then*

(i) $|f(z)| \leq \dfrac{|z|}{(1 - |z|)^{1/\beta}}$

(ii) $|f''(0)| \leq 2/\beta$.

For $\beta = 1$, we find

(i') $|f(z)| \leq \dfrac{|z|}{(1-|z|)}$

(ii') $|f''(0)| \leq 2$.

Example 3.6. *Let $k_\beta(z) = z(1+z)^{-1/\beta}$, $0 < \beta \leq 1$. Then $\frac{zk'_\beta(z)}{k_\beta(z)} = A_\beta$, where $A_\beta(z) = 1 - \frac{1}{\beta}\frac{z}{1+z}$. Since A_β maps $\mathbb{D}$ onto* $\operatorname{Re} w > 1 - \frac{1}{2\beta}$. *One can see that k_β belongs $\mathcal{S}^*(\alpha)$ if and only if $\beta \geq \frac{1}{2(1-\alpha)}$. If $1/\beta > 2$ then k_β is not univalent in $\mathbb{D}$.*

The subject related to Jack's lemma has been discussed by Örnek [15] in a recent paper. Recently, Mateljević [9] has extended Örnek's result and obtained the following.

Theorem 3.7. *If f belongs $\mathcal{S}^*(\alpha)$, $0 \leq \alpha < 1$, and $1/\beta = 2(1-\alpha)$, then*

(i) $|f(z)| \leq \dfrac{|z|}{(1-|z|)^{1/\beta}}$

(ii) $|f''(0)| \leq 2/\beta$.

In particular, it can be seen that Ornek's result (i') $|f(z)| \leq \frac{|z|}{(1-|z|)}$ and (ii')$|f''(0)| \leq 2$ if f belongs to the class $\mathcal{S}^*(1/2)$. For convex functions (i') holds. Since convex functions are in $\mathcal{S}^*(1/2)$, this result is a generalization of corresponding one for convex functions.

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