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Strict Common Fixed Points of Nonlinear Mappings via δ-Distances and an Application

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Abstract. The purpose of this paper is to prove some strict common fixed point theorems for weakly compatible hybrid pairs of nonlinear mappings defined on semi-metric spaces employing implicit relations which unify, extend and generalize several results from the literature. As an application, we prove a general common fixed point theorem for integral type contractive conditions. Finally, we give an example to demonstrate the main result of the paper.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction and Preliminaries

In 1922, the Polish mathematician, S. Banach proved a classical theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways, and by now there exists an extensive literature on and around this classical theorem. Fixed point theorems for hybrid pair of set-valued and single-valued mappings is a relatively new development and have numerous applications in science, engineering, economics and game theory (e.g. [15]).

A semi-metric d in respect of a non-empty set X is a function $d : X \times X \to [0, \infty)$ which satisfies $d(x, y) = d(y, x)$ and $d(x, y) = 0 \Leftrightarrow x = y$ (for all $x, y \in X$). If d is a semi-metric on a set X, then for $x \in X$ and $\epsilon > 0$, we write $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. A topology $\tau(d)$ on X is given by the sets U (along with empty set) in which for each $x \in U$, one can find some $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. A set $S \subset X$ is a neighbourhood of $x \in X$ if and only if there is a U containing x such that $x \in U \subset S$. Thus a semi-metric space (X, d) is a topological space whose topology $\tau(d)$ on X is induced by a semi-metric d. A semi-metric d is said to be a potent semi-metric (cf. [9]) if for each $x \in X$ and for each $\epsilon > 0$, $B(x, \epsilon)$ is a neighbourhood of x in the topology $\tau(d)$. Notice that $\lim\limits_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $\tau(d)$. The distinction between a semi-metric and a potent semi-metric is apparent as one can easily construct a semi-metric d such that $B(x, \epsilon)$ need not be a neighbourhood of x in $\tau(d)$. As semi-metric spaces are not essentially Hausdorff, therefore in order

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to prove fixed point theorems, some additional axioms are required. The following axioms are relevant to our present presentation which are available in Aliouche [10], Cho et al. [20], Galvin and Shore [28], Hicks and Rhoades [29] and Wilson [66]. From now on semi-metric as well as potent semi-metric spaces will be denoted by (X, d) whereas a nonempty arbitrary set will be denoted by Y.

(W_3) :[66] Given $\{x_n\}, x$ and y in X with $d(x_n, x) \to 0$ and $d(x_n, y) \to 0$ imply $x = y$.
(W_4) :[66] Given $\{x_n\}, \{y_n\}$ and an x in X with $d(x_n, x) \to 0$ and $d(x_n, y_n) \to 0$ imply $d(y_n, x) \to 0$.
(HE) :[10] Given $\{x_n\}, \{y_n\}$ and an x in X with $d(x_n, x) \to 0$ and $d(y_n, x) \to 0$ imply $d(x_n, y_n) \to 0$.
($1C$) :[20] A semi-metric d is said to be 1-continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y)$.
(CC) :[28] A semi-metric d is said to be continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, y) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$ where $\{x_n\}$, $\{y_n\}$ are sequences in X and $x, y \in X$.

Clearly, the continuity (i.e. (CC)) of a semi-metric is a stronger property than ($1C$) (or 1-continuity) i.e. (CC) implies ($1C$) but not conversely. Also (W_4) implies (W_3) and ($1C$) implies (W_3) but converse implications are not true. All other possible implications amongst (W_3), ($1C$) and (HE) are not true in general whose nice illustrations via demonstrative examples are available in Cho et al. [20]. But (CC) implies all the remaining four conditions namely: (W_3), (W_4), (HE) and ($1C$). For more details on these axioms, one can see [9] while for details on Cauchy sequences and the completeness in semi-metric spaces, one can be referred to [18, 28].

We define $CB(X) = \{C|C$ is a nonempty closed and bounded subset of X$\}$, $B(X) = \{C|C$ is a nonempty bounded subset of X$\}$ and the functions $\delta(A, B)$ and $D(A, B)$ by $\delta(A, B) = \sup\{d(a, b) : a \in A,\ b \in B\}$ and $D(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}$ for all $A, B \in B(X)$. If A consists of a single point a, we write $\delta(A, B) = \delta(a, B)$. If B also consists of a single point b, we write $\delta(A, B) = d(a, b)$. It follows immediately from the definition of δ that

$\delta(A, B) = \delta(B, A) \geq 0$,
$\delta(A, B) = 0$ iff $A = B = \{a\}$,
$\delta(A, B) \leq \delta(A, C) + \delta(C, B)$,
$\delta(A, A) = diam(A)$ for all $A, B, C \in B(X)$.

Lemma 1.1 [26]. Let $\{A_n\}$ and $\{B_n\}$ be two sequences in $B(X)$ both converging in $B(X)$ to the sets A and B respectively. Then

$$\lim_{n\to\infty} \delta(A_n, B_n) = \delta(A, B).$$

Definition 1.1. Let $A : X \to X$ and $S : X \to B(X)$, then a point $x \in X$ is said to be a

(i) coincidence point (strict coincidence point) of A and S if $Ax \in Sx$ ($Sx = \{Ax\}$),
(ii) fixed point of S if $x \in Sx$,
(iii) stationary point or strict fixed point of S if $Sx = \{x\}$.

In the sequel, $C(A, S)$ and $SC(A, S)$ denote the set of coincidence points and strict coincidence points of mappings A and S respectively.

Definition 1.2. Let $A : X \to X$ and $S : X \to B(X)$ wherein (X, d) is a semi-metric space. Then following [36, 61], we say that the pair (A, S) is

(i) weakly commuting on X if for any $x \in X$

$$\delta(SAx, ASx) \leq \max\{\delta(Ax, Sx),\ diamASx\},$$

(ii) quasi-commuting on X if for any $x \in X$

$$ASx \subset SAx,$$

(iii) slightly commuting on X if for any $x \in X$

$$\delta(SAx, ASx) \leq \max\{\delta(Ax, Sx),\ diamSx\},$$

(iv) R-weakly commuting (cf. [50]) if there exists some $R > 0$ such that $\delta(ASx, SAx) \leq R.\delta(Ax, Sx)$ for all $x \in X$.

Pant [51] also defined the pair (A, S) to be pointwise R-weakly commuting if for given $x \in X$, there exists an $R > 0$ such that (iv) holds.

Clearly two commuting mappings satisfy (i)-(iii) but reverse implications are not true in general. In [36], it is demonstrated by suitable examples that the foregoing three concepts are mutually independent and none of them implies the other two.

Definition 1.3.[42] A pair (A, S) of hybrid mappings with $A : X \to X$ and $S : X \to B(X)$ is said to be δ-compatible if $\lim_{n\to\infty} \delta(SAx_n, ASx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $ASx_n \in B(X)$, $Ax_n \to t$ and $Sx_n \to \{t\}$ as $n \to \infty$ for some $t \in X$.

Definition 1.4.[41] A pair (A, S) of hybrid mappings with $A : X \to X$ and $S : X \to B(X)$ of a nonempty set X is said to be weakly compatible if $ASx = SAx$ for all $x \in C(A, S)$ (or $x \in SC(A, S)$).

Thus, the maps A and S are pointwise R-weakly commuting iff they are weakly compatible. Moreover, if the pair (A, S) is δ-compatible, then it is weakly compatible, but the converse is not true in general (see [43]).

Definition 1.5.[12] A pair (A, S) with $A : X \to X$ and $S : X \to B(X)$ of hybrid mappings of a nonempty set X is said to be occasionally weakly compatible (OWC) if $SAx = ASx$ for some $x \in C(A, S)$ (or $x \in SC(A, S)$).

Remark 1.1.[12] If A and S are weakly compatible, then they are occasionally weakly compatible, but the converse implication is not true in general as exhibited by the following example.

Example 1.1. Let $X = [1, \infty)$ and define $A, S : X \to X$ by: $Ax = 3x - 2$ and $Sx = x^2$. We have $Ax = Sx$ iff $x = 1$ and $x = 2$ and $AS(1) = SA(1) = 1$, but $AS(2) \neq SA(2)$. Therefore, A and S are occasionally weakly compatible, but not weakly compatible.

Remark 1.2. Every mapping $A : X \to X$ and the identity mapping of X (i.e. id_X) are weakly compatible, while $A : X \to X$ and id_X are OWC iff A has a fixed point.

Lemma 1.2.[44] Let X be a set, A, S are OWC single-valued self-mappings of X. If A and S have a unique point of coincidence $w = Ax = Sx$, then w is the unique common fixed point of A and S.

In the recent years, several authors proved common fixed point theorems for OWC mappings (cf. [3, 11, 44]). In [33], Imdad et al. pointed out that OWC is not a proper generalization of weak compatibility (WC) when $C(A, S)$ is empty set as definition of weak compatibility is vacuously satisfied while for OWC, $C(A, S)$ should be nonempty. Pant and Pant [52] redefined OWC and term it as conditionally commuting pair under the additional hypothesis that $C(A, S)$ is nonempty. Recently Dorić et al. [24] showed that a pair of OWC single-valued mappings reduces to weakly compatible mappings in the presence of unique point of coincidence (or unique common fixed point) which amounts to say that no such generalization can be obtained for single-valued mappings (also see Proposition 2 and Corollary 3 from [45]). But for pairs of hybrid mappings, situation is different wherein OWC is a strictly weaker condition than weak compatibility. So, one can obtain more general results by using OWC in the case of pair of hybrid mappings.

We give following result (without proof) which establishes relationship between OWC and WC for hybrid pair of single-valued and set-valued mappings.

Lemma 1.3. Let $A : X \to X$ and $S : X \to B(X)$ be a pair of hybrid mappings which has a unique strict point of coincidence (or a unique strict common fixed point). Then the pair (A, S) is OWC iff it is WC.

On the lines of Aamri and Moutawakil [1], we can have the following:

Definition 1.6. A hybrid pair of mappings with $A : X \to X$ and $S : X \to B(X)$ is said to satisfy the property

(E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = t, \text{ and } \lim_{n\to\infty} Sx_n = \{t\}, \text{ for some } t \in X.$$

Clearly, a pair of δ-compatible mappings as well as δ-non-compatible mappings satisfy the property (E.A).

On the lines of Pant [51], we give the following for a pair of hybrid mappings:

Definition 1.7. A hybrid pair (A, S) of mappings with $A : X \to X$ and $S : X \to B(X)$ is said to be reciprocally continuous iff $\lim_{n\to\infty} ASx_n = \{At\}$ and $\lim_{n\to\infty} SAx_n = St$, for every sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} Ax_n = t$ and $\lim_{n\to\infty} Sx_n = \{t\}$, for some $t \in X$.

Clearly any pair of continuous mappings is reciprocally continuous but converse need not be true in general.

Motivated from Bouhadjera and Godet-Thobie [16], we define the following definitions:

Definition 1.8. A hybrid pair (A, S) of mappings with $A : X \to X$ and $S : X \to B(X)$ is said to be δ-subcompatible iff there exists a sequence $\{x_n\} \in X$ such that $\lim_{n\to\infty} \delta(ASx_n, SAx_n) = 0$ with $\lim_{n\to\infty} Ax_n = t$ and $\lim_{n\to\infty} Sx_n = \{t\}$, for some $t \in X$.

Definition 1.9. A hybrid pair of mappings (A, S) with $A : X \to X$ and $S : X \to B(X)$ is said to be subsequential continuous iff there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty} ASx_n = \{At\} \quad \text{and} \quad \lim_{n\to\infty} SAx_n = St$$

with $\lim_{n\to\infty} Ax_n = t$ and $\lim_{n\to\infty} Sx_n = \{t\}$, for some $t \in X$.

If A and S are both continuous or reciprocally continuous, then they are obviously subsequentially continuous. But, there do exist pairs of subsequentially continuous mappings which are neither continuous nor reciprocally continuous.

Definition 1.10. An altering distance function is a mapping $\varphi : [0, \infty) \to [0, \infty)$ which satisfies the following conditions:

(φ_1) : φ is increasing and continuous, and

(φ_2) : $\varphi(t) = 0$ if and only if $t = 0$.

For fixed point theorems involving altering distances in metric spaces, one can be referred to [46, 56, 59] besides some other ones.

Definition 1.11.[34] Two families of self mappings $\{A_i\}$ and $\{B_k\}$ are said to be pairwise commuting if

(i) $A_iA_j = A_jA_i; i, j \in \{1, 2, ...m\}$,

(ii) $B_kB_l = B_lB_k; k, l \in \{1, 2, ...n\}$,

(iii) $A_iB_k = B_kA_i; i \in \{1, 2...m\}, k \in \{1, 2, ...n\}$.

2. Implicit Functions

Popa [54] initiated the idea of implicit functions in metric fixed point theory. Thereafter, several authors (e.g. [8, 13, 38, 49]) utilized this technique as it remains an effective tool to prove unified fixed point theorems besides being general enough to yield unknown contraction conditions at the same time. In order to define our implicit function, let Ψ be the family of lower semi-continuous functions $F : \mathfrak{R}_+^6 \to \mathfrak{R}$ satisfying the following conditions.

(F_1) : F is decreasing in variables t_2 to t_6,

$(F_2) : F(t, 0, t, 0, 0, t) > 0$, for all $t > 0$,

$(F_3) : F(t, 0, 0, t, t, 0) > 0$, for all $t > 0$,

$(F_4) : F(t, t, 0, 0, t, t) > 0$, for all $t > 0$.

Example 2.1. Define $F(t_1, t_2, \cdots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \cdots, t_6) = t_1 - \psi(\max\{t_2, t_3, t_4, t_5, t_6\})$$

where $\psi : \Re_+ \to \Re$ is an upper semi-continuous function such that $\psi(0) = 0$ and $\psi(t) < t$ for all $t > 0$.

Example 2.2. Define $F(t_1, t_2, \cdots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \cdots, t_6) = t_1 - \psi(t_2, t_3, t_4, t_5, t_6)$$

where $\psi : \Re_+^5 \to \Re$ is an upper semi-continuous function such that $\max\{\psi(0, t, 0, 0, t), \psi(0, 0, t, t, 0), \psi(t, 0, 0, t, t)\} < t$ for each $t > 0$.

Example 2.3. Define $F(t_1, t_2, \cdots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \cdots, t_6) = t_1^2 - \psi(t_2^2, t_3 t_4, t_5 t_6, t_3 t_6, t_4 t_5)$$

where $\psi : \Re_+^5 \to \Re$ is an upper semi-continuous function such that $\max\{\psi(0, 0, 0, t, 0), \psi(0, 0, 0, 0, t), \psi(t, 0, t, 0, 0)\} < t$ for each $t > 0$.

Example 2.4. Define $F(t_1, t_2, \cdots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \cdots, t_6) = t_1^2 - \alpha \max\{t_2^2, t_3^2, t_4^2\} - \beta \max\{t_3 t_5, t_4 t_6\} - \gamma t_5 t_6$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \gamma < 1$.

Example 2.5. Define $F(t_1, t_2, \cdots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \cdots, t_6) = t_1 - \alpha t_2 - \beta \max\{t_3, t_4\} - \gamma \max\{t_3 + t_4, t_5 + t_6\}$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + 2\gamma < 1$.

Example 2.6. Define $F(t_1, t_2, \cdots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \cdots, t_6) = \begin{cases} t_1 - \alpha t_2 - \beta \dfrac{t_3^2 + t_4^2}{t_3 + t_4} - \gamma(t_5 + t_6), & \text{if } t_3 + t_4 \neq 0 \\ t_1 - t_2, & \text{if } t_3 + t_4 = 0 \end{cases}$$

where $\alpha, \beta, \gamma \geq 0$ and $\beta + \gamma < 1$.

Example 2.7. Define $F(t_1, t_2, \cdots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \cdots, t_6) = \begin{cases} t_1^p - k t_2^p - \dfrac{t_3 t_4^p + t_5 t_6^p}{t_3 + t_4}, & \text{if } t_3 + t_4 \neq 0 \\ t_1 - t_2, & \text{if } t_3 + t_4 = 0 \end{cases}$$

where $p \geq 1$ and $0 \leq k < \infty$.

Example 2.8. Define $F(t_1, t_2, \cdots, t_6) : \Re_+^6 \to \Re$ as

$$F(t_1, t_2, \cdots, t_6) = t_1 - \min\{\max\{t_3, t_4\}, t_5, t_6\} - \psi(\max\{t_2, t_3, t_4, t_5, t_6\}),$$

wherein $\psi : \Re_+ \to \Re_+$ is an upper semi-continuous function such that $\psi(t) < t$ for all $t > 0$.

Consider the set of functions, $\Phi = \{\phi : \mathfrak{R}_+ \to \mathfrak{R}_+$ is a Lebesgue integrable mapping which is summable and satisfies $\int_0^{\epsilon} \phi(t)dt > 0$ for all $\epsilon > 0\}$. Now, we give examples which satisfy inequalities of integral type.

Example 2.9. Define $F(t_1, t_2, \cdots, t_6) : \mathfrak{R}_+^6 \to \mathfrak{R}$ as

$$F(t_1, t_2, \cdots, t_6) = \int_0^{t_1} \phi(t)dt - \psi\left(\int_0^{\max\{t_2,t_3,t_4,t_5,t_6\}} \phi(t)dt\right)$$

where $\psi : \mathfrak{R}_+ \to \mathfrak{R}_+$ is an upper semi-continuous function such that $\psi(t) < t$ for all $t > 0$ and $\phi \in \Phi$.

Example 2.10. Define $F(t_1, t_2, \cdots, t_6) : \mathfrak{R}_+^6 \to \mathfrak{R}$ as

$$F(t_1, t_2, \cdots, t_6) = \int_0^{t_1} \phi(t)dt - \psi\left(\max\left\{\int_0^{t_2} \phi(t)dt, \int_0^{t_3} \phi(t)dt, \int_0^{t_4} \phi(t)dt, \int_0^{t_5} \phi(t)dt, \int_0^{t_6} \phi(t)dt\right\}\right)$$

where $\psi : \mathfrak{R}_+ \to \mathfrak{R}_+$ is an upper semi-continuous function such that $\psi(t) < t$ for all $t > 0$ and $\phi \in \Phi$.

Example 2.11. Define $F(t_1, t_2, \cdots, t_6) : \mathfrak{R}_+^6 \to \mathfrak{R}$ as

$$F(t_1, t_2, \cdots, t_6) = \Big(\int_0^{t_1} \phi(t)dt\Big)^p - \psi\Big(a\Big(\int_0^{t_2} \phi(t)dt\Big)^p - (1-a)\max\Big\{\alpha\Big(\int_0^{t_3} \phi(t)dt\Big)^p,$$

$$\beta\Big(\int_0^{t_4} \phi(t)dt\Big)^p, \Big(\int_0^{t_3} \phi(t)dt\Big)^{\frac{p}{2}}.\Big(\int_0^{t_6} \phi(t)dt\Big)^{\frac{p}{2}}, \Big(\int_0^{t_5} \phi(t)dt\Big)^{\frac{p}{2}}.\Big(\int_0^{t_6} \phi(t)dt\Big)^{\frac{p}{2}}\Big\}\Big)$$

where $\psi : \mathfrak{R}_+ \to \mathfrak{R}_+$ is an upper semi-continuous function such that $\psi(t) < t$ for all $t > 0$, $0 \leq a, \alpha, \beta \leq 1$, $p \geq 1$ and $\phi \in \Phi$.

Example 2.12. Define $F(t_1, t_2, \cdots, t_6) : \mathfrak{R}_+^6 \to \mathfrak{R}$ as

$$F(t_1, t_2, \cdots, t_6) = \int_0^{t_1} \phi(t)dt - \alpha \max\left\{\int_0^{t_2} \phi(t)dt, \int_0^{t_3} \phi(t)dt, \int_0^{t_4} \phi(t)dt\right\}$$

$$-(1-\alpha)\left(a\int_0^{t_5} \phi(t)dt + b\int_0^{t_6} \phi(t)dt\right)$$

where $0 \leq \alpha < 1$, $a, b \geq 0$, $a + b \leq 1$ and $\phi \in \Phi$.

Example 2.13. Define $F(t_1, t_2, \cdots, t_6) : \mathfrak{R}_+^6 \to \mathfrak{R}$ as

$$F(t_1, t_2, \cdots, t_6) = \int_0^{t_1} \phi(t)dt - \psi\left(\max\left\{\int_0^{t_2} \phi(t)dt, \int_0^{t_3} \phi(t)dt, \int_0^{t_4} \phi(t)dt\right\}\right) - \frac{1}{2}\left(\int_0^{t_5} \phi(t)dt + \int_0^{t_6} \phi(t)dt\right)$$

where $\psi : \mathfrak{R}_+ \to \mathfrak{R}_+$ is an upper semi-continuous function such that $\psi(t) < t$ for all $t > 0$ and $\phi \in \Phi$.

Verification of requirements $(F_1), (F_2), (F_3)$ and (F_4) in respect of Examples 2.1-2.13 are straight forward, hence details are not included.

3. Main Results

We begin with the following two results without completeness and weak compatibility requirements.

Theorem 3.1. *Let $A, B : X \to X$ and $S, T : X \to B(X)$ wherein X be a nonempty set equipped with a semi-metric d which enjoys (CC). If the pairs (A, S) and (B, T) are subcompatible and reciprocally continuous, then the pairs (A, S) and (B, T) have strict coincidence point. Further, if the pairs (A, S) and (B, T) satisfy the condition:*

$$F(\delta(Sx, Ty), d(Ax, By), \delta(Ax, Sx), \delta(By, Ty), \delta(Sx, By), \delta(Ty, Ax)) < 0, \tag{3.1.1}$$

for all $x, y \in X$, $F \in \Psi$, wherein F satisfies F_1 and F_4, then A, B, S and T have a unique strict common fixed point.

Proof. Since the pair (A, S) is subcompatible, there exists a sequence $\{x_n\} \in X$ such that

$$\lim_{n\to\infty} \delta(ASx_n, SAx_n) = 0$$

with

$$\lim_{n\to\infty} Ax_n = t, \ \lim_{n\to\infty} Sx_n = \{t\}.$$

As the pair (A, S) is reciprocally continuous

$$\lim_{n\to\infty} ASx_n = \{At\}, \ \lim_{n\to\infty} SAx_n = St$$

so that $\{At\} = St$ which shows that t is a strict coincidence point.
As the pair (B, T) is subcompatible, there exists a sequence $\{y_n\} \in X$ such that

$$\lim_{n\to\infty} \delta(BTy_n, TBy_n) = 0$$

with

$$\lim_{n\to\infty} By_n = t', \ \lim_{n\to\infty} Ty_n = \{t'\}.$$

As the pair (B, T) is reciprocally continuous,

$$\lim_{n\to\infty} BTy_n = \{Bt'\}, \ \lim_{n\to\infty} TBy_n = Tt'$$

so that $\{Bt'\} = Tt'$ which shows that t' is a strict coincidence point.

Now, we prove that $t = t'$. For this, setting $x = x_n$ and $y = y_n$ in this (3.1.1), we get

$$F(\delta(Sx_n, Ty_n), d(Ax_n, By_n), \delta(Ax_n, Sx_n), \delta(By_n, Ty_n), \delta(Sx_n, By_n), \delta(Ty_n, Ax_n)) < 0$$

which on making $n \to \infty$ (due to F_1 and (CC)) gives rise

$$F(\delta(t, t'), \delta(t, t'), 0, 0, \delta(t, t'), \delta(t, t')) \leq 0,$$

a contradiction to (F_4). Hence $t = t'$.
Now we assert that t is a strict common fixed point of the pair (B, T). Suppose that $Bt \neq t$, then using (3.1.1), we have

$$F(\delta(Sx_n, Tt), d(Ax_n, Bt), \delta(Ax_n, Sx_n), \delta(Bt, Tt), \delta(Sx_n, Bt), \delta(Tt, Ax_n)) < 0.$$

Making use of (F_1),(CC) and letting $n \to \infty$, we get

$$F(\delta(t, Bt), \delta(t, Bt), 0, 0, \delta(t, Bt), \delta(Bt, t)) \leq 0$$

which is a contradiction to (F_4). Therefore, $Bt = t$ which shows that t is a strict common fixed point of the pair (B, T).
Again, suppose that $At \neq t$, then using (3.1.1), we have

$$F(\delta(St, Tt), d(At, Bt), \delta(At, St), \delta(Bt, Tt), \delta(St, Bt), \delta(Tt, At)) < 0,$$

which on using (F_1) gives rise

$$F(\delta(At, t), \delta(At, t), 0, 0, \delta(At, t), \delta(At, t)) < 0,$$

a contradiction to (F_4). Therefore, $At = \{t\}$ which shows that t is a strict common fixed point of the pair (A, S). Thus, $\{t\}$ is a strict common fixed point of both the pairs (A, S) and (B, T). Uniqueness of common fixed point is an easy consequence of inequality (3.1.1) in view of condition (F_4). This completes the proof.

Theorem 3.2. Let $A, B : X \to X$ and $S, T : X \to B(X)$ wherein X be a nonempty set equipped with a semi-metric d which enjoys (CC). If the pairs (A, S) and (B, T) are subsequentially continuous and compatible, then the pairs (A, S) and (B, T) have strict coincidence point. Further, suppose that the pairs (A, S) and (B, T) satisfy condition (3.1.1), then A, B, S and T have a unique strict common fixed point.

Proof. Since the pair (A, S) is subsequentially continuous, there exists a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} Ax_n = t$, $\lim_{n\to\infty} Sx_n = \{t\}$ such that

$$\lim_{n\to\infty} ASx_n = \{At\} \text{ and } \lim_{n\to\infty} SAx_n = St.$$

In view of the compatibility of the pair (A, S), we have, $St = \{At\}$ which shows that t is a strict coincidence point of A and S.

Since the pair (B, T) is also subsequentially continuous, there exists a sequence $\{y_n\}$ in X with $\lim_{n\to\infty} By_n = t'$, $\lim_{n\to\infty} Ty_n = \{t'\}$ such that

$$\lim_{n\to\infty} BTy_n = \{Bt'\} \text{ and } \lim_{n\to\infty} TBy_n = Tt'.$$

In view of the compatibility of the pair (B, T), we have, $Tt' = \{Bt'\}$ which shows that t' is a strict coincidence point of B and T. The rest of the proof can be completed on the lines of Theorem 3.1.

Remark 3.1. Notice that Theorems 3.1 and 3.2 never require conditions on closedness, completeness and containment among ranges of the involves mappings. These results generalize some relevant results of the existing literature (e.g. [11, 30, 36, 61]). Also, notice that we never require any contraction condition up to coincidence points.

Our remaining results employ altering distances. Our first result of this kind runs as follows:

Theorem 3.3. Let $A, B : X \to X$ and $S, T : X \to B(X)$ wherein X be a nonempty set equipped with a semi-metric d enjoying (W_3) and (HE). Suppose that

(a) the pair (A, S) (or (B, T)) has the property $(E.A)$,

(b) $S(X) \subset B(X)$ (or $T(X) \subset A(X)$),

(c) $A(X)$ (or $B(X)$) is a closed subset of X and

(d) for all $x, y \in X$, $F \in \Psi$ and $\varphi(t)$ is an altering distance such that

$$F\Big(\varphi(\delta(Sx, Ty)), \varphi(d(Ax, By)), \varphi(\delta(Ax, Sx)), \varphi(\delta(By, Ty)), \varphi(\delta(Sx, By)), \varphi(\delta(Ty, Ax))\Big) < 0. \tag{3.3.1}$$

Then the pairs (A, S) and (B, T) have strict coincidence point. If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique strict common fixed point.

Proof. If the pair (A, S) enjoys the property $(E.A)$, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = t, \ \lim_{n\to\infty} Sx_n = \{t\}, \quad \text{for some } t \in X.$$

Since $S(X) \subset B(X)$, hence for each $\{x_n\}$ there exists $\{y_n\}$ in X such that $By_n = Sx_n$. Therefore, $\lim_{n\to\infty} By_n = t \in \{t\} = \lim_{n\to\infty} Sx_n$. Thus, in all we have $Ax_n \to t, Sx_n \to \{t\}$ and $By_n \to t$. Now, we assert that $Ty_n \to \{t\}$. If not, then using (3.3.1), we have

$$F\Big(\varphi(\delta(Sx_n, Ty_n)), \varphi(d(Ax_n, By_n)), \varphi(\delta(Ax_n, Sx_n)), \varphi(\delta(By_n, Ty_n)), \varphi(\delta(Sx_n, By_n)), \varphi(\delta(Ty_n, Ax_n))\Big) < 0$$

which on letting $n \to \infty$ and making use of (W_3) and (HE) gives rise

$$F\Big(\varphi(\delta(t, \lim_{n\to\infty} Ty_n)), 0, 0, \varphi(\delta(\lim_{n\to\infty} Ty_n, t)), \varphi(\delta(t, \lim_{n\to\infty} Ty_n)), 0\Big) \leq 0,$$

a contradiction to (F_2). Hence $\lim_{n\to\infty} Ty_n \to \{t\}$ so that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = t,\ \lim_{n\to\infty} Sx_n = \{t\} = \lim_{n\to\infty} Ty_n, \quad \text{for some } t \in X.$$

If $A(X)$ is a closed subset of X, then $\lim_{n\to\infty} Ax_n = t \in A(X)$. Therefore, there exists a point $u \in X$ such that $Au = t$. Now we assert that $\{Au\} = Su$. If not, then using (3.3.1), we have

$$F\Big(\varphi(\delta(Ty_n, Su)), \varphi(d(By_n, Au)), \varphi(\delta(Au, Su)), \varphi(\delta(By_n, Ty_n)), \varphi(\delta(Ty_n, Au)), \varphi(\delta(Su, By_n))\Big) < 0$$

which on letting $n \to \infty$ and making use of (W_3) and (HE) gives rise

$$F\Big(\varphi(\delta(Au, Su)), \varphi(d(Au, t)), \varphi(\delta(Au, Su)), \varphi(\delta(t, t)), \varphi(\delta(Au, t)), \varphi(\delta(t, Au))\Big) \leq 0$$

or

$$F\Big(\varphi(\delta(Au, Su)), 0, \varphi(\delta(Au, Su)), 0, 0, \varphi(\delta(Au, Su))\Big) \leq 0,$$

a contradiction to (F_3). Hence $\{Au\} = Su = \{t\}$. Thus, u is a strict coincidence point of the pair (A, S).

Since $S(X) \subset B(X)$ and $Su \in S(X)$, there exists $w \in X$ such that $\{Bw\} = Su = \{t\}$. Now we assert that $\{Bw\} = Tw$. If not, then again using (3.3.1), we have

$$F\Big(\varphi(\delta(Su, Tw)), \varphi(d(Au, Bw)), \varphi(\delta(Au, Su)), \varphi(\delta(Bw, Tw)), \varphi(\delta(Su, Bw)), \varphi(\delta(Tw, Au))\Big) < 0$$

or

$$F\Big(\varphi(\delta(t, Tw)), \varphi(d(t, Bw)), \varphi(\delta(t, t)), \varphi(\delta(Bw, Tw)), \varphi(\delta(t, Bw)), \varphi(\delta(Tw, t))\Big) < 0$$

or

$$F\Big(\varphi(\delta(Bw, Tw)), 0, 0, \varphi(\delta(Bw, Tw)), \varphi(\delta(Bw, Tw)), 0\Big) < 0$$

a contradiction to (F_2). Hence $\{Bw\} = Tw = \{t\}$, which shows that w is a strict coincidence point of the pair (B, T).

Since the pair (A, S) is weakly compatible, there exists a point u at which the pair (A, S) commutes i.e. $At = ASu = SAu = St$. Now we assert that t is a strict common fixed point of the pair (A, S). Suppose that $At \neq t$, then using (3.3.1), we have

$$F\Big(\varphi(\delta(St, Tu)), \varphi(d(At, Bu)), \varphi(\delta(At, St)), \varphi(\delta(Bu, Tu)), \varphi(\delta(St, Bu)), \varphi(\delta(Tu, At))\Big) < 0$$

which (in view of (F_1)) gives rise

$$F\Big(\varphi(\delta(At, t)), \varphi(\delta(At, t)), 0, 0, \varphi(\delta(At, t)), \varphi(\delta(t, At))\Big) < 0$$

a contradiction to (F_4). Therefore, $At = t$ which shows that t is a strict common fixed point of the pair (A, S).

Since the pair (B, T) is also weakly compatible, then there exists a point w at which the pair (B, T) commutes i.e. $Bt = BTw = TBw = Tt$. Suppose that $Bt \neq t$, then using (3.3.1), we get

$$F\Big(\varphi(\delta(Sw, Tt)), \varphi(d(Aw, Bt)), \varphi(\delta(Aw, Sw)), \varphi(\delta(Bt, Tt)), \varphi(\delta(Sw, Bt)), \varphi(\delta(Tt, Aw))\Big) < 0$$

which (in view of (F_1)) gives rise

$$F\Big(\varphi(\delta(Bt, t)), \varphi(\delta(Bt, t)), 0, 0, \varphi(\delta(Bt, t)), \varphi(\delta(t, Bt))\Big) < 0$$

a contradiction to (F_4). Therefore, $Bt = t$ which shows that $\{t\}$ is a strict common fixed point of the pair (B, T). Thus $\{t\}$ is a strict common fixed point of both the pairs (A, S) and (B, T).

Uniqueness of common fixed point is an easy consequence of inequality (3.3.1) in view of condition (F_4). This completes the proof.

Remark 3.2. Theorem 3.3 generalizes the corresponding relevant theorems of [11, 30, 36, 61].

By restricting A,B,S and T suitably, one can derive corollaries involving two as well as three mappings. Here, it may be pointed out that any result for two mappings is itself a new result. For the sake of brevity, we opt to mention just one such corollary by restricting Theorem 3.3 to a pair of mappings A and S.

Corollary 3.1. Let $A : X \to X$ and $S : X \to B(X)$ wherein X be a nonempty set equipped with a semi-metric d which enjoys (W_3) and (HE). Suppose that

(a) the pair (A, S) share the property $(E.A)$,

(b) $A(X)$ is a closed subset of X and

(c) for all $x, y \in X$, $F \in \Psi$ and $\varphi(t)$ is an altering distance

$$F\Big(\varphi(\delta(Sx, Sy)), \varphi(d(Ax, Ay)), \varphi(\delta(Ax, Sx)), \varphi(\delta(Ay, Sy)), \varphi(\delta(Sx, Ay)), \varphi(\delta(Sy, Ax))\Big) < 0.$$

Then the pair (A, S) has strict coincidence point. If the pair (A, S) is weakly compatible, then A and S have a unique strict common fixed point.

Theorem 3.4. The conclusions of Theorem 3.3 remain true if the condition (b) of Theorem 3.3 is replaced by following.

(b') $\overline{S(X)} \subset B(X)$ or $\overline{T(X)} \subset A(X)$.

Theorem 3.5. The conclusions of Theorem 3.3 remain true if condition (3.3.1) is replaced by the following.

$$F\Big(\varphi(\delta(Sx, Ty)), \varphi(d(Ax, By)), \varphi(\delta(Ax, Sx)), \varphi(\delta(By, Ty)), \varphi(D(Sx, By)), \varphi(D(Ty, Ax))\Big) < 0. \tag{3.5.1}$$

Proof. In view of (F_1), condition (3.5.1) implies condition (3.3.1).

Theorem 3.6. The conclusions of Theorem 3.3 remain true if condition (3.3.1) is replaced by the following.

$$F\Big(\varphi(\delta(Sx, Ty)), \varphi(d(Ax, By)), \varphi(D(Ax, Sx)), \varphi(D(By, Ty)), \varphi(D(Sx, By)), \varphi(D(Ty, Ax))\Big) < 0. \tag{3.6.1}$$

Proof. In view of (F_1), condition (3.6.1) implies condition (3.3.1).

Remark 3.3. Theorem 3.6 generalizes a multitude of the corresponding theorems contained in [2, 5, 6, 13, 16, 22, 25–27, 42, 54, 57, 58, 60, 63, 64].

Theorem 3.7. The conclusions of Theorem 3.3 remain true if condition (3.3.1) is replaced by the following.

$$F\Big(\varphi(\delta(Sx, Ty)), \varphi(d(Ax, By)), \varphi(D(Ax, Sx)), \varphi(D(By, Ty)), \varphi(\delta(Sx, By)), \varphi(\delta(Ty, Ax))\Big) < 0. \tag{3.7.1}$$

Proof. In view of (F_1), condition (3.7.1) implies condition (3.3.1).

Remark 3.4. In Theorem 3.3, as F has the decreasing property in t_2, t_3 and t_4, so above theorem generalizes the corresponding theorems of [3, 11].

Corollary 3.2. The conclusions of Theorem 3.3 remain true if inequality (3.3.1) is replaced by one of the following contraction conditions. For all $x, y \in X$, $F \in \Psi$ and $\varphi(t)$ is an altering distance

(I) $\varphi(\delta(Sx, Ty)) < \psi(\max\{\varphi(d(Ax, By)), \varphi(\delta(Ax, Sx)), \varphi(\delta(By, Ty)), \varphi(\delta(Sx, By)), \varphi(\delta(Ty, Ax))\})$ where $\psi : \Re_+ \to \Re$ is an upper semi-continuous function such that $\psi(0) = 0$ and $\psi(t) < t$ for all $t > 0$.

(II) $\varphi(\delta(Sx, Ty)) < \psi\Big(\varphi(d(Ax, By)), \varphi(\delta(Ax, Sx)), \varphi(\delta(By, Ty)), \varphi(\delta(Sx, By)), \varphi(\delta(Ty, Ax))\Big)$ where $\psi : \Re_+^5 \to \Re$ is an upper semi-continuous function such that $\max\{\psi(0, t, 0, 0, t), \psi(0, 0, t, t, 0), \psi(t, 0, 0, t, t)\} < t$ for each $t > 0$.

(III) $\varphi(\delta(Sx,Ty)^2) < \psi\Big(\varphi(d(Ax,By)^2), \varphi(\delta(Ax,Sx))\varphi(\delta(By,Ty)), \varphi(\delta(Sx,By))\varphi(\delta(Ty,Ax)),$

$$\varphi(\delta(Ax,Sx))\varphi(\delta(Ty,Ax)), \varphi(\delta(By,Ty))\varphi(\delta(Sx,By))\Big)$$

where $\psi : \mathfrak{R}_+^5 \to \mathfrak{R}$ is an upper semi-continuous function such that $\max\{\psi(0,0,0,t,0), \psi(0,0,0,0,t), \psi(t,0,t,0,0)\} < t$ for each $t > 0$.

(IV) $\varphi(\delta(Sx,Ty)^2) < \alpha \max\{\varphi(d(Ax,By)^2), \varphi(\delta(Ax,Sx)^2), \varphi(\delta(By,Ty)^2)\} + \beta \max\{\varphi(\delta(Ax,Sx))\varphi(\delta(Sx,By)),$

$$\varphi(\delta(By,Ty))\varphi(\delta(Ty,Ax))\} + \gamma(\varphi(\delta(Sx,By))\varphi(\delta(Ty,Ax)))$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \gamma < 1$.

(V) $\varphi(\delta(Sx,Ty)) < \alpha\varphi(d(Ax,By)) + \beta \max\{\varphi(\delta(Ax,Sx)), \varphi(\delta(By,Ty))\} + \gamma \max\{\varphi(\delta(Ax,Sx)) + \varphi(\delta(By,Ty)),$

$$\varphi(\delta(Sx,By)) + \varphi(\delta(Ty,Ax))\}$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + 2\gamma < 1$.

(VI) $\varphi(\delta(Sx,Ty)) < \alpha\varphi(d(Ax,By)) + \beta \dfrac{\varphi(\delta(Ax,Sx)^2) + \varphi(\delta(By,Ty)^2)}{\varphi(\delta(Ax,Sx)) + \varphi(\delta(By,Ty)}) + \gamma(\varphi(\delta(Sx,By)) + \varphi(\delta(Ty,Ax))),$

if $\varphi(\delta(Ax,Sx)) + \varphi(\delta(By,Ty)) \neq 0$,

$\varphi(\delta(Sx,Ty)) < \alpha\varphi(d(Ax,By))$, if $\varphi(\delta(Ax,Sx)) + \varphi(\delta(By,Ty)) = 0$

where $\alpha, \beta, \gamma \geq 0$ and $\beta + \gamma < 1$.

(VII) $\varphi(\delta(Sx,Ty)^p) < k\varphi(d(Ax,By)^p) + \dfrac{\varphi(\varphi(\delta(Ax,Sx))\varphi(\delta(By,Ty)^p) + \varphi(\delta(Sx,By))\varphi(\delta(Ty,Ax)^p)}{\varphi(\delta(Ax,Sx)) + \varphi(\delta(By,Ty))}$

if $\varphi(\delta(Ax,Sx)) + \varphi(\delta(By,Ty)) \neq 0$,

$\varphi(\delta(Sx,Ty)^p) < k\varphi(d(Ax,By)^p)$, if $\varphi(\delta(Ax,Sx)) + \varphi(\delta(By,Ty)) = 0$

where $p \geq 1$ and $0 \leq k < \infty$.

(VIII) $\varphi(\delta(Sx,Ty)) < \min\{\max\{\varphi(\delta(Ax,Sx)), \varphi(\delta(By,Ty))\}, \varphi(\delta(Sx,By)), \varphi(\delta(Ty,Ax))\}$

$$+\psi\{\max\{\varphi(d(Ax,By)), \varphi(\delta(Ax,Sx)), \varphi(\delta(By,Ty)), \varphi(\delta(Sx,By)), \varphi(\delta(Ty,Ax))\}\},$$

wherein $\psi : \mathfrak{R}_+ \to \mathfrak{R}_+$ satisfying $\psi(t) < t$ for all $t > 0$ and ψ is upper semi-continuous.

Proof. The proof follows from Theorem 3.3 in view of Examples 2.1–2.8.

Remark 3.5. Corollaries corresponding to contraction conditions (I–VII) are new results as such corollaries extend and generalize corresponding relevant results contained in [2, 5, 6, 13, 16, 22, 25–27, 42, 54, 57, 58, 60, 63, 64].

Remark 3.6. We can also outline corollaries corresponding to Corollary 3.2 in respect of Theorems 3.5-3.7 so as to get further new results.

As an application of Theorem 3.3, we have the following result for four finite families of self mappings.

Theorem 3.8. Let $\{A_1, A_2, \ldots, A_m\}$, $\{B_1, B_2, \ldots, B_p\} : X \to X$ and $\{S_1, S_2, \ldots, S_n\}$ and $\{T_1, T_2, \ldots, T_q\} : X \to B(X)$ be four finite families of mappings of a semi-metric space (X, d) wherein d enjoys (W_3) and (HE) with $A = A_1A_2\ldots A_m$, $B = B_1B_2\ldots B_p$, $S = S_1S_2, \ldots S_n$ and $T = T_1T_2\ldots T_q$ satisfying condition (3.3.1) and the pair (A, S) or (B, T) enjoys the property $(E.A)$. If $S(X) \subset B(X)$ (or $T(X) \subset A(X)$) and $A(X)$ (or $B(X)$) is a closed subset of X, then

(a) the pair (A, S) has a strict coincidence point,

(b) the pair (B, T) has a strict coincidence point.

Moreover, if families $(\{A_i\}, \{S_r\})$ and $(\{B_k\}, \{T_t\})$ are pairwise commuting, then (for all i, r, k and t) A_i, B_k, S_r and T_t have a strict common fixed point.

Proof. Proof can be completed on the lines of Imdad et al. ([38], Theorem 2.2).

Remark 3.7. By restricting four families as $\{A_1, A_2\}, \{B_1, B_2\}, \{S_1\}$ and $\{T_1\}$ in Theorem 3.8, we deduce a substantial but partial generalization of the main results of Imdad and Khan [37] as such a result will deduce stronger commutativity condition besides relaxing continuity requirements and weakening completeness requirement of the space to the closedness of suitable subspaces.

By setting $A_1 = A_2 = \ldots = A_m = G$, $B_1 = B_2 = \ldots = B_p = H$, $S_1 = S_2 = \ldots = S_n = I$ and $T_1 = T_2 = \ldots = T_q = J$ in Theorem 3.8, we deduce the following:

Corollary 3.3. Let $G, H : X \to X$ and $I, J : X \to B(X)$ wherein semi-metric d enjoys (W_3) and (HE), pair (G^m, I^n) (or (H^p, J^q)) enjoys the property $(E.A)$, $I^n(X) \subset H^p(X)$ (or $J^q(X) \subset G^m(X)$) and satisfying the condition

$$F(\varphi(\delta(I^n x, J^q y)), \varphi(d(G^m x, H^p y)), \varphi(\delta(G^m x, I^n x)), \varphi(\delta(H^p y, J^q y)), \varphi(\delta(I^n x, H^p y)), \varphi(\delta(J^q y, G^m x))) < 0$$

for all $x, y \in X$, $F \in \Psi$ and $\varphi(t)$ is an altering distance where m, n, p and q are fixed positive integers. If $G^m(X)$ (or $H^p(X)$) is a closed subset of X, then G, H, I and J have a strict common fixed point provided $GI = IG$ and $HJ = JH$.

Remark 3.8. Corollary 3.3 is a slight but partial generalization of Theorem 3.3 as the commutativity requirements (i.e. $GI = IG$ and $HJ = JH$) in this corollary are relatively stronger as compared to weak compatibility in Theorem 3.3.

4. An Application

In [17], Branciari established the following result as a generalization of Banach fixed point theorem for integral type contractions.

Theorem 4.1. Let (X, d) be a complete metric space and $f : X \to X$ be a mapping such that for all $x, y \in X$and $k \in (0, 1)$

$$\int_0^{d(fx,fy)} \phi(t)dt \leq k \int_0^{d(x,y)} \phi(t)dt, \tag{4.1.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a Lebesgue measurable mapping (i.e. with finite integral) on each compact subset of $[0, \infty)$ such that for $\epsilon > 0$, $\int_0^{\epsilon} \phi(t)dt > 0$. Then f has a unique fixed point $z \in X$ and for all $x \in X$, $\lim_{n\to\infty} f^n x = z$.

Several common fixed point theorems in metric and semi-metric spaces for compatible, weakly compatible and OWC mappings satisfying contractive conditions of integral type are proved (e.g. [11, 23, 47–49, 55]). Later, Suzuki [62] proved that integral type contractions are Meir-Keeler contractions. He also showed that Meir-Keeler contractions of integral type are still Meir-Keeler contractions. Jachymski [39] also proved that most contractive conditions of integral type given recently by many authors coincide with classical ones. But he gave a new contractive condition of integral type which is independent of classical ones. Most recently Popa and Mocanu [55, 56] obtained integral type contractions via altering distance function and proved general common fixed point results for integral type contractive conditions.

In what follows, we further attempt to obtain common fixed point theorems for integral type contractions via altering distances, i.e. results involving integral type contractions are not new ones, but can be obtained via classical altering distances functions.

Lemma 4.1. The function $\varphi(t) = \int_0^t \phi(x)dx$, where $\phi(x)$ is (as in Theorem 4.1) an altering distance function.

Proof. By definitions of $\varphi(t)$ and $\phi(x)$, it follows that $\varphi(t)$ is increasing and $\varphi(t) = 0$ if and only if $t = 0$. By Lemma 2.5 (cf. [49]), $\varphi(t)$ is continuous.

Now, we prove a common fixed point theorem for the pairs of hybrid mappings satisfying integral type contractive condition.

Theorem 4.2. Let $A, B : X \to X$ and $S, T : X \to B(X)$ wherein X be nonempty set equipped with a semi-metric d which enjoy (W_3) and (HE). Suppose that

(a) the pair (A, S) (or (B, T)) share the property $(E.A)$,

(b) $S(X) \subset B(X)$ (or $T(X) \subset A(X)$),

(c) $A(X)$ (or $B(X)$) is a closed subset of X and

$$F\Big(\int_0^{\delta(Sx,Ty)} \phi(t)dt, \int_0^{d(Ax,By)} \phi(t)dt, \int_0^{\delta(Ax,Sx)} \phi(t)dt, \int_0^{\delta(By,Ty)} \phi(t)dt, \int_0^{\delta(Sx,By)} \phi(t)dt, \int_0^{\delta(Ty,Ax)} \phi(t)dt\Big) < 0 \tag{4.2.1}$$

for all $x, y \in X$, $F \in \Psi$ and $\phi \in \Phi$. If the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique strict common fixed point.

Proof. In view of Lemma 4.1, we have $\varphi(\delta(Sx, Ty)) = \int_0^{\delta(Sx,Ty)} \phi(t)dt$, $\varphi(d(Ax, By)) = \int_0^{d(Ax,By)} \phi(t)dt$, $\varphi(\delta(Ax, Sx)) = \int_0^{\delta(Ax,Sx)} \phi(t)dt$, $\varphi(\delta(By, Ty)) = \int_0^{\delta(By,Ty)} \phi(t)dt$, $\varphi(\delta(Sx, By)) = \int_0^{\delta(Sx,By)} \phi(t)dt$ and $\varphi(\delta(Ty, Ax)) = \int_0^{\delta(Ty,Ax)} \phi(t)dt$. Then by inequality (4.2.1), we have

$$F\Big(\varphi(\delta(Sx, Ty)), \varphi(d(Ax, By)), \varphi(\delta(Ax, Sx)), \varphi(\delta(By, Ty)), \varphi(\delta(Sx, By)), \varphi(\delta(Ty, Ax))\Big) < 0.$$

In view of Lemma 4.1, $\varphi(t)$ is an altering distance function. Hence all the conditions of Theorem 3.3 are satisfied and therefore conclusions of Theorem 4.2 follow from Theorem 3.3. This completes the proof.

Corollary 4.1. The conclusions of Theorem 4.2 remain true if inequality (4.2.1) is replaced by one of the following contraction conditions: (for all $x, y \in X$, $F \in \Psi$ and $\phi \in \Phi$)

(I) $\int_0^{\delta(Sx,Ty)} \phi(t)dt < \psi\left(\int_0^{\max\{d(Ax,By),\delta(Ax,Sx),\delta(By,Ty),\delta(Sx,By),\delta(Ty,Ax)\}} \phi(t)dt\right)$
where $\psi : \Re_+ \to \Re_+$ is an upper semi-continuous function satisfying $\psi(t) < t$ for all $t > 0$,

(II) $\int_0^{\delta(Sx,Ty)} \phi(t)dt < \psi\Big(\max\Big\{\int_0^{d(Ax,By)} \phi(t)dt, \int_0^{\delta(Ax,Sx)} \phi(t)dt, \int_0^{\delta(By,Ty)} \phi(t)dt, \int_0^{\delta(Sx,By)} \phi(t)dt, \int_0^{\delta(Ty,Ax)} \phi(t)dt\Big\}\Big)$
where $\psi : \Re_+ \to \Re_+$ is an upper semi-continuous function satisfying $\psi(t) < t$ for all $t > 0$,

(III) $$\Big(\int_0^{\delta(Sx,Ty)} \phi(t)dt\Big)^p < \psi\Big(a\Big(\int_0^{d(Ax,By)} \phi(t)dt\Big)^p + (1-a)\max\Big\{\alpha\Big(\int_0^{\delta(Ax,Sx)} \phi(t)dt\Big)^p, \beta\Big(\int_0^{\delta(By,Ty)} \phi(t)dt\Big)^p,$$
$$\Big(\int_0^{\delta(Ax,Sx)} \phi(t)dt\Big)^{\frac{p}{2}}.\Big(\int_0^{\delta(Ty,Ax)} \phi(t)dt\Big)^{\frac{p}{2}}, \Big(\int_0^{\delta(Sx,By)} \phi(t)dt\Big)^{\frac{p}{2}}.\Big(\int_0^{\delta(Ty,Ax)} \phi(t)dt\Big)^{\frac{p}{2}}\Big\}\Big)$$
where $\psi : \Re_+ \to \Re_+$ is an upper semi-continuous function satisfying $\psi(t) < t$ for all $t > 0$, $0 \le a, \alpha, \beta \ge 1$ and $p \ge 1$,

(IV) $$\int_0^{\delta(Sx,Ty)} \phi(t)dt < \alpha \max\Big\{\int_0^{d(Ax,By)} \phi(t)dt, \int_0^{\delta(Ax,Sx)} \phi(t)dt, \int_0^{\delta(By,Ty)} \phi(t)dt\Big\}$$
$$+(1-\alpha)\left(a\int_0^{\delta(Sx,By)} \phi(t)dt + b\int_0^{\delta(Ty,Ax)} \phi(t)dt\right)$$
$0 \le \alpha < 1$, $a, b \ge 0$ and $a + b \le 1$,

(V) $\int_0^{\delta(Sx,Ty)} \phi(t)dt < \psi\left(\max\left\{\int_0^{d(Ax,By)} \phi(t)dt, \int_0^{\delta(By,Ty)} \phi(t)dt, \int_0^{\delta(Sx,By)} \phi(t)dt\right\}\right)$

$$+\frac{1}{2}\left(\int_0^{\delta(Sx,By)} \phi(t)dt + \int_0^{\delta(Ty,Ax)} \phi(t)dt\right)$$

where $\psi : \Re_+ \to \Re_+$ is an upper semi-continuous function satisfying $\psi(t) < t$ for all $t > 0$.

Proof. The proof follows from Theorem 4.2 and Examples 2.9–2.13.

Remark 4.1. Theorem 4.2 and Corollary 4.1 extend and generalize several relevant results especially those contained in [10, 11, 17, 23, 48, 53, 55, 65].

5. An Illustrative Example

Example 5.1. Let $X = [0, 1]$ equipped with semi-metric $d(x, y) = (x - y)^2$. Define mappings $A = B : X \to X$ and $S = T : X \to B(X)$ by

$$A(x) = \frac{x}{2} \quad \text{and} \quad S(x) = \left[0, \frac{x}{x+4}\right] \qquad \forall\ x \in X.$$

Then we have that $S(X) = [0, \frac{1}{5}] \subseteq [0, \frac{1}{2}] = A(X)$ and $AS(0) = SA(0) = \{0\}$. Thus pair (A, S) is weakly compatible. Moreover, $A(X) = [0, \frac{1}{2}]$ is closed in X and pair (A, S) satisfies the property (E.A.) (consider $\{x_n\} = \left\{\frac{1}{n}\right\}$) as

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \{0\}.$$

Now, we begin to verify Condition (3.3.1) of Theorem 3.3. Consider the map F as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \psi(\max\{t_2, t_3, t_4, t_5, t_6\})$$

and define $\psi : \Re^+ \to \Re$ by $\psi(t) = \frac{t}{4}$ for all $t \geq 0$ and φ is a suitable altering distance.

If $x, y \in (0, 1]$ and $x \geq y$ (or $y \geq x$), then

$$\varphi(\delta(Sx, Sy)) = \varphi(\max\{\left(\frac{x}{x+4}\right)^2, \left(\frac{y}{y+4}\right)^2\}) < \varphi(\max\{\left(\frac{x}{4}\right)^2, \left(\frac{y}{4}\right)^2\})$$

$$= \frac{1}{4}\varphi(\left(\frac{x}{2}\right)^2) = \frac{1}{4}\varphi(\delta(Ax, Sy))$$

$$\leq \frac{1}{4}\left(\max\left\{\varphi(d(Ax, Ay)), \varphi(\delta(Ax, Sx)), \varphi(\delta(Ay, Sy)), \varphi(\delta(Sx, Ay)), \varphi(\delta(Sy, Ax))\right\}\right)$$

$$= \psi\left(\max\left\{\varphi(d(Ax, Ay)), \varphi(\delta(Ax, Sx)), \varphi(\delta(Ay, Sy)), \varphi(\delta(Sx, Ay)), \varphi(\delta(Sy, Ax))\right\}\right).$$

Therefore all the conditions of Theorem 3.3 are satisfied. Here, 0 is a coincidence as well as a unique strict common fixed point of the pair (A, S).

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