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From Graphical Metric Spaces to Fixed Point Theory in Binary Related Distance Spaces

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Abstract. Very recently, many fixed point results have been introduced in the setting of graphical metric spaces. Due to their intimate links, such works also deal with metric spaces endowed with partial orders. As the reachability relationship in any directed graph (containing all cycles) is a reflexive transitive relation (that is, a preorder), but it is not necessarily a partial order, results on graphical metric spaces are independent from statements on ordered metric spaces. The main aim of this paper is to show that fixed point theorems in the setting of graphical metric spaces can be directly deduced from their corresponding results on measurable spaces endowed with a binary relation. Finally, we also describe the main advantages of involving this last class of spaces.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

Very recently, some manuscripts in the field of fixed point theory have appeared in the setting of graphical metric spaces. The condition of having a graph on the underlying space is mainly interesting when the graph has a mathematical/practical significance (for instance, possible ways between some places in a city). When having a graph on a set, a new possibility appears: we can move from a point to another point following an edge of the graph. And, when we repeat these movements a finite number of times, we are considering a (*directed* throughout the edges of the graph) *path* from a point to another point. This especial characteristic permit us to introduce a new version of the triangle inequality for graphical metric spaces: it must only be satisfied for points placed on a directed path. And, more specifically, such triangle inequality bounds the distance between the extremes of the path (the third point must be an interior point of the path). As a consequence, graphical metric spaces generalize metric spaces because we do not need to assume that the triangle inequality holds for all three points of the space. But graphical metric spaces have another important advantage in the field of fixed point theory.

From a mathematical point of view, the set of all edges of a graph on a set is a reflexive binary relation on the own set. We remark that reflexivity naturally appears in manuscripts dealing in this topic, but it is

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not necessary. In the past, many contractivity conditions were considered by assuming that they only held for pairs of points that were comparable by the binary relation (we especially recall here the pioneering manuscripts [18, 20, 29], in which a partial order is involved). **Prof. Ćirić successfully contributed to this study** (see [10–12, 17]). But, in most of cases, the considered binary relations were transitive. In this sense, if the contractivity condition holds for comparable points x and y, and for y and z, then it necessarily holds for x and z. Most of results in fixed point theory decisively use this fact as a key argument of their proofs. However, graphical metric spaces have highlighted that, under some kind of contractivity conditions, transitivity can be avoided. If the contractivity condition holds for two edges (x, y) and (y, z) in a graphical metric space, then it is not necessary that it also holds for (x, z) (in fact, it is possible that this pair is not an edge of the graph).

Many researchers have working for weakening the conditions on the binary relation, having in mind the notion of partial order. For instance, in [21], the authors started to explore this line of research by using preordered sets, avoiding the antisymmetric condition. The subsequent papers [2, 6, 8, 9, 16, 22–24, 26, 27] also deal with binary relations that do not satisfy any condition of the definition of partial orders. To get success, the type of considered contractivity condition is essential: some of them are as general that we cannot omit the transitivity.

This manuscript has two main aims: on the one hand, we show how subtle must be the arguments employed in fixed point theory when we handle graphical metric spaces. Researchers run the risk of applying the triangle inequality when two points are not connected by a directed path. To do this, we illustrate a counterexample for a result recently appeared in Shukla *et al.* [28]. On the other hand, we extend some fixed point theorems in this setting to a kind of weak metric spaces endowed with an arbitrary binary relation. In fact, we are convinced that fixed point theory on graphical metric spaces can be reduced to their corresponding results in a more general context by involving binary relations.

2. Preliminaries

From now on, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ stands for the set of all nonnegative integers and $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$. Henceforth, let X be a nonempty set and let $T : X \to X$ be a mapping from X into itself. A sequence $\{x_n\} \subseteq X$ is *infinite* if $x_n \neq x_m$ for all $n, m \in \mathbb{N}$. A *fixed point of* T is a point $x \in X$ such that $Tx = x$. We denote by $\mathrm{Fix}(T)$ the set of all fixed points of T. The *iterates* of the self-mapping T are the mappings $\{T^n : X \to X\}_{n\in\mathbb{N}}$ defined by

$$T^0 = I_X, \quad T^1 = T, \quad T^2 = T \circ T, \quad T^{n+1} = T \circ T^n \quad \text{for all } n \geq 2.$$

Given a point $x_0 \in X$, the *Picard sequence of* T *based on* x_0 is the sequence $\{x_n\}_{n\in\mathbb{N}}$ given by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Notice that $x_n = T^n x_0$ for all $n \in \mathbb{N}$.

A *binary relation on* X is a nonempty subset $\mathcal{S}$ of the Cartesian product $X \times X$. For simplicity, we denote $x\mathcal{S}y$ if $(x, y) \in \mathcal{S}$. A sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is $\mathcal{S}$*-nondecreasing* if $x_n\mathcal{S}x_{n+1}$ for all $n \in \mathbb{N}$. Two points x and y are $\mathcal{S}$*-comparable* if $x\mathcal{S}y$ or $y\mathcal{S}x$. We denote it by $x\widetilde{\mathcal{S}}y$. Clearly, $\widetilde{\mathcal{S}}$ is the least symmetric binary relation on X that contains $\mathcal{S}$.

A binary relation $\mathcal{S}$ on X is *reflexive* if $x\mathcal{S}x$ for all $x \in X$; it is *transitive* if $x\mathcal{S}z$ for all $x, y, z \in X$ such that $x\mathcal{S}y$ and $y\mathcal{S}z$; and it is *antisymmetric* if $x\mathcal{S}y$ and $y\mathcal{S}x$ imply $x = y$. A *preorder* (or a *quasiorder*) is a reflexive, transitive binary relation. And a *partial order* is an antisymmetric preorder. We denote by $\mathcal{S}_X$ the trivial preorder on X given by $x\mathcal{S}_X y$ for all $x, y \in X$.

If X is endowed with a binary relation $\mathcal{S}$, we say that T is $\mathcal{S}$*-nondecreasing* if $Tx\mathcal{S}Ty$ for all $x, y \in X$ such that $x\mathcal{S}y$.

Following [1, 4, 7], a function $\phi : [0, \infty) \longrightarrow [0, \infty)$ is a *(c)-comparison function* if ϕ is nondecreasing and $\sum_{n\in\mathbb{N}} \phi^n(t) < \infty$ for all $t > 0$. We denote by $\mathcal{F}^{(c)}_{\mathrm{com}}$ the family of all (c)-comparison functions. Examples of (c)-comparison functions are $\phi_\lambda(t) = \lambda\, t$ for all $t \in [0, \infty)$, where $\lambda \in [0, 1)$. It is easy to deduce that a (c)-comparison function must satisfy: (1) $\phi(t) < t$ for all $t \in (0, \infty)$; (2) $\phi(0) = 0$; (3) ϕ is continuous at $t = 0$; (4) if $\phi(t) \geq t$, then $t = 0$; (5) $\phi^m(t) \leq \phi^n(t) \leq t$ for all $t \in [0, \infty)$ and all $n, m \in \mathbb{N}$ such that $n \leq m$; (6) ϕ^n is nondecreasing for all $n \in \mathbb{N}$; and (7) $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in [0, \infty)$.

Let X be a nonempty set and let Δ denote the diagonal of the cartesian product $X \times X$. Let G be a graph, without parallel edges, such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, that is, $\Delta \subseteq E(G)$. Then X is said to be endowed with a graph $G = (V(G), E(G))$.

If x and y are vertices in a graph G, then a *path (or directed path) from* x to y of length $r \in \mathbb{N}$ is a finite sequence $z_0, z_1, \ldots, z_r \in V(G)$ of $r+1$ vertices of G such that $z_0 = x$, $z_r = y$ and $(z_{i-1}, z_i) \in E(G)$ for all $i \in \{1, 2, \ldots, r\}$. A graph is *connected* if there is a path between any two vertices. Furthermore, two vertices x and y of a directed graph are *connected* if there is a path from x to y and a path from y to x. A graph G is *weakly connected* if, treating all its edges as being undirected, there is a path from every vertex to every other vertex.

If $x \in V(G)$ and $r \in \mathbb{N}$, let $[x]_G^r$ be the set of all points $y \in X$ such that there is a directed path from x to y of length r belonging to G. In such a case we will write $(xPy)_G$. We write $z \in (xPy)_G$ if z is included in some directed path from x to y. A sequence $\{x_n\} \subseteq X$ is said to be *G-termwise* if $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, and $\{x_n\}$ is *G-termwise connected* if $(x_n P x_{n+1})_G$ for all $n \in \mathbb{N}$ (that is, $x_{n+1} \in [x_n]_G^{r_n}$ for all $n \in \mathbb{N}$, where $r_n \in \mathbb{N}$).

Definition 2.1 (Shukla *et al.* [28], Definition 2.1). *Let X be a nonempty set endowed with a graph G and $d_G : X \times X \to \mathbb{R}$ be a function satisfying the following conditions:*

($GM1$) $d_G(x, y) \geq 0$ *for all* $x, y \in X$;

($GM2$) $d_G(x, y) = 0$ *if and only if* $x = y$;

($GM3$) $d_G(x, y) = d_G(y, x)$ *for all* $x, y \in X$;

($GM4$) $(xPy)_G$, $z \in (xPy)_G$ *implies* $d_G(x, y) \leq d_G(x, z) + d_G(z, y)$ *for all* $x, y \in X$.

The the mapping d_G is called a graphical metric on X, *and the pair (X, d_G) is called a* graphical metric space.

If (X, d_G) is a graphical metric space, then $E(G) \subseteq X \times X$ is a reflexive binary relation on X. To be coherent with our notation, we will denote it by $\mathcal{S}_G$, that is,

$$x \mathcal{S}_G y \quad \text{if} \quad (x, y) \in E(G). \tag{1}$$

Definition 2.2. *In a graphical metric space (X, d_G), a sequence $\{x_n\} \subseteq X$ is d_G-*Cauchy *if $\lim_{n,m\to\infty} d_G(x_n, x_m) = 0$, and it is d_G-*convergent to $x \in X$ *if $\lim_{n\to\infty} d_G(x_n, x) = 0$ (in such a case, we will write $\{x_n\} \xrightarrow{d_G} x$ and we will say that x is a d_G-*limit of $\{x_n\}$*). We say that (X, d_G) is* complete *if every d_G-Cauchy sequence in X is d_G-convergent to a point of X.*

The following notion of completeness is slightly different to the previous one.

Definition 2.3. *Let (X, d_G) be a graphical metric space and let G' be a subgraph of G such that $V(G') = X$. We say that (X, d_G) is G'-*complete *if every G'-termwise connected d_G-Cauchy sequence in X d_G-converges in X.*

Definition 2.4 (Shukla *et al.* [28], Definition 3.1). *Let (X, d_G) be a graphical metric space, $T : X \to X$ be a mapping and G' be a subgraph of G such that $\Delta \subseteq E(G')$. Then T is called a (G, G')-graphical contraction on X if the following conditions hold:*

($GC1$) T preserves edges in $E(G')$, *that is, $(x, y) \in E(G')$ implies $(Tx, Ty) \in E(G')$;*

($GC2$) *there exists $\lambda \in [0, 1)$ such that*

$$d_G(Tx, Ty) \leq \lambda\, d_G(x, y) \quad \text{for all } x, y \in X \text{ with } (x, y) \in E(G').$$

We must point out that, to our opinion, it is not convenient to impose condition ($GC1$) for a contraction. Probably, this condition derives from Definition 2.1 given by Jachymski in [13], where the author also imposed this axiom for *Banach G-contractions*. However, this assumption means that many classical contractions are not contractions in this sense. For instance, if $X = \mathbb{R}$ is endowed with the graph G such that $V(G) = \mathbb{R}$ and $(x, y) \in E(G) \Leftrightarrow y \leq x$, then the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by $Tx = -x/2$ for all $x \in \mathbb{R}$ is not a contraction in the sense of Definition 2.4, which is a little bit odd. Condition (GC_1) can be interpreted as a nondecreasing assumption w.r.t. the binary relation $\mathcal{S}_G$ on X and, from our point of view, it should be removed from Definition 2.4. Anyhow, it could only have a sense if we suppose that the domain of the metric d_G is $E(G)$ rather than $X \times X$.

Definition 2.5 (Shukla *et al.* [28], Definition 3.7). *Let (X, d_G) be a graphical metric space and $T : X \to X$ be a mapping. Then, the quadruple (X, d_G, G', T) is said to have the* property (S) *if:*

(S) *whenever a G'-termwise connected T-Picard sequence $\{x_n\}$ has two limits x^* and y^*, where $x^* \in X$, $y^* \in T(X)$, then $x^* = y^*$.*

3. Some Commentaries about Fixed Point Theory in Graphical Metric Spaces

As we have commented in the introduction, graphical metric spaces have proved to be of interest in the field of fixed point theory. They have highlighted that we can consider contractivity conditions that only must hold for binary relations that are not necessarily transitive. If the contractivity condition is strong enough, transitivity can be avoided. Other advantage of graphical metric space is the fact that we have only to check that the triangle inequality is satisfied for three points placed in a directed path of the graph. But this advantage can become a drawback because we have not a control on the distance between two points that are not in a directed path. This fact could lead researchers to make mistakes when working in this setting, especially when we try to use the generalized version of the triangle inequality (see Remark 4.14 below). This is the case we are going to describe.

In [28], the authors proved the following theorem.

Theorem 3.1 (Shukla *et al.* [28], Theorem 3.10). *Let (X, d_G) be a G'-complete graphical metric space and $T : X \to X$ be a (G, G')-graphical contraction. Suppose that the following conditions hold:*

(A) *there exists $x_0 \in X$ such that $Tx_0 \in [x_0]^l_{G'}$ for some $l \in \mathbb{N}$;*

(B) *if a G'-termwise connected T-Picard sequence $\{x_n\}$ converges in X, then there exist a limit $z \in X$ of $\{x_n\}$ and $n_0 \in \mathbb{N}$ such that $(x_n, z) \in E(G')$ or $(z, x_n) \in E(G')$ for all $n > n_0$.*

Then, there exists $x^* \in X$ such that the T-Picard sequence $\{x_n\}$ with initial value $x_0 \in X$, is G'-termwise connected and converges to both, x^* and Tx^*. In addition, if the quadruple (X, d_G, G', T) has the property (S), then T has a fixed point in X.

After that, the same authors studied the uniqueness of the fixed point. Let

$$X_T = \{ x \in X : (x, Tx) \in E(G') \}.$$

Theorem 3.2 (Shukla *et al.* [28], Theorem 3.13). *Let (X, d_G) be a G'-complete graphical metric space and $T : X \to X$ be a (G, G')-graphical contraction. Suppose that all the conditions of Theorem 3.10 are satisfied, then T has a fixed point. In addition, if X_T is weakly connected (as a subgraph of G'), then the fixed point of T is unique.*

Here we are going to show that weakly connectedness of X_T is not strong enough to guarantee that the fixed point of T is unique, that is, Theorem 3.2 is false.

Let $X = [-1, 0) \cup (0, 1] \cup \{ \omega^-, \omega^+ \}$ where ω^- and ω^+ are two distinct points such that $([-1, 0) \cup (0, 1]) \cap \{ \omega^-, \omega^+ \} = \varnothing$. Let $G = G'$ be the graph such that $V(G) = V(G') = X$ and

$$\begin{aligned} E(G) = E(G') = \Delta &\cup \{ (x, y) \in X \times X : 0 < y < x \leq 1 \} \\ &\cup \{ (x, y) \in X \times X : -1 \leq x < y < 0 \} \\ &\cup \{ (x, -x) \in X \times X : 0 < x \leq 1 \} \\ &\cup \{ (\omega^-, x) \in X \times X : -1 \leq x < 0 \} \\ &\cup \{ (x, \omega^+) \in X \times X : 0 < x \leq 1 \}. \end{aligned}$$

Let $d_G : X \times X \to [0,\infty)$ be the function given by:

$$d_G(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x,y \in [-1,0) \cup (0,1] \text{ are such that} \\ & \qquad y < 0 < x < -y \quad \text{or} \quad x < 0 < y < -x, \\ \left|x-y\right|, & \text{if } x,y \in [-1,0) \cup (0,1]\,,\ x \neq y \text{ in other case,} \\ y, & \text{if } x = \omega^+ \text{ and } y \in (0,1]\,, \\ x, & \text{if } y = \omega^+ \text{ and } x \in (0,1]\,, \\ 1+\left|y\right| = 1-y, & \text{if } x = \omega^+ \text{ and } y \in [-1,0)\,, \\ 1+\left|x\right| = 1-x, & \text{if } y = \omega^+ \text{ and } x \in [-1,0)\,, \\ 1+y, & \text{if } x = \omega^- \text{ and } y \in (0,1]\,, \\ 1+x, & \text{if } y = \omega^- \text{ and } x \in (0,1]\,, \\ \left|y\right| = -y, & \text{if } x = \omega^- \text{ and } y \in [-1,0)\,, \\ \left|x\right| = -x, & \text{if } y = \omega^- \text{ and } x \in [-1,0)\,, \\ 2, & \text{if } \{x,y\} = \{\omega^-,\omega^+\}\,. \end{cases}$$

Let $T : X \to X$ be given by

$$Tx = \begin{cases} x/2, & \text{if } x \in [-1,0) \cup (0,1]\,, \\ x, & \text{if } x \in \{\omega^-,\omega^+\}\,. \end{cases}$$

We are going to show that all hypotheses of Theorem 3.2 hold and, although $X_T = X$ is weakly connected, the contraction T has two distinct fixed points (which are ω^- and ω^+).

1. Unless the diagonal Δ, there are only five kind of edges:

$$(\oplus,\oplus)\,,\ (\ominus,\ominus)\,,\ (\oplus,\ominus)\,,\ (\omega^-,\ominus) \text{ and } (\oplus,\omega^+)\,.$$

2. Let $x,y \in X$ be such that $(x,y) \in E(G)$.

$$\begin{cases} \bullet \text{ If } y \in (0,1], \text{ then } x \in (0,1] \text{ and } 0 < y \leq x \leq 1. \\ \bullet \text{ If } x \in [-1,0)\,, \text{ then } y \in [-1,0) \text{ and } -1 \leq x \leq y < 0. \end{cases}$$

3. If $x \in X$ and $(x,\omega^-) \in E(G)$, then $x = \omega^-$.
4. If $x \in X$, $(\omega^-,x) \in E(G)$ and $x \neq \omega^-$, then $x \in [-1,0)$.
5. If $x \in X$ and $(\omega^+,x) \in E(G)$, then $x = \omega^+$.
6. If $x \in X$, $(x,\omega^+) \in E(G)$ and $x \neq \omega^+$, then $x \in (0,1]$.
7. If $x,y \in [-1,0) \cup (0,1]$ are such that $(x,y) \in E(G)$ and x and y have different sign, then x is positive, y is negative and $|x| = \left|y\right|$ (in particular, $y = -x < 0 < x$).

$$\left.\begin{array}{c} x,y \in [-1,0) \cup (0,1] \\ (x,y) \in E(G) \\ x\,y < 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} y = -x < 0 < x \\ (x,y) = (\oplus,\ominus) \\ |x| = \left|y\right| \end{array}\right.$$

8. If $x \in (0,1]$ then $(\omega^-,-x), (-1,-x)\,, (1,-1)\,, (1,x)\,, (x,\omega^+) \in E(G)$.
9. The graph G is weakly connected because, treating all its edges as being undirected, there is a path from every vertex to every other vertex.
10. $X_T = X$.
11. The set $X_T = X$ is weakly connected because X it is.
12. Let $z_0, z_1, \ldots, z_r \in X$ be a nontrivial ($r > 0$) directed path in G such that $z_{i-1} \neq z_i$ for all $i \in \{1,2,\ldots,r\}$.

$$\begin{cases} \bullet \text{ If } \omega^- \in \{z_0,z_1,\ldots,z_r\}, \text{ then } z_0 = \omega^-, \quad z_1,z_2,\ldots,z_r \in [-1,0) \quad \text{and} \\ \qquad -1 \leq z_1 < z_2 < \ldots < z_{r-1} < z_r < 0. \\ \qquad \text{In particular, } 0 < |z_r| < |z_{r-1}| < \ldots < |z_2| < |z_1|\,. \\ \bullet \text{ If } \omega^+ \in \{z_0,z_1,\ldots,z_r\}, \text{ then } z_r = \omega^+, \quad z_0,z_1,\ldots,z_{r-1} \in (0,1] \quad \text{and} \\ \qquad 0 < z_{r-1} < z_{r-2} < \ldots < z_1 < z_0 \leq 1. \\ \qquad \text{In particular, } 0 < |z_{r-1}| < |z_{r-2}| < \ldots < |z_1| < |z_0|\,. \end{cases}$$

13. *There is not a directed path on G joining ω^- and ω^+ (neither from ω^- to ω^+ nor viceversa).*
14. Let $z_0, z_1, \ldots, z_r \in X$ be a directed path in G (vertices can be repeated).

$$\begin{cases} \bullet \text{ If } \omega^- \in \{z_0, z_1, \ldots, z_r\}, \text{ then there is } i_0 \in \{0,1,\ldots,r\} \text{ such that} \\ \qquad z_i = \omega^- \text{ for all } i \le i_0 \text{ and } z_i \in [-1,0) \text{ for all } i > i_0. \\ \bullet \text{ If } \omega^+ \in \{z_0, z_1, \ldots, z_r\}, \text{ then there is } i_0 \in \{0,1,\ldots,r\} \text{ such that} \\ \qquad z_i = \omega^+ \text{ for all } i \ge i_0 \text{ and } z_i \in (0,1] \text{ for all } i < i_0. \end{cases}$$

15. Let $z_0, z_1, \ldots, z_r \in X$ be a directed path in G (vertices can be repeated).

$$\begin{cases} \bullet \text{ If } \omega^- \in \{z_0, z_1, \ldots, z_r\}, \text{ then } \{\, z_0, z_1, \ldots, z_r \,\} = \{\, \omega^-, \omega^-, \ldots, \omega^-, \ominus, \ominus, \ldots \ominus \,\} \\ \qquad \text{(where negative terms can exist or not).} \\ \bullet \text{ If } \omega^+ \in \{z_0, z_1, \ldots, z_r\}, \text{ then } \{\, z_0, z_1, \ldots, z_r \,\} = \{\, \oplus, \oplus, \ldots, \oplus, \omega^+, \omega^+, \ldots, \omega^+ \,\} \\ \qquad \text{(where positive terms can exist or not).} \end{cases}$$

16. If a directed path $z_0, z_1, \ldots, z_r \in X$ in G contains a positive term and a negative term, then $z_0, z_1, \ldots, z_r \in [-1,0) \cup (0,1]$.
17. If $z_0, z_1, \ldots, z_r \in X$ is a directed path in G such that $z_{i-1} \neq z_i$ for all $i \in \{1, 2, \ldots, r\}$, then ω^- can only be placed at the first position (z_0), and ω^+ can only be placed at the last position (z_r). The rest of points of the directed path are included in $[-1,0) \cup (0,1]$.
18. Let $z_0, z_1, \ldots, z_r \in [-1,0) \cup (0,1]$ be a directed path in G. Then one, and only one, of the following cases holds:

$$\begin{cases} \bullet\ z_0, z_1, \ldots, z_r \in (0,1] \text{ and } 0 < z_r \le z_{r-1} \le \ldots \le z_1 \le z_0 \le 1; \\ \bullet\ z_0, z_1, \ldots, z_r \in [-1,0) \text{ and } -1 \le z_0 \le z_1 \le \ldots \le z_{r-1} \le z_r < 0; \\ \bullet \text{ there is } i_0 \in \{1, 2, \ldots, r\} \text{ such that} \\ \qquad -1 \le z_{i_0} \le z_{i_0+1} \le \ldots \le z_{r-1} \le z_r < 0 < -z_{i_0} = z_{i_0-1} \le z_{i_0-2} \le \ldots \le z_1 \le z_0 \le 1. \end{cases}$$

19. Let $z_0, z_1, \ldots, z_r \in [-1,0) \cup (0,1]$ be a directed path in G. Then the ordered sequence $\{z_0, z_1, \ldots, z_r\}$ contains, at most, a unique change of sign. Furthermore,

$$0 < |\, z_r \,| \le |\, z_{r-1} \,| \le \ldots \le |\, z_1 \,| \le |\, z_0 \,|.$$

20. Let $z_0, z_1, \ldots, z_r \in [-1,0) \cup (0,1]$ be a directed path in G. If the finite sequence $z_0, z_1, \ldots, z_r$ contains a negative term z_{i_0}, then $z_{i_0}, z_{i_0+1}, \ldots, z_r$ are all negative numbers and

$$-1 \le z_{i_0} < z_{i_0+1} < \ldots < z_{r-1} < z_r < 0.$$

21. If a directed path $z_0, z_1, \ldots, z_r \in [-1,0) \cup (0,1]$ contains positive and negative terms, then the first terms are positive, and, from the first negative term and so on, the rest are all negative, satisfying

$$\begin{aligned} &-1 \le z_{i_0} \le z_{i_0+1} \le \ldots \le z_{r-1} \le z_r < 0 < -z_{i_0} = z_{i_0-1} \le z_{i_0-2} \le \ldots \le z_1 \le z_0 \le 1 \\ &\quad \text{and} \quad 0 < |\, z_r \,| \le |\, z_{r-1} \,| \le \ldots \le |\, z_1 \,| \le |\, z_0 \,|. \end{aligned}$$

$$\{\, z_0, z_1, \ldots, z_r \,\} = \{\, \oplus, \oplus, \ldots, \oplus, \ominus, \ominus, \ldots \ominus \,\}.$$

22. If $x, y, z \in X$ are three distinct points such that $y \in (xPz)_G$, then one, and only one, of the following cases holds:

$$\begin{cases} \left. \begin{array}{l} \bullet\ x, y, z \in (0,1] \text{ and } 0 < z < y < x; \\ \bullet\ x, y \in (0,1],\ z \in [-1,0) \text{ and } z < 0 < -z \le y < x; \\ \bullet\ x \in (0,1],\ y, z \in [-1,0) \text{ and } y < z < 0 < -z < -y \le x; \\ \bullet\ x, y, z \in [-1,0) \text{ and } x < y < z < 0; \end{array} \right\} \begin{array}{c} \text{In these cases,} \\ x, y, z \in [-1,0) \cup (0,1] \\ \text{and } 0 < |\, z \,| \le \left|\, y \,\right| \le |\, x \,|. \end{array} \\ \\ \bullet\ z = \omega^+,\ x, y \in (0,1] \text{ and } 0 < y < x \le 1; \\ \bullet\ x = \omega^-,\ y, z \in [-1,0) \text{ and } -1 \le y < z < 0. \end{cases}$$

23. *(X, d_G) is a graphical metric space.*
24. If $\{x_n\}_{n\in\mathbb{N}} \subset X$ is a G-termwise connected sequence, then there is a sequence $\{y_n\}_{n\in\mathbb{N}} \subset X$ such that $(y_n, y_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is a subsequence of $\{y_n\}$ (that is, $x_n = y_{\sigma(n)}$ for all $n \in \mathbb{N}$, where $\sigma : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function such that $\sigma(0) = 0$, so $x_0 = y_0$).
25. Let $\{x_n\}_{n\in\mathbb{N}} \subset X$ be a G-termwise connected sequence. Then one, and only one, of the following cases holds:

$$\begin{cases} (a) \text{ there is } n_0 \in \mathbb{N} \text{ such that } x_n = \omega^+ \text{ for all } n \geq n_0 \text{ and } x_n \in (0,1] \text{ for all } n < n_0; \\ (b)\ x_n = \omega^- \text{ for all } n \in \mathbb{N}; \\ (c) \text{ there is } n_0 \in \mathbb{N} \text{ such that } x_n = \omega^- \text{ for all } n \leq n_0 \text{ and } x_n \in [-1,0) \text{ for all } n > n_0 \\ \qquad \text{(in this case, } -1 \leq x_n \leq x_{n+1} < 0 \text{ for all } n > n_0); \\ (d)\ -1 \leq x_n \leq x_{n+1} < 0 \text{ for all } n \in \mathbb{N}; \\ (e)\ 0 < x_{n+1} < x_n \leq 1 \text{ for all } n \in \mathbb{N}; \\ (f) \text{ there is } n_0 \in \mathbb{N} \text{ such that } \quad 0 < x_{n_0} < x_{n_0-1} \leq \ldots \leq x_1 \leq x_0 \leq 1 \quad \text{and} \\ \qquad -1 \leq x_n \leq x_{n+1} < 0 \text{ for all } n > n_0. \end{cases}$$

26. If $\{x_n\} \subset X$ is a d_G-Cauchy sequence, then there is $n_0 \in \mathbb{N}$ such that

$$\begin{cases} \bullet \text{ either } x_n \in (0,1] \cup \{\omega^+\} \text{ for all } n \geq n_0; \\ \bullet \text{ or } x_n \in [-1,0) \cup \{\omega^-\} \text{ for all } n \geq n_0. \end{cases}$$

27. Let $\{x_n\} \subset X$ be a d_G-Cauchy sequence and let $n_0 \in \mathbb{N}$ be such that either $x_n \in (0,1] \cup \{\omega^+\}$ for all $n \geq n_0$ or $x_n \in [-1,0) \cup \{\omega^-\}$ for all $n \geq n_0$. If $\{\, x_n : n \geq n_0 \,\} \cap [-1,1]$ contains infinite terms, then it is a d_E-Cauchy sequence, where d_E is the Euclidean metric given by $d_E(x,y) = \left| x - y \right|$ for all $x, y \in X$.
28. *The space (X, d_G) is (G, G')-complete.*
29. If $\{x_n\}$ is a Picard sequence of T, then:

$$\begin{cases} \bullet \text{ if } x_0 = \omega^-, \text{ then } x_n = \omega^- \text{ for all } n \in \mathbb{N} \text{ and} \\ \qquad \text{this sequence only } d_G\text{-converges to } \omega^-; \\ \bullet \text{ if } x_0 = \omega^+, \text{ then } x_n = \omega^+ \text{ for all } n \in \mathbb{N} \text{ and} \\ \qquad \text{this sequence only } d_G\text{-converges to } \omega^+; \\ \bullet \text{ if } x_0 \in (0,1], \text{ then } x_n = x_0/2^n \in (0,1] \text{ for all } n \in \mathbb{N} \text{ and} \\ \qquad \text{this sequence only } d_G\text{-converges to } \omega^+; \\ \bullet \text{ if } x_0 \in [-1,0), \text{ then } x_n = x_0/2^n \in [-1,0) \text{ for all } n \in \mathbb{N} \text{ and} \\ \qquad \text{this sequence only } d_G\text{-converges to } \omega^-. \end{cases}$$

    Furthermore, all these sequences are also G-termwise connected.
30. *Condition (B): if a G-termwise connected T-Picard sequence $\{x_n\}$ converges in X, then there exist a limit $z \in X$ of $\{x_n\}$ and $n_0 \in N$ such that $(x_n, z) \in E(G)$ or $(z, x_n) \in E(G)$ for all $n > n_0$.*
31. *Condition (S): whenever a G-termwise connected, T-Picard sequence $\{x_n\}$ has two d-limits x^* and y^*, where $x^* \in X$ and $y^* \in T(X)$, then $x^* = y^*$.*
32. *The mapping T is a (G, G')-graphical contraction with $\lambda = 1/2$.*
33. If $x_0 = 1$, then $(x_0, Tx_0) = (1, 1/2) \in E(G)$, so $Tx_0 \in [x_0]_G^1$.

As a consequence, all hypotheses of Theorem 3.2 are satisfied. However, T has two distinct fixed points: ω^- and ω^+. The mistake in the proof of Theorem 3.13 in [28] is the fact that, although there is an undirected path from ω^- to ω^+, there is not a directed path from ω^- to ω^+. Thus, we cannot use a triangle inequality such that

$$d_G(\omega^-, \omega^+) \leq \sum_{i=1}^{n} d_G(T^n z_{i-1}, T^n z_i)$$

in order to prove that $d_G(\omega^-, \omega^+) = 0$. In fact, given two arbitrary fixed points x^* and y^* of T, the authors used the fact that $(x^* P y^*)_{G'}$ or $(y^* P z^*)_{G'}$, which corresponds to hypothesis (U) in Theorem 5.3 below.

4. Binary Related Distance Spaces

One of the most well known techniques to extend the Banach contractive mapping principle is to consider abstract metric spaces that generalize the idea of metric space. To do that, one may omit some of the properties that characterize a metric space (see, for instance, [14, 23]). By removing the triangle inequality in a metric space, we obtain the notion of *symmetric space* (see [5]). However, it is difficult to handle sequences when the triangle inequality is avoided. Some problems arise: the limit of a convergent sequence need not to be unique, there are convergent sequences that are not Cauchy, etc. Graphical metric spaces are no strangers to these problems (see [28]). Matthews' partial metric spaces [19] were introduced by omitting the idea that the self-distance from a point to itself must be zero. The more conditions we remove, the greater the difficulties.

In this section we introduce a kind of extended metric spaces by using a binary relation on the underlying set. In this way, we generalize the notion of graphical metric space by involving a modified triangle inequality that only holds for points in an appropriate path. We also study some of the first properties of this class of spaces.

From now on, let $\mathcal{S}$ be an arbitrary binary relation on X. Given two points $x, y \in X$, an *$\mathcal{S}$-path* (or *directed $\mathcal{S}$-path*) *from x to y* is a set of $r+1$ points $z_0, z_1, \ldots, z_r \in X$ (where $r \in \mathbb{N}^*$ is the *length* of the path) satisfying $z_0 = x$, $z_r = y$ and $z_{i-1}\mathcal{S}z_i$ for all $i \in \{1, 2, \ldots, r\}$. Given $x_0 \in X$ we will denote by $[x_0]_{\mathcal{S}}$ the set of all points $y \in X$ such that there exists an $\mathcal{S}$-path from x_0 to y.

Definition 4.1. *We will say that a sequence $\{x_n\} \subseteq X$ is $\mathcal{S}$-*nondecreasing-connected *if $x_{n+1} \in [x_n]_{\mathcal{S}}$ for all $n \in \mathbb{N}$.*

Remark 4.2. *The reader can observe that we are going to use the name "$\mathcal{S}$-*nondecreasing*" to refer a property in which the involved sequence $\{x_n\}$ satisfies $x_n\mathcal{S}x_{n+1}$ for all $n \in \mathbb{N}$, and the name "$\mathcal{S}$-*nondecreasing-connected*" when the sequence satisfies $x_{n+1} \in [x_n]_{\mathcal{S}}$ for all $n \in \mathbb{N}$. In the setting of graphical metric spaces, a sequence in a "G-*termwise*" property must verify $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, while in a "G-*termwise-connected*" property, the sequence will satisfy $x_{n+1} \in [x_n]_G^{r_n}$ for all $n \in \mathbb{N}$ (where $r_n \in \mathbb{N}$), that is, $(x_nPx_{n+1})_G$.*

Proposition 4.3. *If $\mathcal{S}$ is a binary relation on X, then a sequence $\{x_n\} \subseteq X$ is $\mathcal{S}$-nondecreasing-connected if, and only if, there exists an $\mathcal{S}$-nondecreasing sequence $\{y_n\} \subseteq X$ such that $\{x_n\}$ is a subsequence of $\{y_n\}$ (that is, $x_n = y_{\sigma(n)}$ for all $n \in \mathbb{N}$, where $\sigma : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function such that $\sigma(0) = 0$).*

Proof. Suppose that $\{x_n\}$ is a $\mathcal{S}$-nondecreasing-connected sequence. Given $n \in \mathbb{N}$, as $x_{n+1} \in [x_n]_{\mathcal{S}}$, there exists a directed $\mathcal{S}$-path $z_0^n, z_1^n, \ldots, z_{k(n)}^n \in X$ from $z_0^n = x_n$ to $z_{k(n)}^n = x_{n+1} = z_0^{n+1}$. By joining the consecutive directed paths

$$\left\{x_0 = z_0^0, z_1^0, \ldots, z_{k(0)}^0 = x_1 = z_0^1\right\} \cup \left\{z_1^1, z_2^1 \ldots, z_{k(1)}^1 = x_2 = z_0^2\right\} \cup \left\{z_1^2, z_2^2 \ldots, z_{k(2)}^2 = x_3 = z_0^3\right\} \cup \ldots$$

we obtain the consecutive terms of the sequence $\{y_n\}$ such that $y_n\mathcal{S}y_{n+1}$ for all $n \in \mathbb{N}$ and verifying that $\{x_n\}$ is a subsequence of $\{y_n\}$. The converse is similar. □

Obviously, every $\mathcal{S}$-nondecreasing sequence is also $\mathcal{S}$-nondecreasing-connected. In general, the converse is false, but we must point out that it holds if $\mathcal{S}$ is transitive.

In the following definitions, let $d : X \times X \to \mathbb{R}$ be a function (with no metric structure).

Definition 4.4. *(cf. [22]) Given a function $d : X \times X \to \mathbb{R}$, a sequence $\{x_n\} \subseteq X$ is:*

- d-Cauchy *if* $\lim_{n,m\to\infty} d(x_n, x_m) = 0$*;*
- d-convergent to $x \in X$ *if* $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$ *(in such a case, we will write $\{x_n\} \xrightarrow{d} x$ and we will say that x is a d-*limit of *$\{x_n\}$);*

We say that (X, d) is complete *if every d-Cauchy sequence in X is d-convergent to a point of X.*

Definition 4.5. *We will say that a mapping $T : X \to X$ is* $(d, \mathcal{S})$-nondecreasing-connected-continuous at $x_0 \in X$ *if $\{Tx_n\} \xrightarrow{d} Tx_0$ for all $\mathcal{S}$-nondecreasing-connected sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \xrightarrow{d} x_0$. And T is* $(d, \mathcal{S})$-nondecreasing-connected-continuous *if it is $(d, \mathcal{S})$-nondecreasing-connected-continuous at each point $x \in X$.*

Definition 4.6. *A nonempty subset $A \subseteq X$ is* $(d, \mathcal{S})$-nondecreasing-connected-precomplete *if for all $\mathcal{S}$-nondecreasing-connected, d-Cauchy sequence $\{a_n\} \subseteq A$ there exists $x \in X$ such that $\{a_n\}$ d-converges to x.*

Remark 4.7. *Every nonempty subset of a complete (or $(d, \mathcal{S})$-nondecreasing-connected-precomplete) BRDS is also $(d, \mathcal{S})$-nondecreasing-connected-precomplete.*

Notice that the d-limit of such sequence $\{a_n\}$ need not to belong to A.

Definition 4.8. *A* binary related distance space *(briefly, a* BRDS*) is a triple $(X, d, \mathcal{S})$ where $\mathcal{S}$ is an arbitrary binary relation on X and $d : X \times X \to [0, \infty)$ is a function such that the following conditions are fulfilled:*

(B_1) *If $x, y \in X$ verify $d(x, y) = 0$ and $y \in [x]_{\mathcal{S}}$, then $x = y$.*

(B_2) *$d(x, y) = d(y, x)$.*

(B_3) *$d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ such that $y \in [x]_{\mathcal{S}}$ and $z \in [y]_{\mathcal{S}}$.*

Remark 4.9. *Some of the results we are going to introduce can be proved in a more general setting, by replacing condition (B_1) by the following assumption:*

$\left(B_1'\right)$ *If $x, y \in X$ verify $d(x, y) = 0$ and $x\mathcal{S}y$, then $x = y$.*

Remark 4.10. *Every metric space (X, d) is a BRDS whatever the binary relation $\mathcal{S}$ on X.*

Lemma 4.11. *Every graphical metric space (X, d_G) is a BRDS under the reflexive binary relation $\mathcal{S}_G$ defined by (1).*

The notion of BRDS extends the notion of graphical metric space. For instance, in a BRDS, the binary relation $\mathcal{S}$ is not necessarily reflexive, and condition "$d(x, x) = 0$ for all $x \in X$" is avoided.

Remark 4.12. *One can easily consider a topology on every BRDS by defining that a subset A of X is* closed *if A satisfies the following condition: if $\{a_n\} \subseteq A$ and $\{a_n\} \xrightarrow{d} x \in X$, then $x \in A$. Clearly, $\varnothing$ and X are closed, the arbitrary intersection of closed sets is closed, and the finite union of closed sets is closed. Nevertheless, this topology need not to be either T_2 (see Remark 2.13 in [28] in graphical metric spaces) or T_1 (because it is not ensured that singleton $\{x\}$ is closed when $d(x, x) > 0$).*

Remark 4.13. *In [13], Example 2.1, Jachymski advised that condition $\Delta \subseteq E(G)$ was stated in order that all constant mappings were G-contractions. Concretely, his definition of G-*contraction *$T : X \to X$ in metric spaces endowed with a graph G included two aspects:*

$$\begin{cases} \bullet\ T \text{ preserves edges of } G\text{, that is, if } (x, y) \in E(G)\text{, then } (Tx, Ty) \in E(G)\,; \\ \bullet \text{ there is } \lambda \in (0, 1) \text{ such that } d(Tx, Ty) \leq \lambda\, d(x, y) \text{ for all } (x, y) \in E(G)\,. \end{cases}$$

A constant mapping from a metric space into itself always satisfies the second condition, so condition $\Delta \subseteq E(G)$ was imposed for guaranteeing that constant mappings preserved edges of G. This condition holds in BRDS if we assume that the binary relation $\mathcal{S}$ is reflexive. However, we are not interested in constant mappings (for which we know that they always have a unique fixed point). We prefer to develop our study in a general setting in order to cover the widest possible set of self-mapping although, in some particular cases, we are not considering the family of constant mappings. This is coherent with the fact that we have omitted the condition "$d(x, x) = 0$ for all $x \in X$": if there are two points $x_0, y_0 \in X$ such that $d(x_0, x_0) > 0$, $d(y_0, y_0) = 0$ and $y_0\mathcal{S}y_0$, then the constant mapping $Tx \equiv x_0$ does not satisfy the second condition of a G-contraction. This is the reason why we are not definitively interested in constant mappings. Nevertheless we point out that under some additional conditions (like "$\mathcal{S}$ is reflexive" and "$d(x, x) = 0$ for all $x \in X$"), our study is also applicable to constant mappings.

Remark 4.14. *It can be proved, by induction methodology, that if $(X, d, \mathcal{S})$ is a BRDS and $x_0, x_1, \ldots, x_n \in X$ ($n \geq 2$) are points such that $x_i \in [x_{i-1}]_{\mathcal{S}}$ for all $i \in \{1, 2, \ldots, n\}$, then*

$$d(x_0, x_n) \leq \sum_{i=1}^{n} d(x_{i-1}, x_i).$$

In particular, this inequality holds if $x_0, x_1, \ldots, x_n$ is a directed $\mathcal{S}$-path in X. However, this inequality may be false if the considered points are not in an appropriate directed $\mathcal{S}$-path.

Let us show that the notion of BRDS properly extends the notion of graphical metric space.

Lemma 4.15. *Let (X, d_G) be a graphical metric space and let ω be a point such that $\omega \notin X$. Let $X_\omega = X \cup \{\omega\}$ and let define $D : X_\omega \times X_\omega \to [0, \infty)$ by:*

$$D(x, y) = \begin{cases} d_G(x, y), & \text{if } x, y \in X, \\ 1, & \text{if } x = y = \omega, \\ 0, & \text{if } x \neq y \text{ and } \omega \in \{x, y\}. \end{cases}$$

Also consider the binary relation $\mathcal{S}$ on X_ω given by:

$$x\mathcal{S}y \quad \text{if} \quad (x, y \in X \text{ and } (x, y) \in E(G)).$$

Then $(X_\omega, D, \mathcal{S})$ is a BRDS that satisfies condition $\left(B_1'\right)$, but (X_ω, D) is not a graphical metric space.

Proof. On the one hand, let $x, y \in X_\omega$ be such that $D(x, y) = 0$ and $y \in [x]_{\mathcal{S}}$. Then there is an $\mathcal{S}$-path $z_0 = x, z_1, \ldots, z_r = y \in X_\omega$ from x to y. Since $z_{i-1}\mathcal{S}z_i$ for all $i \in \{1, 2, \ldots, r\}$, but ω is not $\mathcal{S}$-comparable to points of X, then $z_0, z_1, \ldots, z_r \in X$. In particular, $x, y \in X$. Therefore, $d_G(x, y) = D(x, y) = 0$, so $x = y$ by condition $(GM2)$. On the other hand, D is clearly symmetric. Finally, let $x, y, z \in X_\omega$ be such that $y \in [x]_{\mathcal{S}}$ and $z \in [y]_{\mathcal{S}}$. Since ω is not $\mathcal{S}$-comparable to any other point in X_ω (even to itself), then ω does not belong to any path from x to y and from y to z by involving $\mathcal{S}$-comparable points. Hence, all points in such paths are in X (in particular, $x, y, z \in X$), and all edges are in $E(G)$. Therefore, $(xPz)_G$ and $y \in (xPz)_G$. By $(GM4)$, $d_G(x, z) \leq d_G(x, y) + d_G(y, z)$, so $D(x, z) \leq D(x, y) + D(y, z)$ and (B_3) holds. Thus, $(X_\omega, D, \mathcal{S})$ is a BRDS that also satisfies condition $\left(B_1'\right)$. However, (X_ω, D) is not a graphical metric space because $D(\omega, \omega) = 1 \neq 0$. Furthermore, if $x \in X$, then $D(x, \omega) = 0$, but $x \neq \omega$. □

The following result is a way to reduce our arguments to infinite sequences.

Proposition 4.16. *Let $(X, d, \mathcal{S})$ be a BRDS and let $\{x_n\} \subseteq X$ be a d-Cauchy T-Picard sequence on X such that $x_{n+1} \in [x_n]_{\mathcal{S}}$ for all $n \in \mathbb{N}$. Then either $\{x_n\}$ is infinite or $\{x_n\}$ contains a fixed point ω of T (in the last case, $d(\omega, \omega) = 0$ and $\{x_n\}$ d-converges to ω).*

Proof. Suppose that the Picard sequence $\{x_n\}$ of the operator $T : X \to X$ is not infinite. Then there are $n_0, m_0 \in \mathbb{N}$ such that $n_0 < m_0$ and $x_{n_0} = x_{m_0}$. Let $h_0 = m_0 - n_0 \in \mathbb{N}^*$. It is easy to prove by induction that, for all $k \in \mathbb{N}$, $x_{n_0+k} = x_{m_0+k}$ and $x_{p_0+k h_0} = x_{p_0}$ for all $p_0 \geq n_0$. As the terms of $\{x_n\}$ are repeated in a cyclic way from n_0 and so on, $\{x_n : n \geq n_0\} = \{x_{n_0}, x_{n_0+1}, \ldots, x_{m_0-1}\}$. Let $\varepsilon_0 = \max\left(\left\{d\left(x_i, x_j\right) : i, j \in \{n_0, n_0+1, \ldots, m_0-1\}\right\}\right)$. To show that $\varepsilon_0 = 0$, suppose that $\varepsilon_0 > 0$. Let $i_0, j_0 \in \{n_0, n_0+1, \ldots, m_0-1\}$ be such that $\varepsilon_0 = d\left(x_{i_0}, x_{j_0}\right)$. As $\{x_n\}$ is a d-Cauchy sequence, there is $r_0 \geq n_0$ such that $d(x_n, x_m) < \varepsilon_0/2$ for all $n, m \geq r_0$. Let $k_0 \in \mathbb{N}$ be an integer number such that $n_0 + k_0 h_0 \geq r_0$. Therefore

$$\varepsilon_0 = d\left(x_{i_0}, x_{j_0}\right) = d\left(x_{i_0+k_0 h_0}, x_{j_0+k_0 h_0}\right) \leq \frac{\varepsilon_0}{2},$$

which is a contradiction. Hence, $\varepsilon_0 = 0$ and $d\left(x_i, x_j\right) = 0$ for all $i, j \in \{n_0, n_0+1, \ldots, m_0-1\}$. As the sequence is cyclic, $d(x_n, x_m) = 0$ for all $n, m \geq n_0$. In particular, since $Tx_{n_0} = x_{n_0+1} \in [x_{n_0}]_{\mathcal{S}}$ and $d(x_{n_0}, Tx_{n_0}) = d(x_{n_0}, x_{n_0+1}) = 0$, condition (B_1) guarantees that $Tx_{n_0} = x_{n_0}$, so $\omega = x_{n_0}$ is a fixed point of T. In this case, $d(\omega, \omega) = d(x_{n_0}, Tx_{n_0}) = 0$. Furthermore, $x_n = \omega$ for all $n \geq n_0$, so $d(x_n, \omega) = d(\omega, \omega) = 0$, which means that $\{x_n\}$ d-converges to ω. □

5. Fixed Point Theory in Binary Related Distance Spaces

In this section we introduce some fixed point theorems in the setting of binary related distance spaces. The contractivity conditions play a key role in the following results because we will not assume any *a priori* condition on the binary relation.

5.1. Berinde-Rus type fixed point theory in binary related distance spaces

In our first main result, we describe the behavior of a Picard sequence starting from an appropriate initial condition.

Theorem 5.1. *Let $(X, d, \mathcal{S})$ be a BRDS and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing self-mapping such that there is $\phi \in \mathcal{F}_{\mathrm{com}}^{(c)}$ satisfying*

$$d(Tx, Ty) \leq \phi(d(x, y)) \quad \text{for all } x, y \in X \text{ such that } x\mathcal{S}y. \tag{2}$$

Suppose that there exists $x_0 \in X$ such that $x_0 \in [x_0]_{\mathcal{S}}$, $Tx_0 \in [x_0]_{\mathcal{S}}$ and $\mathcal{O}_T(x_0)$ is $(d, \mathcal{S})$-nondecreasing-connected-precomplete. Also suppose that at least one of the following conditions hold:

(*a*) *T is $(d, \mathcal{S})$-nondecreasing-connected-continuous.*

(*b*) *For all $\mathcal{S}$-nondecreasing-connected, d-Cauchy, d-convergent, infinite, Picard sequence $\{x_n\}$ of T, there exist a limit $\omega \in X$ of $\{x_n\}$ and $n_0 \in \mathbb{N}$ such that $x_n\mathcal{S}\omega$ or $\omega\mathcal{S}x_n$ for all $n \geq n_0$.*

Then there exists a point $\omega \in X$ such that the Picard sequence of T based on x_0 is $\mathcal{S}$-nondecreasing-connected and it converges, at the same time, to ω and to $T\omega$.

The previous theorem improves Theorem 3.1 in several senses:

- BRDS are more general than graphical metric spaces. In particular, the condition "$d(x, x) = 0$ for all $x \in X$" is omitted.
- The binary relation need not satisfy any condition (in particular, it is not necessarily reflexive).
- The contractivity condition is weaker because $\phi \in \mathcal{F}_{\mathrm{com}}^{(c)}$.
- Condition (*b*) in Theorem 5.1 is weaker than condition (*b*) in Theorem 3.1 because we consider a restrictive family of sequences.
- We introduce an alternative condition to (*b*) which was not considered in Theorem 3.1.

Proof. Let $\{x_n\}$ be the Picard sequence of T based on the point $x_0 \in X$ such that $x_0 \in [x_0]_{\mathcal{S}}$ and $Tx_0 \in [x_0]_{\mathcal{S}}$. Since $x_1 = Tx_0 \in [x_0]_{\mathcal{S}}$, there is a finite path $z_0, z_1, \ldots, z_r \in X$ from $z_0 = x_0$ to $z_r = x_1 = Tx_0$ (of length r) such that $z_{i-1}\mathcal{S}z_i$ for all $i \in \{1, 2, \ldots, r\}$. Let $t_0 = \max_{1 \leq i \leq r} d(z_{i-1}, z_i)$. If $t_0 = 0$, then $d(z_{i-1}, z_i) = 0$ for all $i \in \{1, 2, \ldots, r\}$. By (B_1), $z_{i-1} = z_i$ for all i, so $Tx_0 = x_1 = z_r = z_0 = x_0$. In this case, x_0 is a fixed point of T and the sequence $\{x_n\}$ satisfies $x_n = x_0$ for all $n \in \mathbb{N}$. As $d(x_0, x_0) = d(z_0, z_1) = 0$, then $\{x_n\}$ d-converges to x_0 and to Tx_0, and the proof is finished.

On the contrary case, assume that $t_0 > 0$. As $z_{i-1}\mathcal{S}z_i$ for all $i \in \{1, 2, \ldots, r\}$ and T is $\mathcal{S}$-nondecreasing, then $Tz_{i-1}\mathcal{S}Tz_i$ for all $i \in \{1, 2, \ldots, r\}$. By induction,

$$(T^n z_{i-1})\mathcal{S}(T^n z_i) \quad \text{for all } i \in \{1, 2, \ldots, r\} \text{ and all } n \in \mathbb{N}. \tag{3}$$

In particular, $\{x_n = T^n z_0, T^n z_1, \ldots, T^n z_r = T^n Tx_0 = x_{n+1}\}$ is a finite path (of length r) from x_n to x_{n+1} whose terms are $\mathcal{S}$-increasingly comparable, so $x_{n+1} \in [x_n]_{\mathcal{S}}$ for all $n \in \mathbb{N}$, and the sequence $\{x_n\}$ is $\mathcal{S}$-nondecreasing-connected. Moreover, applying the contractivity condition (2), for all $i \in \{1, 2, \ldots, r\}$ and all $n \in \mathbb{N}$,

$$\begin{aligned} d(T^n z_{i-1}, T^n z_i) &\leq \phi\left(d\left(T^{n-1} z_{i-1}, T^{n-1} z_i\right)\right) \leq \phi^2\left(d\left(T^{n-2} z_{i-1}, T^{n-2} z_i\right)\right) \\ &\leq \ldots \leq \phi^n\left(d(z_{i-1}, z_i)\right). \end{aligned}$$

As ϕ^n is nondecreasing,

$$d\left(T^n z_{i-1}, T^n z_i\right) \leq \phi^n\left(d\left(z_{i-1}, z_i\right)\right) \leq \phi^n\left(\max_{1 \leq j \leq r} d\left(z_{j-1}, z_j\right)\right) = \phi^n\left(t_0\right)$$

Thus, for $k \in \mathbb{N}^*$,

$$\begin{aligned} d\left(x_{k-1}, x_k\right) &= d\left(T^{k-1} x_0, T^k x_0\right) = d\left(T^{k-1} x_0, T^{k-1} T x_0\right) = d\left(T^{k-1} z_0, T^{k-1} z_r\right) \\ &\leq \sum_{i=1}^{r} d\left(T^{k-1} z_{i-1}, T^{k-1} z_i\right) \leq \sum_{i=1}^{r} \phi^{k-1}\left(t_0\right) = r\,\phi^{k-1}\left(t_0\right). \end{aligned}$$

As $\lim_{n\to\infty} \phi^n\left(t_0\right) = 0$, then $\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0$. Next, let $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$ with $m \geq 2$. In this case,

$$d\left(x_n, x_{n+m}\right) \leq \sum_{k=n+1}^{n+m} d\left(x_{k-1}, x_k\right) \leq \sum_{k=n+1}^{n+m} r\,\phi^{k-1}\left(t_0\right) \leq r \sum_{k=n}^{\infty} \phi^k\left(t_0\right).$$

As $\phi \in \mathcal{F}_{\mathrm{com}}^{(c)}$, the values of $d\left(x_n, x_{n+m}\right)$ are arbitrarily near to zero for n large enough. Similarly, from $x_0 \in [x_0]_{\mathcal{S}}$, we can also deduce that the values of $d\left(x_n, x_n\right)$ are arbitrarily near to zero for n large enough. Hence, $\{x_n\}$ is a d-Cauchy sequence. If $\{x_n\}$ is not infinite, Proposition 4.16 shows that it contains a fixed point of T and $\{x_n\}$ d-converges to such fixed point, so the proof is finished. On the contrary case, assume that $\{x_n\}$ is infinite. In this case, $\{x_n\}$ is a d-Cauchy, infinite, Picard sequence $\{x_n\}$ of T. As $O_T\left(x_0\right)$ is $(d, \mathcal{S})$-nondecreasing-connected-precomplete, there is $\omega \in X$ such that $\{x_n\}$ d-converges to ω. Next, we consider two cases.

Case (a). In this case, as $\{x_n\}$ is an $\mathcal{S}$-nondecreasing-connected sequence such that $\{x_n\} \overset{d}{\to} x_0$, then $\{x_{n+1} = T x_n\} \overset{d}{\to} T x_0$, so $\{x_n\}$ d-converges, at the same time, to ω and to $T\omega$.

Case (b). In this case, $\{x_n\}$ is a d-Cauchy, d-convergent, infinite, Picard sequence of T, so condition (b) guarantees that there exist a limit $\omega \in X$ of $\{x_n\}$ and $n_0 \in \mathbb{N}$ such that $x_n \mathcal{S} \omega$ or $\omega \mathcal{S} x_n$ for all $n \geq n_0$. In any case, the contractivity condition (2) yields

$$d\left(x_n, T\omega\right) = d\left(T x_{n-1}, T\omega\right) \leq \phi\left(d\left(x_{n-1}, \omega\right)\right) \quad \text{for all } n \geq n_0 + 1.$$

As ϕ is continuous at $t = 0$ and $\phi(0) = 0$, then $\{x_n\}$ also d-converges to $T\omega$. □

Theorem 5.2. *Under the hypothesis of Theorem 5.1, additionally assume that the following property holds:*

(S) *whenever an $\mathcal{S}$-nondecreasing-connected, d-Cauchy, d-convergent, infinite, Picard sequence $\{x_n\}$ of T has two d-limits $\omega \in X$ and $\omega^* \in T(X)$, then $\omega = \omega^*$.*

Then T has a fixed point.

Furthermore, if $\omega, \omega' \in \mathrm{Fix}(T)$ satisfy $\omega \in [\omega']_{\mathcal{S}}$, then $d\left(\omega, \omega'\right) = 0$ and $\omega = \omega'$. In particular, $d(\omega, \omega) = 0$ for all $\omega \in \mathrm{Fix}(T)$ such that $\omega \in [\omega]_{\mathcal{S}}$.

Proof. We have proved that if $\{x_n = T^n x_0\}_{n\in\mathbb{N}}$ is not infinite, then it contains a fixed point of T. In other case, it converges, at the same time, to ω and to $T\omega$, so condition (S) leads to $T\omega = \omega$.

Next let $\omega, \omega' \in \mathrm{Fix}(T)$ be two fixed points of T such that $\omega \in [\omega']_{\mathcal{S}}$. Then there is a finite path $z_0, z_1, \ldots, z_r \in X$ from $z_0 = \omega$ to $z_r = \omega'$ (of length r) such that $z_{i-1} \mathcal{S} z_i$ for all $i \in \{1, 2, \ldots, r\}$. As we proved in (3), as T is $\mathcal{S}$-nondecreasing, for all $n \in \mathbb{N}$, $T^n z_0, T^n z_1, \ldots, T^n z_r \in X$ is a finite path from $T^n z_0 = T^n \omega = \omega$ to $T^n z_r = T^n \omega' = \omega'$ (of length r) such that $(T^n z_{i-1}) \mathcal{S} (T^n z_i)$ for all $i \in \{1, 2, \ldots, r\}$. Applying the contractivity condition (2), for all $i \in \{1, 2, \ldots, r\}$ and all $n \in \mathbb{N}$,

$$\begin{aligned} d\left(T^n z_{i-1}, T^n z_i\right) &\leq \phi\left(d\left(T^{n-1} z_{i-1}, T^{n-1} z_i\right)\right) \leq \phi^2\left(d\left(T^{n-2} z_{i-1}, T^{n-2} z_i\right)\right) \\ &\leq \ldots \leq \phi^n\left(d\left(z_{i-1}, z_i\right)\right). \end{aligned}$$

If $t_0 = \max_{1\le j\le r} d\left(z_{j-1}, z_j\right)$, as ϕ^n is nondecreasing,

$$d\left(T^n z_{i-1}, T^n z_i\right) \le \phi^n\left(d\left(z_{i-1}, z_i\right)\right) \le \phi^n\left(\max_{1\le j\le r} d\left(z_{j-1}, z_j\right)\right) = \phi^n\left(t_0\right).$$

Thus, for $n \in \mathbb{N}^*$,

$$\begin{aligned} d\left(\omega, \omega'\right) &= d\left(T^n\omega, T^n\omega'\right) = d\left(T^n z_0, T^n z_r\right) \\ &\le \sum_{i=1}^{r} d\left(T^n z_{i-1}, T^n z_i\right) \le \sum_{i=1}^{r} \phi^n\left(t_0\right) = r\,\phi^n\left(t_0\right). \end{aligned}$$

If $t_0 = 0$, then $\phi^n(t_0) = 0$, so $d(\omega, \omega') = 0$. If $t_0 > 0$, then $\lim_{n\to\infty} \phi^n(t_0) = 0$, and we may also deduce that $d(\omega, \omega') = 0$. In any case, $d(\omega, \omega') = 0$. From (B_1) we conclude that $\omega = \omega'$. □

In the following result we study the uniqueness of the fixed point under an additional condition.

Theorem 5.3. *Let $(X, d, \mathcal{S})$ be a BRDS and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing self-mapping such that there is $\phi \in \mathcal{F}^{(c)}_{\mathrm{com}}$ satisfying*

$$d(Tx, Ty) \le \phi(d(x, y)) \quad \text{for all } x, y \in X \text{ such that } x\mathcal{S}y. \tag{4}$$

Suppose that there exists $x_0 \in X$ such that $x_0 \in [x_0]_{\mathcal{S}}$, $Tx_0 \in [x_0]_{\mathcal{S}}$ and $\mathcal{O}_T(x_0)$ is $(d, \mathcal{S})$-nondecreasing-connected-precomplete. Also suppose that at least one of the following conditions hold:

(a) T is $(d, \mathcal{S})$-nondecreasing-connected-continuous.

(b) For all $\mathcal{S}$-nondecreasing-connected, d-Cauchy, d-convergent, infinite, Picard sequence $\{x_n\}$ of T, there exist a limit $\omega \in X$ of $\{x_n\}$ and $n_0 \in \mathbb{N}$ such that $x_n\mathcal{S}\omega$ or $\omega\mathcal{S}x_n$ for all $n \ge n_0$.

Additionally assume that the following properties holds:

(S) whenever an $\mathcal{S}$-nondecreasing-connected, d-Cauchy, d-convergent, infinite, Picard sequence $\{x_n\}$ of T has two d-limits $\omega \in X$ and $\omega^* \in T(X)$, then $\omega = \omega^*$;

(U) for all $\omega, \omega' \in \mathrm{Fix}(T)$ we have that $\omega \in [\omega']_{\mathcal{S}}$ or $\omega' \in [\omega]_{\mathcal{S}}$.

Then T has a unique fixed point.

Proof. Theorem 5.2 guarantees that T has at least a fixed point. To study the uniqueness, let $\omega, \omega' \in \mathrm{Fix}(T)$ be two fixed points of T. By condition (U), we have that $\omega \in [\omega']_{\mathcal{S}}$ or $\omega' \in [\omega]_{\mathcal{S}}$. In any case, Theorem 5.2 also guarantees that $\omega = \omega'$, so T has a unique fixed point. □

The following result is another way to guarantee existence and uniqueness of fixed points. Let define

$$\mathcal{X}_{T,\mathcal{S}} = \{\, x \in X : x\mathcal{S}Tx \,\}.$$

Clearly, if $\mathcal{S}$ is reflexive on $\mathrm{Fix}(T)$, then $\mathrm{Fix}(T) \subseteq \mathcal{X}_{T,\mathcal{S}}$.

Theorem 5.4. *Let $(X, d, \mathcal{S})$ be a BRDS and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing self-mapping such that there is a function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) < t$ for all $t > 0$ and*

$$d(Tx, Ty) \le \phi(d(x, y)) \quad \text{for all } x, y \in X \text{ such that } x\mathcal{S}y. \tag{5}$$

Suppose that $\mathcal{X}_{T,\mathcal{S}}$ is nonempty and that the function

$$f : \mathcal{X}_{T,\mathcal{S}} \to [0, \infty), \quad x \mapsto f(x) = d(x, Tx)$$

has an absolute minimum at $\omega \in \mathcal{X}_{T,\mathcal{S}}$. Then ω is a fixed point of T and $d(\omega, \omega) = 0$.

Furthermore, if $\phi \in \mathcal{F}^{(c)}_{\mathrm{com}}$ and property (U) of Theorem 5.3 holds, then T has a unique fixed point.

Proof. We prove that $d(\omega, T\omega) = 0$ by contradiction. Suppose that $d(\omega, T\omega) > 0$. Since ω is an absolute minimum of f, then

$$d(\omega, T\omega) \leq d(x, Tx) \quad \text{for all } x \in \mathcal{X}_{T,\mathcal{S}}.$$

Let $\omega' = T\omega$. Since $\omega \in \mathcal{X}_{T,\mathcal{S}}$, $\omega \mathcal{S} T\omega$. As T is $\mathcal{S}$-nondecreasing, $T\omega \mathcal{S} T^2\omega$, so $\omega' \mathcal{S} T\omega'$, which means that $\omega' \in \mathcal{X}_{T,\mathcal{S}}$. Then $d(\omega, T\omega) \leq d(\omega', T\omega')$. Nevertheless, by (5),

$$d(\omega', T\omega') = d\left(T\omega, T^2\omega\right) \leq \phi(d(\omega, T\omega)) < d(\omega, T\omega).$$

This contradiction proves that $d(\omega, T\omega) = 0$. Hence axiom (B_1) guarantees that $\omega = T\omega$. In particular $\mathrm{Fix}(T) \neq \varnothing$. Furthermore $d(\omega, \omega) = d(\omega, T\omega) = 0$. Finally, the uniqueness of the fixed point follows by repeating the arguments of the proof of Theorem 5.3. □

Remark 5.5. *Very recently, Samet [25] have shown the equivalence between general theorems in metric spaces endowed with a partial order $\preccurlyeq$ with their corresponding results in metric spaces. To do that, he considered the complete metric space given as the closure of the orbit of the initial condition x_0, that is, $\mathcal{Z} = \overline{O_T(x_0)}$. In this case, we only need to suppose that the contractivity condition holds in a proper subset of X.*

5.2. *Banach type fixed point theory in graphical metric spaces*

In this subsection we obtain some consequences of the above obtained results in the setting of graphical metric spaces. Before that, we translate some notions on graphical spaces to BRDS. As a framework, let (X, d_G) be a graphical metric space, let G' be a subgraph of G such that $V(G') = X$ and let $T : X \to X$ be a mapping. Consider on X the binary relation $\mathcal{S}_{G'}$ given, for $x, y \in X$, by

$$x \mathcal{S}_{G'} y \quad \text{if} \quad (x, y) \in E(G'). \tag{6}$$

Proposition 5.6. *The following properties hold.*

1. *$(X, d_G, \mathcal{S}_{G'})$ is a BRDS.*
2. *If T is a (G, G')-graphical contraction with constant $\lambda \in [0, 1)$, then T is $\mathcal{S}_{G'}$-nondecreasing and*

   $$d(Tx, Ty) \leq \lambda\, d(x, y) \quad \text{for all } x, y \in X \text{ such that } x \mathcal{S}_{G'} y.$$

3. *If there exists a point $x_0 \in X$ such that $Tx_0 \in [x_0]^l_{G'}$ (for some $l \in \mathbb{N}$), then $x_0 \in [x_0]_{\mathcal{S}_{G'}}$ and $Tx_0 \in [x_0]_{\mathcal{S}_{G'}}$.*
4. *If (X, d_G) is G'-complete, then any nonempty subset A of X is $(d_G, \mathcal{S}_{G'})$-nondecreasing-connected-precomplete (in particular, $O_T(x_0)$ is).*
5. *If property (B) in Theorem 3.1 holds, then assumption (b) in Theorem 5.2 also holds.*
6. *If the quadruple (X, d_G, G', T) satisfies the property (S) in Theorem 3.1, then condition (S) in Theorem 5.2 also holds.*

Proof. Item 1 follows from Lemma 4.11 and items 2, 3, 5 and 6 are trivial. Let us prove item 4. Suppose that (X, d_G) is G'-complete and let $A \subseteq X$ be a nonempty subset. To prove that A is $(d_G, \mathcal{S}_{G'})$-nondecreasing-connected-precomplete, let $\{a_n\} \subseteq A$ be an $\mathcal{S}_{G'}$-nondecreasing-connected, d-Cauchy sequence. Proposition 4.3 guarantees that there exists an $\mathcal{S}_{G'}$-nondecreasing sequence $\{y_n\} \subseteq X$ such that $\{a_n\}$ is a subsequence of $\{y_n\}$, that is, there exists an strictly increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma(0) = 0$ and $a_n = y_{\sigma(n)}$ for all $n \in \mathbb{N}$. Since $\{y_n\}$ is $\mathcal{S}_{G'}$-nondecreasing, then $y_k \mathcal{S}_{G'} y_{k+1}$ for all $k \in \mathbb{N}$, that is, $(y_k, y_{k+1}) \in E(G')$ for all $k \in \mathbb{N}$. Then $a_n = y_{\sigma(n)}, y_{\sigma(n)+1}, y_{\sigma(n)+2}, \ldots, y_{\sigma(n+1)} = a_{n+1}$ is a directed path on G' from each a_n to a_{n+1}, that is, $a_{n+1} \in [a_n]_{G'}$ for all $n \in \mathbb{N}$. As a result, $\{a_n\}$ is a G'-termwise connected sequence. As (X, d_G) is G'-complete and $\{a_n\}$ is a G'-termwise connected, d_G-Cauchy sequence, there is $\omega \in X$ such that $\{a_n\}$ d_G-converges to ω. Therefore, A is $(d_G, \mathcal{S}_{G'})$-nondecreasing-connected-precomplete. □

As a consequence of the previous proposition, we deduce that the main result in [28] (in the setting of graphical metric spaces) trivially follows from Theorem 5.2 (in the setting of BRDS).

Theorem 5.7. *Theorem 3.1 is an immediate consequence of Theorem 5.2.*

Proof. Under the hypothesis of Theorem 3.1, let $\mathcal{S} = \mathcal{S}_{G'}$ the binary relation on X defined in (6). Proposition 5.6 guarantees that all hypotheses of Theorem 5.2 are fulfilled, so the existence of a fixed point of T is ensured. □

Notice that we cannot establish that Theorem 3.2 is a consequence of Theorem 5.3 because, as we have demonstrated in Section 3, the weakly connectedness of X_T is not strong enough to guarantee that the fixed point of T is unique. To guarantee uniqueness of the fixed point of T, we need to assume that every two possible fixed points of T are the extremes of a directed path (see Theorem 5.3).

In order to introduce a version of our main results in the setting of graphical metric spaces, we need the following notion.

Definition 5.8. *Let (X, d_G) be a graphical metric space, let G' be a subgraph of G such that $V(G') = X$ and let $T : X \to X$ be a mapping. We will say that the mapping T is (d_G, G')-*termwise-connected-continuous *at $x_0 \in X$ if $\{Tx_n\} \overset{d_G}{\to} Tx_0$ for all G'-termwise connected sequence $\{x_n\} \subseteq X$ such that $\{x_n\} \overset{d_G}{\to} x_0$. And T is (d_G, G')-*termwise-connected-continuous *if it is (d_G, G')-termwise-connected-continuous at each point $x \in X$.*

Proposition 5.9. *If T is (d_G, G')-termwise-connected-continuous at $x_0 \in X$, then T is $(d_G, \mathcal{S}_{G'})$-nondecreasing-connected-continuous at x_0.*

Corollary 5.10. *Let (X, d_G) be a G'-complete graphical metric space and let $T : X \to X$ be a mapping. Suppose that the following conditions are fulfilled:*

(A) *there exists $x_0 \in X$ such that $Tx_0 \in [x_0]^{\ell}_{G'}$ for some $\ell \in \mathbb{N}$;*

(C) *if $(x, y) \in E(G')$ then $(Tx, Ty) \in E(G')$ (that is, T preserves the edges of G');*

(D) *there is $\phi \in \mathcal{F}^{(c)}_{\mathrm{com}}$ such that*

$$d(Tx, Ty) \leq \phi(d(x, y)) \quad \text{for all } x, y \in X \text{ such that } (x, y) \in E(G').$$

Also suppose that at least one of the following conditions holds:

(a_{gr}) *T is (d_G, G')-termwise-connected-continuous.*

(b_{gr}) *For all G'-termwise connected, d_G-Cauchy, d_G-convergent, infinite, Picard sequence $\{x_n\}$ of T, there exist a d_G-limit $\omega \in X$ of $\{x_n\}$ and $n_0 \in \mathbb{N}$ such that $(x_n, \omega) \in E(G')$ or $(\omega, x_n) \in E(G')$ for all $n > n_0$.*

Then, there exists $\omega \in X$ such that the T-Picard sequence $\{x_n\}$ with initial value $x_0 \in X$ is G'-termwise connected and it d_G-converges at the same time to both, ω and $T\omega$. In addition to this, suppose that:

(S_{gr}) *whenever a G'-termwise connected, d_G-Cauchy, d_G-convergent, infinite, Picard sequence $\{x_n\}$ of T has two d_G-limits $\omega \in X$ and $\omega^* \in T(X)$, then $\omega = \omega^*$.*

Then T has a fixed point in X.
Furthermore, if $\omega, \omega' \in \mathrm{Fix}(T)$ satisfy $\omega \in [\omega']_{G'}$, then $\omega = \omega'$.

Proof. It easily follows by applying Theorem 5.2 to the BRDS $(X, d_G, \mathcal{S}_{G'})$ and taking into account properties of Propositions 5.6 and 5.9. □

Corollary 5.11. *Under the hypothesis of Corollary 5.10, additionally assume that the following property holds:*

(U_{gr}) *for all $\omega, \omega' \in \mathrm{Fix}(T)$ we have that $\omega \in [\omega']_{G'}$ or $\omega' \in [\omega]_{G'}$.*

Then T has a unique fixed point.

Proof. It follows from Theorem 5.3 because $[x]_{G'} = [x]_{\mathcal{S}_{G'}}$ for all $x \in X$. □

5.3. *Banach type fixed point results in binary related distance spaces*

The following results are particularizations of Theorems 5.2 and 5.3 by using the (c)-comparison function $\phi_\lambda(t) = \lambda\, t$ for all $t \in [0,\infty)$, where $\lambda \in [0,1)$, which originates the classical Banach type contractivity condition (that only has to be verified for $\mathcal{S}$-comparable points).

Corollary 5.12. *Let $(X,d,\mathcal{S})$ be a BRDS and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing self-mapping such that there is $\lambda \in [0,1)$ satisfying*

$$d(Tx,Ty) \leq \lambda\, d(x,y) \quad \text{for all } x,y \in X \text{ such that } x\mathcal{S}y.$$

Suppose that there exists $x_0 \in X$ such that $x_0 \in [x_0]_{\mathcal{S}}$, $Tx_0 \in [x_0]_{\mathcal{S}}$ and $\mathcal{O}_T(x_0)$ is $(d,\mathcal{S})$-nondecreasing-connected-precomplete. Assume that

(S) *whenever an $\mathcal{S}$-nondecreasing-connected, d-Cauchy, d-convergent, infinite, Picard sequence $\{x_n\}$ of T has two d-limits $\omega \in X$ and $\omega^* \in T(X)$, then $\omega = \omega^*$.*

Suppose that at least one of the following conditions hold:

(a) *T is $(d,\mathcal{S})$-nondecreasing-connected-continuous.*

(b) *For all $\mathcal{S}$-nondecreasing-connected, d-Cauchy, d-convergent, infinite, Picard sequence $\{x_n\}$ of T, there exist a limit $\omega \in X$ of $\{x_n\}$ and $n_0 \in \mathbb{N}$ such that $x_n\mathcal{S}\omega$ or $\omega\mathcal{S}x_n$ for all $n \geq n_0$.*

Then T has a fixed point.

Furthermore, if $\omega,\omega' \in \mathrm{Fix}(T)$ satisfy $\omega \in [\omega']_{\mathcal{S}}$, then $d(\omega,\omega') = 0$ and $\omega = \omega'$. In particular, $d(\omega,\omega) = 0$ for all $\omega \in \mathrm{Fix}(T)$ such that $\omega \in [\omega]_{\mathcal{S}}$.

Corollary 5.13. *Under the hypothesis of Corollary 5.12, additionally assume that the following property holds:*

(U) *for all $\omega,\omega' \in \mathrm{Fix}(T)$ we have that $\omega \in [\omega']_{\mathcal{S}}$ or $\omega' \in [\omega]_{\mathcal{S}}$.*

Then T has a unique fixed point.

5.4. *A Caristi type fixed point theorem in binary related distance spaces*

In this subsection we introduce a Caristi type fixed point theorem in the setting of binary related distance metric spaces.

Theorem 5.14. *Let $(X,d,\mathcal{S})$ be a BRDS and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing and $(d,\mathcal{S})$-nondecreasing-continuous mapping such that $T(X)$ is $\mathcal{S}$-nondecreasing-precomplete. Suppose that there is a function $\phi : X \to [0,\infty)$ such that*

$$\max\{d(x,x), d(x,Tx)\} \leq \phi(x) - \phi(Tx) \quad \text{for all } x \in X \text{ such that } x\mathcal{S}Tx. \tag{7}$$

Additionally, assume that:

(S') *whenever an $\mathcal{S}$-nondecreasing, d-Cauchy, d-convergent, infinite, Picard sequence $\{x_n\}$ of T has two d-limits $\omega \in X$ and $\omega^* \in T(X)$, then $\omega = \omega^*$.*

If there is $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$, then T has a fixed point.

Proof. Let $\{x_n\}$ be the Picard sequence of T based on the point $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$. As T is $\mathcal{S}$-nondecreasing, $x_n\mathcal{S}x_{n+1}$ for all $n \in \mathbb{N}$. Using the contractivity condition (7), for all $n \in \mathbb{N}$,

$$\begin{aligned} 0 &\leq \max\{d(x_n,x_n), d(x_n,x_{n+1})\} = \max\{d(x_n,x_n), d(x_n,Tx_n)\} \\ &\leq \phi(x_n) - \phi(Tx_n) = \phi(x_n) - \phi(x_{n+1}). \end{aligned}$$

In particular, $0 \leq \phi(x_{n+1}) \leq \phi(x_n)$ for all $n \in \mathbb{N}$, so the sequence $\{\phi(x_n)\}$ is non-increasing and bounded below. Hence, it is d_E-convergent, so it is also d_E-Cauchy. Let us show that $\{x_n\}$ is a d-Cauchy sequence. Let $\varepsilon > 0$. As $\{\phi(x_n)\}$ is d_E-Cauchy, there is $n_0 \in \mathbb{N}$ such that $\left|\phi(x_n) - \phi(x_{n+m})\right| < \varepsilon$ for all $n \geq n_0$ and all $m \in \mathbb{N}$. Thus, given $n \geq n_0$ and $m \in \mathbb{N}^*$, since $x_n, x_{n+1}, \ldots, x_{n+m}$ is a finite path such that $x_{n+k}\mathcal{S}x_{n+k+1}$ for all $k \in \{0, 1, \ldots, m-1\}$,

$$\begin{aligned} d(x_n, x_{n+m}) &\leq \sum_{i=1}^{m} d(x_{i-1}, x_i) = \sum_{i=1}^{m} d(x_{i-1}, Tx_{i-1}) \leq \sum_{i=1}^{m} \left[\phi(x_{i-1}) - \phi(Tx_{i-1})\right] \\ &= \phi(x_n) - \phi(x_{n+m}) \leq \left|\phi(x_n) - \phi(x_{n+m})\right| < \varepsilon. \end{aligned}$$

Similarly, if $n \geq n_0$,

$$\begin{aligned} d(x_n, x_n) &\leq \max\{d(x_n, x_n), d(x_n, x_{n+1})\} \leq \phi(x_n) - \phi(x_{n+1}) \\ &\leq \left|\phi(x_n) - \phi(x_{n+1})\right| < \varepsilon. \end{aligned}$$

As a consequence, $\{x_n\}$ is a d-Cauchy sequence. Taking into account that $T(X)$ is $\mathcal{S}$-nondecreasing-precomplete, $\{x_n : n \geq 1\} \subseteq T(X)$ and $\{x_n\}$ is $\mathcal{S}$-nondecreasing, there is $z_0 \in X$ such that $\{x_n\}$ d-converges to z_0. Furthermore, as T is $(d, \mathcal{S})$-nondecreasing-continuous, $\{x_{n+1} = Tx_n\}$ also d-converges to $Tz_0 \in T(X)$. By Proposition 4.16, either $\{x_n\}$ is infinite or $\{x_n\}$ contains a fixed point ω of T. In the second case, the proof is finished. If we suppose that $\{x_n\}$ is infinite, we conclude that $Tz_0 = z_0$ by condition (S'). In any case, T has a fixed point. □

Corollary 5.15. *(Alfuraidan and Khamsi [3], Theorem 3) Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a distance d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and monotone increasing mapping. Assume that there exists a lower semi-continuous function $\phi : X \to [0, \infty)$ such that*

$$d(x, Tx) \leq \phi(x) - \phi(Tx) \quad \textit{whenever } Tx \preccurlyeq x.$$

Then T has a fixed point if, and only if, there exists $x_0 \in X$ such that $Tx_0 \preccurlyeq x_0$.

Proof. Let $\mathcal{S}$ be the binary relation on X such that $x\mathcal{S}y$ if $y \preccurlyeq x$. If T has a fixed point ω, then $T\omega\mathcal{S}\omega = \omega\mathcal{S}\omega$ because $\preccurlyeq$ is reflexive. For the converse, we have only to apply Theorem 5.14. As (X, d) is a complete metric space, then $(X, d, \mathcal{S})$ is a BRDS such that $T(X)$ is $\mathcal{S}$-nondecreasing-precomplete. Furthermore, T is $\mathcal{S}$-nondecreasing, and $Tx_0 \preccurlyeq x_0$ implies that $x_0\mathcal{S}Tx_0$. Moreover, as T is a G-Caristi mapping, then T satisfies (7). As the limit of a convergent sequence in a metric space is unique, condition (S') of Theorem 5.14 immediately holds. Hence, such result guarantees that T has a fixed point. □

In the setting of metric spaces endowed with a graph, we highlight the following Caristi's type result.

Corollary 5.16. *(Alfuraidan and Khamsi [3], Theorem 8) Let G be an oriented graph on the set X with $E(G)$ containing all loops and suppose that there exists a distance d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous mapping satisfying:*

- *T is G-edge preserving, that is, $(Tx, Ty) \in E(G)$ for all $x, y \in X$ such that $(x, y) \in E(G)$;*
- *T is a G-Caristi mapping, that is, there is a lower semi-continuous function $\phi : X \to [0, \infty)$ such that*

$$d(x, Tx) \leq \phi(x) - \phi(Tx) \quad \textit{whenever } (Tx, x) \in E(G).$$

Then T has a fixed point if, and only if, there exists $x_0 \in X$ such that $(Tx_0, x_0) \in E(G)$.

Proof. We may repeat here all the argument of the proof of Corollary 5.15 by using the binary relation $\mathcal{S}'_G$ on X such that $x\mathcal{S}'_G y$ if $(y, x) \in E(G)$ (notice that $\mathcal{S}'_G$ does not coincide with $\mathcal{S}_G$ because of the order of the arguments of the edges of G). □

5.5. *Kannan type fixed point theorems in binary related distance spaces*

In [15], Kannan introduced the following contractivity condition in the setting of metric spaces:

$$d(Tx,Ty) \leq \lambda \left(d(x,Tx) + d(y,Ty) \right) \quad \text{for all } x,y \in X,$$

where $\lambda \in [0,1/2)$. In this subsection, we show some Kannan type fixed point theorems in the setting of binary related distance metric spaces.

Theorem 5.17. *Let $(X,d,\mathcal{S})$ be a BRDS and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing and $(d,\mathcal{S})$-nondecreasing-continuous mapping. Suppose that there is a constant $\lambda \in [0,1/2)$ such that*

$$d(Tx,Ty) \leq \lambda \left(d(x,Tx) + d(y,Ty) \right) \quad \text{for all } x,y \in X \text{ such that } x\mathcal{S}y. \tag{8}$$

Additionally, assume that:

(S') *whenever an $\mathcal{S}$-nondecreasing, d-Cauchy, d-convergent, infinite, Picard sequence $\{x_n\}$ of T has two d-limits $\omega \in X$ and $\omega^* \in T(X)$, then $\omega = \omega^*$.*

If there is $x_0 \in X$ such that $x_0\mathcal{S}x_0$, $x_0\mathcal{S}Tx_0$ and $O_T(x_0)$ is $\mathcal{S}$-nondecreasing-precomplete, then T has a fixed point.

Proof. Let $\kappa = \lambda/(1-\lambda) \in [0,1)$ and let $\{x_n\}$ be the Picard sequence of T based on the point $x_0 \in X$ such that $x_0\mathcal{S}x_0$ and $x_0\mathcal{S}Tx_0$. As T is $\mathcal{S}$-nondecreasing, then $x_n\mathcal{S}x_n$ and $x_n\mathcal{S}x_{n+1}$ for all $n \in \mathbb{N}$ (in particular, $\{x_n\}$ is $\mathcal{S}$-nondecreasing). Using the contractivity condition (8), for all $n \in \mathbb{N}$,

$$\begin{aligned} d(x_{n+1},x_{n+2}) = d(Tx_n,Tx_{n+1}) &\leq \lambda \left(d(x_n,Tx_n) + d(x_{n+1},Tx_{n+1}) \right) \\ &\leq \lambda \left(d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) \right). \end{aligned}$$

Then

$$d(x_{n+1},x_{n+2}) \leq \frac{\lambda}{1-\lambda}\, d(x_n,x_{n+1}) = \kappa\, d(x_n,x_{n+1}) \quad \text{for all } n \in \mathbb{N}.$$

In particular,

$$d(x_n,x_{n+1}) \leq \kappa^n\, d(x_0,x_1) \quad \text{for all } n \in \mathbb{N},$$

which implies that $\{d(x_n,x_{n+1})\} \to 0$. Furthermore, as $x_n, x_{n+1}, \ldots, x_{n+m}$ is a directed $\mathcal{S}$-path from x_n to x_{n+m}, we derive that, for all $n \in \mathbb{N}$ and all $m \in \mathbb{N}^*$, $m \geq 2$,

$$\begin{aligned} d(x_n,x_{n+m}) &\leq \sum_{i=n+1}^{n+m} d(x_{i-1},x_i) \leq \sum_{i=n+1}^{n+m} \kappa^{i-1}\, d(x_0,x_1) = d(x_0,x_1) \sum_{i=n+1}^{n+m} \kappa^{i-1} \\ &= d(x_0,x_1)\, \frac{\kappa^n - \kappa^{n+m}}{1-\kappa} \leq \kappa^n\, \frac{d(x_0,x_1)}{1-\kappa}. \end{aligned}$$

Using that $x_0\mathcal{S}x_0$ we similarly obtain that $\{d(x_n,x_n)\} \to 0$, so $\{x_n\}$ is a d-Cauchy sequence. As $O_T(x_0)$ is $\mathcal{S}$-nondecreasing-precomplete, there exists $\omega \in X$ such that $\{x_n\}$ d-converges to ω. If the sequence $\{x_n\}$ contains a fixed point of T, the proof is finished. In other case, Proposition 4.16 guarantees that $\{x_n\}$ is infinite. In this case, as T is a $(d,\mathcal{S})$-nondecreasing-continuous mapping, we also have that $\{x_n\}$ also d-converges to $T\omega$. Finally, by condition (S'), we conclude that $T\omega = \omega$. □

In the following result we omit the condition on the $(d,\mathcal{S})$-nondecreasing-continuity of T. However, we need to include additional hypotheses, as a kind of semi-continuity of the measure d. Notice that, in a BRDS, two basic tools of metric theory fail: the limit of a d-convergent sequence is not necessarily unique and the function $d : X \times X \to [0,\infty)$ is not necessarily unique. In fact, these statements also fail in graphical metric spaces (see Remarks 2.13 and 2.15 in [28]), which are BRDS. Furthermore, a sequence can have infinite distinct limits. Nevertheless, we focus in the case in which either some sequences (such as $\mathcal{S}$-nondecreasing, d-Cauchy, d-convergent, infinite, Picard sequences) can have a unique limit or, if it is not unique, the mapping d is "continuously compatible" with such kind of sequences.

Theorem 5.18. *Let $(X,d,\mathcal{S})$ be a BRDS and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing mapping. Suppose that there is a constant $\lambda \in [0,1/2)$ such that*

$$d(Tx,Ty) \leq \lambda \Big(d(x,Tx) + d(y,Ty) \Big) \quad \text{for all } x,y \in X \text{ such that } x\mathcal{S}y. \tag{9}$$

Additionally, assume that:

(T) *if $\{z_n\} \subseteq X$ is an $\mathcal{S}$-nondecreasing, d-Cauchy, d-convergent, infinite, Picard sequence of T, then there is a d-limit $z \in X$ of $\{z_n\}$ satisfying:*

$$\begin{cases} \bullet \quad d(z,Tz) \leq 2 \limsup_{n\to\infty} d(x_n,Tz), \qquad \text{and} \\ \bullet \quad \text{there is } n_0 \in \mathbb{N} \text{ such that } x_n\mathcal{S}z \text{ or } z\mathcal{S}x_n \text{ for all } n \geq n_0; \end{cases} \tag{10}$$

(S') *whenever an $\mathcal{S}$-nondecreasing, d-Cauchy, d-convergent, infinite, Picard sequence $\{x_n\}$ of T has two d-limits $\omega \in X$ and $\omega^* \in T(X)$, then $\omega = \omega^*$.*

If there is $x_0 \in X$ such that $x_0\mathcal{S}x_0$, $x_0\mathcal{S}Tx_0$ and $\mathcal{O}_T(x_0)$ is $\mathcal{S}$-nondecreasing-precomplete, then T has a fixed point.

Proof. By repeating the arguments of the proof of Theorem 5.17, we deduce that the Picard sequence $\{x_n\}$ of T based on the point $x_0 \in X$ such that $x_0\mathcal{S}x_0$ and $x_0\mathcal{S}Tx_0$ is $\mathcal{S}$-nondecreasing, d-Cauchy and d-convergent. If it contains a fixed point of T, the proof is finished. In other case, Proposition 4.16 guarantees that $\{x_n\}$ is infinite. In this case, condition (T) ensures us that there is a d-limit $\omega \in X$ of $\{x_n\}$ satisfying (10). For $n \geq n_0$, the contractivity condition (9) guarantees that

$$\begin{aligned} d(x_{n+1},T\omega) = d(Tx_n,T\omega) &\leq \lambda \Big(d(x_n,Tx_n) + d(\omega,T\omega) \Big) \\ &\leq \lambda \Big(d(x_n,x_{n+1}) + d(\omega,T\omega) \Big). \end{aligned} \tag{11}$$

As $\{d(x_n,x_{n+1})\} \to 0$, it follows that

$$\limsup_{n\to\infty} d(x_{n+1},T\omega) \leq \lambda\, d(\omega,T\omega).$$

However, from (10),

$$d(\omega,T\omega) \leq 2 \limsup_{n\to\infty} d(x_n,T\omega) \leq 2\lambda\, d(\omega,T\omega).$$

As $\lambda \in [0,1/2)$, then $d(\omega,T\omega) = 0$. In particular, from (11), it follows that $d(x_{n+1},T\omega) \leq \lambda\, d(x_n,x_{n+1})$ for all $n \geq n_0$, so $\{x_n\}$ also d-converges to $T\omega$. Assumption (S) concludes that $T\omega = \omega$. □

Theorem 5.19. *Under the hypothesis of Theorem 5.17 (respectively, Theorem 5.18), also assume that $\omega\mathcal{S}\omega$, $\omega'\mathcal{S}\omega'$ and ($\omega\mathcal{S}\omega'$ or $\omega'\mathcal{S}\omega$) for all $\omega,\omega' \in \mathrm{Fix}(T)$. Then T has a unique fixed point.*

Proof. Let $\omega, \omega' \in \mathrm{Fix}(T)$ be two fixed points of T. By hypothesis, $\omega \mathcal{S} \omega$, $\omega' \mathcal{S} \omega'$ and ($\omega \mathcal{S} \omega'$ or $\omega' \mathcal{S} \omega$). Therefore

$$d(\omega, \omega) = d(T\omega, T\omega) \leq \lambda \left(d(\omega, T\omega) + d(\omega, T\omega) \right) = 2\lambda\, d(\omega, \omega).$$

As $2\lambda < 1$, then $d(\omega, \omega) = 0$. Similarly, $d(\omega', \omega') = 0$. Finally,

$$\begin{aligned} d(\omega, \omega') = d(T\omega, T\omega') &\leq \lambda \left(d(\omega, T\omega) + d(\omega', T\omega') \right) \\ &= \lambda \left(d(\omega, \omega) + d(\omega', \omega') \right) = 0. \end{aligned}$$

By assumption (B_1), we conclude that $\omega = \omega'$. □

Theorems 5.17 and 5.18 can be particularized to the setting of graphical metric spaces. We show the process for the first one.

Definition 5.20. *Let (X, d_G) be a graphical metric space, let G' be a subgraph of G such that $V(G') = X$ and let $T : X \to X$ be a mapping. A nonempty subset $A \subseteq X$ is* (d_G, G, G')-termwise-precomplete *if for all G'-termwise d_G-Cauchy sequence $\{a_n\} \subseteq A$, there exists $x \in X$ such that $\{a_n\}$ d_G-converges to x.*

Corollary 5.21. *Let (X, d_G) be a graphical metric space, let G' be a subgraph of G such that $V(G') = X$ and let $T : X \to X$ be a (d, G, G')-termwise-continuous mapping that preserves edges of G'. Suppose that there is a constant $\lambda \in [0, 1/2)$ such that*

$$d(Tx, Ty) \leq \lambda \left(d(x, Tx) + d(y, Ty) \right) \quad \textit{for all } x, y \in X \textit{ such that } (x, y) \in E(G').$$

Additionally, assume that:

(S') *whenever a G'-termwise, d_G-Cauchy, d_G-convergent, infinite, Picard sequence $\{x_n\}$ of T has two d_G-limits $\omega \in X$ and $\omega^* \in T(X)$, then $\omega = \omega^*$.*

If there is $x_0 \in X$ such that $(x_0, Tx_0) \in E(G')$ and $\mathcal{O}_T(x_0)$ is (d_G, G, G')-termwise-precomplete, then T has a fixed point.

Proof. It is only necessary to apply Theorem 5.17 to the BRDS $(X, d_G, \mathcal{S}_{G'})$ and T. □

In the following result, we modify the family of pairs of points for which the contractivity condition must be satisfied. In this way, the contractivity condition is stronger but the initial condition is weaker.

Theorem 5.22. *Let $(X, d, \mathcal{S})$ be a BRDS and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing and $(d, \mathcal{S})$-nondecreasing-connected-continuous mapping. Suppose that there is a constant $\lambda \in [0, 1/2)$ satisfying:*

$$d(Tx, Ty) \leq \lambda \left(d(x, Tx) + d(y, Ty) \right) \quad \textit{for all } x, y \in X \textit{ such that } y \in [x]_{\mathcal{S}}. \tag{12}$$

Additionally, assume that:

(S''') *whenever an $\mathcal{S}$-nondecreasing-connected, d-Cauchy, d-convergent, infinite, Picard sequence $\{x_n\}$ of T has two d-limits $\omega, \omega^* \in X$, then $\omega = \omega^*$.*

If there is $x_0 \in X$ such that $x_0, Tx_0 \in [x_0]_{\mathcal{S}}$ and $\mathcal{O}_T(x_0)$ is $\mathcal{S}$-nondecreasing-connected-precomplete, then T has a fixed point.

Proof. Let $\{x_n\}$ be the Picard sequence of T based on the point $x_0 \in X$ such that $x_0, Tx_0 \in [x_0]_{\mathcal{S}}$. Then there is a directed $\mathcal{S}$-path $z_0 = x_0, z_1, \ldots, z_r = Tx_0 \in X$ from x_0 to $x_1 = Tx_0$. As $z_{i-1}\mathcal{S}z_i$ for all $i \in \{1, 2, \ldots, r\}$ and T is $\mathcal{S}$-nondecreasing, then $(T^n z_{i-1})\mathcal{S}(T^n z_i)$ for all $i \in \{1, 2, \ldots, r\}$ and all $n \in \mathbb{N}$. In particular, $T^n z_0, T^n z_1, \ldots, T^n z_r \in X$ is a directed $\mathcal{S}$-path from $T^n z_0 = T^n x_0 = x_n$ to $T^n z_r = T^n T x_0 = x_{n+1}$. Hence $x_{n+1} \in [x_n]_{\mathcal{S}}$ for all $n \in \mathbb{N}$, that is, $\{x_n\}$ is an $\mathcal{S}$-nondecreasing-connected sequence in X. The arguments of the proof of Theorem 5.17 show that $\{x_n\}$ is a d-Cauchy sequence (using the constant $\kappa = \lambda/(1-\lambda) \in [0, 1)$). As $O_T(x_0)$ is $\mathcal{S}$-nondecreasing-connected-precomplete, $\{x_n\}$ is d-convergent. If it contains a fixed point of T, the proof is finished. In other case, Proposition 4.16 guarantees that $\{x_n\}$ is infinite. Let ω be a d-limit of $\{x_n\}$. As T is $(d, \mathcal{S})$-nondecreasing-connected-continuous, then $\{x_n\}$ also d-converges to $T\omega$. Hence, by (S'''), we conclude that $T\omega = \omega$. □

Theorem 5.23. *Let $(X, d, \mathcal{S})$ be a BRDS and let $T : X \to X$ be an $\mathcal{S}$-nondecreasing mapping. Suppose that there is a constant $\lambda \in [0, 1/2)$ such that*

$$d(Tx, Ty) \leq \lambda \Big(d(x, Tx) + d(y, Ty) \Big) \quad \text{for all } x, y \in X \text{ such that } y \in [x]_{\mathcal{S}}. \tag{13}$$

Additionally, assume that:

(T') *if $\{z_n\} \subseteq X$ is an $\mathcal{S}$-nondecreasing-connected, d-Cauchy, d-convergent, infinite, Picard sequence of T, then there is a d-limit $z \in X$ of $\{z_n\}$ satisfying:*

$$\begin{cases} \bullet \quad d(z, Tz) \leq 2 \limsup\limits_{n\to\infty} d(x_n, Tz), \qquad \text{and} \\ \bullet \quad \text{there is } n_0 \in \mathbb{N} \text{ such that } x_n \in [\omega]_{\mathcal{S}} \text{ or } \omega \in [x_n]_{\mathcal{S}} \text{ for all } n \geq n_0; \end{cases} \tag{14}$$

(S''') *whenever an $\mathcal{S}$-nondecreasing-connected, d-Cauchy, d-convergent, infinite, Picard sequence $\{x_n\}$ of T has two d-limits $\omega \in X$ and $\omega^* \in T(X)$, then $\omega = \omega^*$.*

If there is $x_0 \in X$ such that $x_0, Tx_0 \in [x_0]_{\mathcal{S}}$ and $O_T(x_0)$ is $\mathcal{S}$-nondecreasing-connected-precomplete, then T has a fixed point.

Proof. Repeating the arguments of the proof of Theorem 5.22, we deduce that the Picard sequence $\{x_n\}$ of T based on the point $x_0 \in X$ such that $x_0, Tx_0 \in [x_0]_{\mathcal{S}}$ is $\mathcal{S}$-nondecreasing-connected, d-Cauchy and d-convergent. If it contains a fixed point of T, the proof is finished. In other case, Proposition 4.16 guarantees that $\{x_n\}$ is infinite. In such a case, by condition (T'), there is a d-limit $\omega \in X$ of $\{x_n\}$ satisfying (14). For $n \geq n_0$, the contractivity condition (13) guarantees that

$$\begin{aligned} d(x_{n+1}, T\omega) = d(Tx_n, T\omega) &\leq \lambda \Big(d(x_n, Tx_n) + d(\omega, T\omega) \Big) \\ &\leq \lambda \Big(d(x_n, x_{n+1}) + d(\omega, T\omega) \Big). \end{aligned} \tag{15}$$

As $\{d(x_n, x_{n+1})\} \to 0$, it follows that

$$\limsup_{n\to\infty} d(x_{n+1}, T\omega) \leq \lambda\, d(\omega, T\omega).$$

However, from (14),

$$d(\omega, T\omega) \leq 2 \limsup_{n\to\infty} d(x_n, T\omega) \leq 2\lambda\, d(\omega, T\omega).$$

As $\lambda \in [0, 1/2)$, then $d(\omega, T\omega) = 0$. In particular, from (15), it follows that $d(x_{n+1}, T\omega) \leq \lambda\, d(x_n, x_{n+1})$ for all $n \geq n_0$, so $\{x_n\}$ also d-converges to $T\omega$. Assumption (S') concludes that $T\omega = \omega$. □

Theorem 5.24. *Under the hypothesis of Theorem 5.22 (respectively, Theorem 5.23), also assume that $\omega \in [\omega]_{\mathcal{S}}$, $\omega' \in [\omega']_{\mathcal{S}}$ and ($\omega \in [\omega']_{\mathcal{S}}$ or $\omega' \in [\omega]_{\mathcal{S}}$) for all $\omega, \omega' \in \mathrm{Fix}(T)$. Then T has a unique fixed point.*

Proof. Let $\omega, \omega' \in \mathrm{Fix}(T)$ be two fixed points of T. By hypothesis, $\omega \in [\omega]_{\mathcal{S}}$, $\omega' \in [\omega']_{\mathcal{S}}$ and ($\omega \in [\omega']_{\mathcal{S}}$ or $\omega' \in [\omega]_{\mathcal{S}}$). Therefore

$$d(\omega,\omega) = d(T\omega,T\omega) \leq \lambda \left(d(\omega,T\omega) + d(\omega,T\omega) \right) = 2\lambda\, d(\omega,\omega).$$

As $2\lambda < 1$, then $d(\omega,\omega) = 0$. Similarly, $d(\omega',\omega') = 0$. Finally,

$$\begin{aligned} d(\omega,\omega') &= d(T\omega,T\omega') \leq \lambda \left(d(\omega,T\omega) + d(\omega',T\omega') \right) \\ &= \lambda \left(d(\omega,\omega) + d(\omega',\omega') \right) = 0. \end{aligned}$$

By assumption (B_1), as $\omega \in [\omega']_{\mathcal{S}}$ or $\omega' \in [\omega]_{\mathcal{S}}$, we conclude that $\omega = \omega'$. □

Remark 5.25. *The first condition in (10) and (14) recalls the following assumption on Jleli and Samet's spaces [14] (see also the extension [23]):*

$(\mathcal{D}_3)$ *there exists $C > 0$ such that*

$$\text{if } x, y \in X \text{ and } \{x_n\} \xrightarrow{\mathcal{D}} x, \quad \text{then} \quad \mathcal{D}(x,y) \leq C \limsup_{n\to\infty} \mathcal{D}(x_n, y).$$

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