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# A Fixed Point Problem Under a Finite Number of Equality Constraints Involving a Ćirić Operator

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**Abstract.** Let $(E, \|\cdot\|)$ be a Banach space with a cone $P$. Let $T, \varphi_i : E \to E$ $(i = 1, 2, \cdots, r)$ be a finite number of mappings. We obtain sufficient conditions for the existence of solutions to the problem

$$\begin{cases} Tx & = & x, \\ \varphi_i(x) & = & 0_E, \; i = 1, 2, \cdots, r, \end{cases}$$

where $0_E$ is the zero vector of $E$, and $T$ is a mapping satisfying a Ćirić-contraction. Some interesting consequences are deduced from our main results.

*To the memory of Professor Lj. Ćirić (1935–2016)*

## 1. Introduction

In 1974, Ćirić [3] established the following fixed point theorem.

**Theorem 1.1 (Ćirić (1974)).** *Let $(M, d)$ be a metric space, and let $T : M \to M$ be a given mapping. Suppose that the following conditions are satisfied:*

*(i) $T$ is orbitally continuous on $M$;*

*(ii) $(M, d)$ is $T$-orbitally complete;*

*(iii) There exists a constant $q \in (0, 1)$ such that*

$$\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(y, Tx)\} \le q d(x, y),$$

*for all $x, y \in M$.*

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*Then, for every $x \in M$, the Picard sequence $\{T^n x\}$ conveges to a fixed point of $T$.*

Note that Theorem 1.1 provides just the existence of at least one fixed point. However, the uniqueness is not satisfied in general. A nice counter-example is presented by Ćirić in [3]. For other related works, we refer the reader to [1, 2, 4, 7, 11, 12, 14] and references therein.

Let $(E, \|\cdot\|)$ be a Banach space with a cone $P$. Let $T, \varphi_i : E \to E$ $(i = 1, 2, \cdots, r)$ be a finite number of mappings. In this paper, we deal with the solvability of the system of operator equations

$$\begin{cases} Tx &= x, \\ \varphi_i(x) &= 0_E, \ i = 1, 2, \cdots, r, \end{cases} \tag{1}$$

where $0_E$ is the zero vector of $E$, and $T$ is mapping satisfying a Ćirić-contraction simialar to (iii). We obtain sufficient conditions for the existence of solutions to (1). Some interesting consequences are deduced from our main results.

At first, let us introduce some concepts that will be used later. This is the aim of the next section.

## 2. Preliminaries

In this paper, the considered Banach space $(E, \|\cdot\|)$ is supposed to be partially ordered by a cone $P$. Recall that a nonempty closed convex set $P \subset E$ is said to be a cone (see [8]) if it satisfies the following conditions:
(P1) $\lambda \geq 0$, $x \in P \Longrightarrow \lambda x \in P$.
(P2) $-x, x \in P \Longrightarrow x = 0_E$.
We define the partial order $\leq_P$ in $E$ induced by the cone $P$ by

$$(x, y) \in E \times E, \quad x \leq_P y \Longleftrightarrow y - x \in P.$$

**Definition 2.1.** *Let $\varphi : E \to E$ be a given mapping. We say that $\varphi$ is $0_E$-level closed from the left, iff the set*

$$lev\varphi_{\geq_P} := \{x \in E : \varphi(x) \geq_p 0_E\}$$

*is nonempy and closed.*

**Definition 2.2.** *Let $\varphi : E \to E$ be a given mapping. We say that $\varphi$ is $0_E$-level closed from the right, iff the set*

$$lev\varphi_{\leq_P} := \{x \in E : \varphi(x) \leq_p 0_E\}$$

*is nonempy and closed.*

**Example 2.3.** *Let $E = \mathbb{R}$ be the set of real numbers. Let $P = [0, \infty)$, and let $\leq_P$ be the standard order in $\mathbb{R}$, denoted by $\leq$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a lower semi-continuous function. Suppose that*

$$\exists\, x_0 \in \mathbb{R} : \ \varphi(x_0) \leq 0.$$

*Then $\varphi$ is 0-level closed from the right. Indeed, the set $lev\varphi_{\leq_P}$ is nonepmty since $x_0 \in lev\varphi_{\leq_P}$. Moreover, if $\{x_n\} \subset lev\varphi_{\leq_P}$ is a convergent sequence to $x \in \mathbb{R}$, then*

$$0 \geq \liminf_{n\to\infty} \varphi(x_n) \geq \varphi(x),$$

*which yields $x \in lev\varphi_{\leq_P}$. Therefore the set $lev\varphi_{\leq_P}$ is closed.*

**Example 2.4.** *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an upper semi-continuous function. Suppose that*

$$\exists\, x_0 \in \mathbb{R} : \ \varphi(x_0) \geq 0.$$

*Then $\varphi$ is 0-level closed from the left. Indeed, the set $lev\varphi_{\geq_P}$ is nonepmty since $x_0 \in lev\varphi_{\geq_P}$. Moreover, if $\{x_n\} \subset lev\varphi_{\geq_P}$ is a convergent sequence to $x \in \mathbb{R}$, then*

$$0 \leq \limsup_{n\to\infty} \varphi(x_n) \leq \varphi(x),$$

*which yields $x \in lev\varphi_{\geq_P}$. Therefore the set $lev\varphi_{\geq_P}$ is closed.*

**Example 2.5.** *Clearly, if $\varphi : E \to E$ is a continuous mapping, then $\varphi$ is $0_E$-level closed from the left, and from the right.*

**Remark 2.6.** *Note that the fact that $\varphi : E \to E$ is $0_E$-level closed from the left, does not imply that $\varphi$ is $0_E$-level closed from the right. Several counter-examples can be obtained. We invite the reader to check this fact by him self.*

**Definition 2.7.** *Let $T : E \to E$ be a given mapping. For a given $x \in E$, we denote by $O(x)$ the orbit of $x$, that is,*

$$O(x) = \{T^n x : n = 0, 1, 2, \cdots\}, \quad x \in E.$$

*We say that $T$ is orbitally contiuous on $F \subseteq E$, iff $T$ is continuous on $O(x)$, for every $x \in F$.*

**Definition 2.8.** *Let $T, \varphi_i : E \to E$ $(i = 1, 2, \cdots, r)$ be a finite number of mappings. We say that $T$ is a Ćirić operator with respect to $\{\varphi_i\}_{i=1}^r$, iff there exists some $q \in (0, 1)$ such that*

$$\min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \le q\|x - y\|,$$

*for every $(x, y) \in E \times E$ such that*

$$\varphi_i(x) \le_P 0_E, \quad i = 1, 2, \cdots, r$$

*and*

$$\varphi_i(y) \ge_P 0_E, \quad i = 1, 2, \cdots, r.$$

**Remark 2.9.** *Let $T, \varphi_i : E \to E$ $(i = 1, 2, \cdots, r)$ be a finite number of mappings. If $T$ is a Ćirić operator with respect to $\{\varphi_i\}_{i=1}^r$, then by symmetry, we have*

$$\min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \le q\|x - y\|,$$

*for every $(x, y) \in E \times E$ such that*

$$\varphi_i(x) \ge_P 0_E, \quad i = 1, 2, \cdots, r$$

*and*

$$\varphi_i(y) \le_P 0_E, \quad i = 1, 2, \cdots, r.$$

Now, we are ready to state and prove our main results.

## 3. Main Results

In order to make easy the lecture of the paper, let us start with the case of one equality constraint.

### 3.1. The case of one equality constraint

We consider system (1) with $r = 1$ and $\varphi_1 = \varphi$.

**Theorem 3.1.** *Let $T, \varphi : E \to E$ be two giving mappings. Suppose that the following conditions are satisfied:*

*(i) $T$ is orbitally continuous on $\mathrm{lev}\varphi_{\le_P}$;*

*(ii) $T$ is a Ćirić operator with respect to $\varphi$, that is,*

$$\min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \le q\|x - y\|,$$

*for every $(x, y) \in E \times E$ such that $\varphi(x) \le_P 0_E$ and $\varphi(y) \ge_P 0_E$;*

*(iii) $\varphi$ is $0_E$-level closed from the left;*

*(iv) There exists $x_0 \in E$ such that $\varphi(x_0) \le_P 0_E$;*

*(v) For every $x \in E$, we have*

$$\varphi(x) \leq_P 0_E \implies \varphi(Tx) \geq_P 0_E$$

*and*

$$\varphi(x) \geq_P 0_E \implies \varphi(Tx) \leq_P 0_E.$$

*Then the Picard sequence $\{T^n x_0\}$ converges to a solution to* (1) *with $r = 1$ and $\varphi_1 = \varphi$.*

*Proof.* Let $x_0 \in E$ be such that

$$\varphi(x_0) \leq_p 0_E.$$

Such a point exists from (iv). From (v), we have

$$\varphi(x_0) \leq_P 0_E \implies \varphi(Tx_0) \geq_P 0_E.$$

Define the sequences $\{x_n\}$ in $E$ by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \cdots$$

Then we have

$$\varphi(x_1) \geq_P 0_E.$$

From (v), we have

$$\varphi(x_1) \geq_P 0_E \implies \varphi(Tx_1) \leq_P 0_E,$$

that is,

$$\varphi(x_2) \leq_P 0_E.$$

Again, using (v), we get from the above inequality that

$$\varphi(x_3) \geq_P 0_E.$$

Then, by induction, we obtain

$$\varphi(x_{2n}) \leq_P 0_E, \ \varphi(x_{2n+1}) \geq_P 0_E, \quad n = 0, 1, 2, \cdots \tag{2}$$

Using (ii) and (2), by symmetry (see Remark 2.9), we obtain

$$\begin{aligned}&\min\{\|Tx_{n-1} - Tx_n\|, \|x_{n-1} - Tx_{n-1}\|, \|x_n - Tx_n\|\} - \min\{\|x_{n-1} - Tx_n\|, \|x_n - Tx_{n-1}\|\}\\ &\leq q\|x_{n-1} - x_n\|, \quad n = 1, 2, 3, \cdots,\end{aligned}$$

that is,

$$\begin{aligned}&\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|, \|x_n - x_{n+1}\|\} - \min\{\|x_{n-1} - x_{n+1}\|, \|x_n - x_n\|\}\\ &\leq q\|x_{n-1} - x_n\|, \quad n = 1, 2, 3, \cdots,\end{aligned}$$

which yields

$$\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|\} \leq q\|x_{n-1} - x_n\|, \quad n = 1, 2, 3, \cdots \tag{3}$$

We distinguish three cases.
Case 1. If $x_{2N} = x_{2N+1}$, for some $N = 0, 1, \cdots$ In this case, we have

$$x_{2N} = Tx_{2N}.$$

Moreover, from (2), we have

$$\varphi(x_{2N}) \leq_P 0_E, \ \varphi(x_{2N}) = \varphi(x_{2N+1}) \geq_P 0_E.$$

Therefore, $\varphi(x_{2N}) \in P$ and $-\varphi(x_{2N}) \in P$. Since $P$ is a cone, we deduce that

$$\varphi(x_{2N}) = 0_E.$$

Hence, $x_{2N} \in E$ is a solution to (1).
Case 2. If $x_{2N+1} = x_{2N+2}$, for some $N = 0, 1, \cdots$ We argue exactly as in case 1 to see that $x_{2N+1}$ is a solution to (1).
Case 3. If $x_n \neq x_{n+1}$, for every $n = 0, 1, \cdots$ In this case, using (3) and the fact that $q \in (0, 1)$, we deduce that

$$\|x_n - x_{n+1}\| \leq q\|x_{n-1} - x_n\|, \quad n = 1, 2, \cdots \tag{4}$$

Since $q \in (0, 1)$, we deduce from (4) that $\{x_n\}$ is a Cauchy sequence in the Banach space $(E, \|\cdot\|)$. Therefore, there exists some $x^* \in E$ such that

$$\lim_{n\to\infty} \|x_n - x^*\| = 0. \tag{5}$$

Since $T$ is orbitally continuous on $\mathrm{lev}\varphi_{\leq_P}$, then $T$ is continuous on $O(x_0)$. Therefore, by (5) we deduce that

$$\lim_{n\to\infty} \|x_{n+1} - Tx^*\| = 0. \tag{6}$$

By the uniqueness of the limit, we obtain

$$Tx^* = x^*. \tag{7}$$

On the other hand, from (2), we have

$$\{x_{2n+1}\} \subset \mathrm{lev}\varphi_{\geq_P}, \quad n = 0, 1, \cdots$$

Since $\varphi$ is $0_E$-level closed from the left, we deduce from (5) that $x^* \in \mathrm{lev}\varphi_{\geq_P}$, that is,

$$\varphi(x^*) \geq_P 0_E.$$

But by (v) and (7), we obtain

$$\varphi(x^*) = \varphi(Tx^*) \leq_P 0_E.$$

Hence, we have $\varphi(x^*) \in P$ and $-\varphi(x^*) \in P$. Since $P$ is a cone, we get

$$\varphi(x^*) = 0_E. \tag{8}$$

Finally, (7) and (8) imply that $x^* \in E$ is a solution to (1). □

**Remark 3.2.** *Observe that the conclusion of Theorem 3.1 is still valid if we replace condition (iii) by the following condition:*
*(iii') $\varphi$ is $0_E$-level closed from the right.*
*Indeed, we have just to replace $\varphi$ by $-\varphi$ in Theorem 3.1 to get the desired result.*

By Remark 3.2, we have the following result.

**Theorem 3.3.** *Let $T, \varphi : E \to E$ be two giving mappings. Suppose that the following conditions are satisfied:*

*(i) $T$ is orbitally continuous on $\mathrm{lev}\varphi_{\leq_P}$;*

*(ii) $T$ is a Ćirić operator with respect to $\varphi$, that is,*

$$\min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \leq q\|x - y\|,$$

*for every $(x, y) \in E \times E$ such that $\varphi(x) \leq_P 0_E$ and $\varphi(y) \geq_P 0_E$;*

*(iii') $\varphi$ is $0_E$-level closed from the right;*

*(iv) There exists $x_0 \in E$ such that $\varphi(x_0) \leq_P 0_E$;*

*(v) For every $x \in E$, we have*

$$\varphi(x) \leq_P 0_E \implies \varphi(Tx) \geq_P 0_E$$

*and*

$$\varphi(x) \geq_P 0_E \implies \varphi(Tx) \leq_P 0_E.$$

*Then the Picard sequence $\{T^n x_0\}$ converges to a solution to* (1) *with $r = 1$ and $\varphi_1 = \varphi$.*

**Remark 3.4.** *Observe that from the proof of Theorem 3.1, Theorems 3.1 and 3.3 are still valid if we replace condition (i) by the following condition:*
*(i') $T$ is orbitally continuous on* $\mathrm{lev}\varphi_{\geq_P}$.

*3.2. The case of a finite number of equality constraints*

Now, we study the solvability of system (1) with $r \geq 2$. We have the following result.

**Theorem 3.5.** *Let $T, \varphi_i : E \to E$ $(i = 1, 2, \cdots, r)$ be a finite number of mappings. Suppose that the following conditions are satisfied:*

*(i) $T$ is orbitally continuous on $\bigcap_{i=1}^r \mathrm{lev}\varphi_{i_{\leq_P}}$;*

*(ii) $T$ is a Ćirić operator with respect to $\{\varphi_i\}_{i=1}^r$;*

*(iii) $\varphi_i$, $i = 1, 2, \cdots, r$, is $0_E$-level closed from the left;*

*(iv) There exists $x_0 \in E$ such that*

$$\varphi_i(x_0) \leq_P 0_E, \quad i = 1, 2, \cdots, r;$$

*(v) For every $x \in E$, we have*

$$\varphi_i(x) \leq_P 0_E,\ i = 1, 2, \cdots, r \implies \varphi_i(Tx) \geq_P 0_E,\ i = 1, 2, \cdots, r$$

*and*

$$\varphi_i(x) \geq_P 0_E,\ i = 1, 2, \cdots, r \implies \varphi_i(Tx) \leq_P 0_E,\ i = 1, 2, \cdots, r.$$

*Then the Picard sequence $\{T^n x_0\}$ converges to a solution to* (1).

*Proof.* Let $x_0 \in E$ be such that

$$\varphi_i(x_0) \leq_p 0_E, \quad i = 1, 2, \cdots, r.$$

Such a point exists from (iv). From (v), we have

$$\varphi_i(Tx_0) \geq_P 0_E, \quad i = 1, 2, \cdots, r.$$

Define the sequences $\{x_n\}$ in $E$ by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \cdots$$

Then we have

$$\varphi_i(x_1) \geq_P 0_E, \quad i = 1, 2, \cdots, r.$$

From (v), we have

$$\varphi_i(Tx_1) \leq_P 0_E, \quad i = 1, 2, \cdots, r,$$

that is,

$$\varphi_i(x_2) \leq_P 0_E, \quad i = 1, 2, \cdots, r.$$

Again, using (v), we get from the above inequalities that

$$\varphi_i(x_3) \geq_P 0_E, \quad i = 1, 2, \cdots, r.$$

Then, by induction, we obtain

$$\varphi_i(x_{2n}) \leq_P 0_E, \quad i = 1, 2, \cdots, r \tag{9}$$

and

$$\varphi_i(x_{2n+1}) \geq_P 0_E, \quad i = 1, 2, \cdots, r. \tag{10}$$

Using (ii), (9), and (10), we obtain

$$\begin{aligned} &\min\{\|Tx_{n-1} - Tx_n\|, \|x_{n-1} - Tx_{n-1}\|, \|x_n - Tx_n\|\} - \min\{\|x_{n-1} - Tx_n\|, \|x_n - Tx_{n-1}\|\} \\ &\leq q\|x_{n-1} - x_n\|, \quad n = 1, 2, 3, \cdots, \end{aligned}$$

that is,

$$\begin{aligned} &\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|, \|x_n - x_{n+1}\|\} - \min\{\|x_{n-1} - x_{n+1}\|, \|x_n - x_n\|\} \\ &\leq q\|x_{n-1} - x_n\|, \quad n = 1, 2, 3, \cdots, \end{aligned}$$

which yields

$$\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|\} \leq q\|x_{n-1} - x_n\|, \quad n = 1, 2, 3, \cdots \tag{11}$$

We distinguish three cases.
Case 1. If $x_{2N} = x_{2N+1}$, for some $N = 0, 1, \cdots$ In this case, we have

$$x_{2N} = Tx_{2N}.$$

Moreover, from (9) and (10), we have

$$\varphi_i(x_{2N}) \leq_P 0_E, \; \varphi_i(x_{2N}) = \varphi_i(x_{2N+1}) \geq_P 0_E, \quad i = 1, 2, \cdots, r.$$

Therefore, $\varphi_i(x_{2N}) \in P$ and $-\varphi_i(x_{2N}) \in P$, for every $i = 1, 2, \cdots, r$. Since $P$ is a cone, we deduce that

$$\varphi_i(x_{2N}) = 0_E, \quad i = 1, 2, \cdots, r.$$

Hence, $x_{2N} \in E$ is a solution to (1).
Case 2. If $x_{2N+1} = x_{2N+2}$, for some $N = 0, 1, \cdots$ We argue exactly as in case 1 to see that $x_{2N+1}$ is a solution to (1).
Case 3. If $x_n \neq x_{n+1}$, for every $n = 0, 1, \cdots$ In this case, using (11) and the fact that $q \in (0, 1)$, we deduce that

$$\|x_n - x_{n+1}\| \leq q\|x_{n-1} - x_n\|, \quad n = 1, 2, \cdots$$

Since $q \in (0, 1)$, we deduce that $\{x_n\}$ is a Cauchy sequence in the Banach space $(E, \|\cdot\|)$. Therefore, there exists some $x^* \in E$ such that

$$\lim_{n\to\infty} \|x_n - x^*\| = 0.$$

Since $T$ is orbitally continuous on $\bigcap_{i=1}^r \mathrm{lev}\varphi_{i_{\leq_P}}$, then $T$ is continuous on $\mathcal{O}(x_0)$. Therefore, we have

$$\lim_{n\to\infty} \|x_{n+1} - Tx^*\| = 0.$$

By the uniqueness of the limit, we obtain

$$Tx^* = x^*.$$

On the other hand, from (10), we have

$$\{x_{2n+1}\} \subset \bigcap_{i=1}^r \mathrm{lev}\varphi_{i_{\geq_P}}, \quad n = 0, 1, \cdots$$

Hence, by (iii) we deduce that $x^* \in \bigcap_{i=1}^{r} \mathrm{lev}\varphi_{i_{\geq_P}}$, that is,

$$\varphi_i(x^*) \geq_P 0_E, \quad i = 1, 2, \cdots, r.$$

But by (v) and the fact that $Tx^* = x^*$, we obtain

$$\varphi_i(x^*) = \varphi_i(Tx^*) \leq_P 0_E, \quad i = 1, 2, \cdots, r.$$

Hence, we have $\varphi_i(x^*) \in P$ and $-\varphi_i(x^*) \in P$, for all $i = 1, 2, \cdots, r$. Since $P$ is a cone, we get

$$\varphi_i(x^*) = 0_E, \quad i = 1, 2, \cdots, r.$$

Therefore, we proved that $x^* \in E$ is a solution to (1). □

**Remark 3.6.** *As in the case of one equality constraint, the conclusion of Theorem 3.5 is still valid if we replace condition (iii) by the following condition:*
*(iii′) $\varphi_i$, $i = 1, 2, \cdots, r$, is $0_E$-level closed from the right.*

**Remark 3.7.** *As in the case of one equality constraint, Theorem 3.5 is still valid if we replace condition (i) by the following condition:*
*(i′) $T$ is orbitally continuous on $\bigcap_{i=1}^{r} \mathrm{lev}\varphi_{i_{\geq_P}}$.*

## 4. Some Consequences

In this section, some interesting consequences following from the above results are presented.

### 4.1. A common fixed point result

Let $T, S : E \to E$ be two giving mappings. We consider the following common fixed point problem: Find $x \in X$ such that

$$\left\{ \begin{array}{rcl} x & = & Tx, \\ x & = & Sx. \end{array} \right. \tag{12}$$

Observe that system (12) is equivalent to

$$\left\{ \begin{array}{rcl} x & = & Tx, \\ \varphi(x) & = & 0_E, \end{array} \right. \tag{13}$$

where $\varphi : E \to E$ is the mapping defined by

$$\varphi(x) = Sx - x, \quad x \in E. \tag{14}$$

Let us define the sets

$$H_1 = \{x \in E : Sx \leq_P x\}$$

and

$$H_2 = \{x \in E : Sx \geq_P x\}.$$

We have the following result.

**Corollary 4.1.** *Let $T, S : E \to E$ be two giving mappings. Suppose that the following conditions are satisfied:*

*(i) $T$ is orbitally continuous on $H_1$;*

*(ii) There exists a constant $q \in (0, 1)$ such that for every $(x, y) \in E \times E$,*

$$Sx \leq_P x,\ Sy \geq_P y \implies \min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \leq q\|x - y\|;$$

*(iii)* $H_2$ *is a closed subset of* $E$*;*

*(iv)* *There exists* $x_0 \in E$ *such that* $Sx_0 \leq_P x_0$*;*

*(v)* *For every* $x \in E$*, we have*

$$Sx \leq_P x \implies S(Tx) \geq_P Tx$$

*and*

$$Sx \geq_P x \implies S(Tx) \leq_P Tx.$$

*Then the Picard sequence* $\{T^n x_0\}$ *converges to a solution to* (12).

*Proof.* Using the fact that (12) is equivalen to (13), and taking in Theorem 3.1 $\varphi : E \to E$, the mapping defined by (14), we obtain immediately the desired result. □

**Remark 4.2.** *The most used techniques for the solvability of* (12) *are based on a compatibility condition introduced by Jungck [9]. Such techniques are interesting and can be useful for the solvability of certain problems (see [5, 6, 9, 10, 13] and references therein). However, two major difficulties arise in the use of such approach. At first, the compatibility condition is not always satisfied, and in some cases it is not easy to check such condition. Moreover, the numerical approximation of the common fixed point is constructed via the axiom of choice using certain inclusions, which makes difficult its numerical implementation. The result given by Corollary 4.1 presents a new approach to study common fixed point problems.*

### 4.2. *A fixed point result*

Taking $S = I_E$ (the identity mapping) in Corollary 4.1, we obtain the following fixed point result, which is similar to Theorem 1.1.

**Corollary 4.3.** *Let* $T : E \to E$ *be a giving mapping. Suppose that the following conditions are satisfied:*

*(i)* $T$ *is orbitally continuous on* $E$*;*

*(ii)* *There exists a constant* $q \in (0, 1)$ *such that for every* $(x, y) \in E \times E$*,*

$$\min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \leq q\|x - y\|.$$

*Then, for every* $x \in E$*, the Picard sequence* $\{T^n x\}$ *converges to a fixed point of* $T$*.*

**Remark 4.4.** *Observe that Corollary 4.3 can be deduced also from Theorem 1.1. Indeed, since E is a banach space, then E is is T-orbitally complete. For the definition of the notion of T-orbitally complete sets, we refer the reader to [3].*

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