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On Strong Summability and Convergence

Eberhard Malkowsky[a]

[a]*Državni Univerzitet u Novom Pazaru, Vuka Karadžića bb, 36300 Novi Pazar, Serbia*

Abstract. We give a survey of the recent results concerning the fundamental topological properties of spaces of stronly summable and convergent sequences, their β– and continuous duals, and the characterizations of classes of linear operators between them. Furthermore we demonstrate how the Hausdorff measure of noncompactness can be used in the characterization of classes of compact operators between the spaces of strongly summable and bounded sequences.

To the memory of Professor Lj. Ćirić (1935–2016)

1. Introduction

The notions of strong summability and convergence play an important role in both classical and modern summability theory and the study of sequence spaces.

The most popular method of summability is that of the arithmetic means, also referred to as the C_1 method, the Cesàro method of order 1. A sequence $x = (x_k)_{k=0}^{\infty}$ of complex numbers is said to be summable C_1 if there exists $\xi \in \mathbb{C}$ such that the sequence $\sigma(x) = (\sigma_n(x))_{n=0}^{\infty}$ of its arithmetic means

$$\sigma_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} x_k \ (n = 0, 1, \dots) \text{ converges to } \xi \text{ as } n \to \infty;$$

this is denoted by $\lim_{C_1} x = \xi$ or $x_k \to \xi(C_1)$ $(k \to \infty)$. It is well known that every convergent sequence is summable C_1 to the same limit, but C_1 summability of a sequence does not imply its convergence, in general, as the simple example of the sequence $((-1)^k)_{k=0}^{\infty}$ shows. According to Hyslop [28], the notion of strong C_1 summability $[C_1]$ was first introduced and applied in the theory of Fourier series by Fekete [23]; a sequence $x = (x_k)_{k=0}^{\infty}$ of complex numbers is said to be strongly summable C_1, if there exists $\xi \in \mathbb{C}$ such that the sequence $\sigma(|x - \xi e|) = (\sigma_n|x - \xi e|)_{n=0}^{\infty}$ converges to 0 as $n \to \infty$, where $e = (1, 1, \dots)$ and

$$\sigma_n(|x - \xi e|) = \frac{1}{n+1}\sum_{k=0}^{n} |x_k - \xi| \ (n = 0, 1, \dots) \, ;$$

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Email address: Eberhard.Malkowsky@math.uni-giessen.de (Eberhard Malkowsky)

this is denoted by $\lim_{[C_1]} x = \xi$ or $x \to \xi([C_1])$. It is clear that $\lim_{[C_1]} x = \xi$ implies $\lim_{C_1} = \xi$, but summability C_1 does not imply summability $[C_1]$, as the simple example of the sequence $x = (x_k)_{k=0}^\infty$ with $x_0 = 0$ and $x_k = \nu(-1)^k$ $(2^\nu \le k \le 2^{\nu+1} - 1; \nu = 0, 1, \dots)$ shows. Also clearly $\lim x = \xi$ implies $\lim_{[C_1]} = \xi$, but summability $[C_1]$ does not imply convergence, as the simple example of the sequence $x = (x_k)_{k=0}^\infty$ with $x_k = \nu$ $(k = 2^\nu)$ and $x_k = 0$ $(k \neq 2^\nu)$ for $\nu = 0, 1, \dots$ shows. Therefore, writing c, c_{C_1} and $c_{[C_1]}$ for the sets of all complex sequences that are convergent, summable C_1 and $[C_1]$, respectively, we have $c \subsetneq c_{[C_1]} \subsetneq c_{C_1}$.

Hyslop [28] generalized the notion of summability $[C_1]$ to that of strong summablity $[C_\alpha]^p$ of order $\alpha > 0$ and index $p > 0$ as follows. (The special case of $p = 1$ was studied by Winn [67].) Let $A_n^\alpha = \binom{n+\alpha}{n}$ and $\sigma_n^\alpha(x)$ $(n = 0, 1, \dots)$ denote the well–known n^{th} Cesàro coefficient of order $\alpha > -1$, and

$$\sigma_n^\alpha(x) = \frac{1}{A_n^\alpha} \sum_{k=0}^n A_{n-k}^{\alpha-1} x_k$$

be the n^{th} Cesàro transform of order α of the sequence $x = (x_k)_{k=0}^\infty$. (We note $\sigma^{(1)}(x) = \sigma(x)$.) Then the sequence x is said to be strongly summable $[C_\alpha]^p$ if there exists $\xi \in \mathbb{C}$ such that

$$\sigma_n\left(\left|\sigma^{\alpha-1}(x) - \xi e\right|^p\right) = \frac{1}{n+1} \sum_{k=0}^n \left|\sigma_k^{\alpha-1}(x) - \xi\right|^p \to 0 \text{ as } n \to \infty; \tag{1}$$

summability $[C_\alpha]^p$ is also denoted by $[C_1, C_{\alpha-1}]^p$. Obviously summability $[C_\alpha]^p$ reduces to summability $[C_1]$ for $\alpha = p = 1$.

Following [28, Theorem 3], Hyslop extended the notion of summability $[C_\alpha]^p$ to $\alpha = 0$ as follows. A sequence $x = (x_k)_{k=0}^\infty$ is said to be summable $[C_0]^p$, or *strongly summable with index* $p > 0$, if $x \in c$ and

$$\lim_{n\to\infty} \sum_{k=0}^n k^p |x_k - x_{k-1}|^p = 0. \tag{2}$$

It was also noted there that the condition in (2) for $p = 1$ does not itself imply convergence, since the sequence $x = (x_k)_{k=0}^\infty$ with $x_0 = x_1 = 0$ and $x_k = \sum_{j=2}^k (j \log j)^{-1}$ is not convergent, but satisfies the condition in (2) for $p = 1$.

Borwein [9] gave a different definition of strong convergence $[H_0]^p$ saying that a sequnece $x = (x_k)_{k=0}^\infty$ is strongly convergent $[H_0]^p$ if there exists $\xi \in \mathbb{C}$ such that

$$\lim_{n\to\infty} \sigma_n\left(\left|\Delta\left((kx_k)_{k=0}^\infty\right) - \xi e\right|^p\right) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^n \left|\Delta_k((kx_k)_{k=0}^\infty) - \xi\right|^p = 0, \tag{3}$$

where $\Delta_k y = y_k - y_{k-1}$ $(k = 0, 1, \dots)$ for each sequence $y = (y_k)_{k=0}^\infty$; throughout we use the convention that each term with a negative subscript is equal to zero. Borwein [9, Theorem 8] showed that his strong convergence $[H_0]^p$ is equivalent to Hyslop's strong convergence $[C_0]^p$ for $p \ge 1$. Kuttner and Thorpe showed that the equivalence of the two notions does not hold for $0 < p < 1$, more precisely ([39, Proposition 1]), if $0 < p < 1$ then $x \to \xi[C_0]^p$ implies $x \to \xi[H_0]^p$, but the converse implication is not true, in general. They used the notation $[c]_p$ $(0 < p < \infty)$ for the set of all sequenecs that are strongly convergent $[H_0]^p$. It was shown in [28, Theorem 4] that strong convergence $[C_0]^p$ implies strong summability $[C_\alpha]^p$ for all $p \ge 1$ and all $\alpha > 0$.

Mòricz [59], generalized Borwein's definition of strong convergence $[H_0]^p$ for $p = 1$ to that of Λ–strong convergence by replacing the terms $1/(n+1)$ and $\Delta_k((kx_k)_{k=0}^\infty)$ by λ_n and $\Delta_k((\lambda_k x_k)_{k=0}^\infty)$, where $\Lambda = (\lambda_k)_{k=0}^\infty$ is an increasing sequence of positive reals tending to infinity.

There are a great number of papers that study inclusion theorems for the various notions of summability and convergence and their generalizations for different parameters, for instance, [9, 24–26, 28, 37]. We will not follow this further.

In this paper, we give a survey of the most important results concerning the basic topological properties of the spaces of strongly summable and bounded sequences and strongly convergent and bounded sequences defined above, their dual spaces, the characterizations of various classes of matrix transformations between them, estimates for the Hausdorff measure of noncompactness of bounded sets, and the characterizations of some classes of compact linear operators between them.

2. Notations and Preliminary Results

First we recall a few standard notations.

Let (X,d) be a metric space. We denote by

$$B_X(x_0,r)=\{x\in X:d(x,x_0)<r\},\ \overline{B}_X(x_0,r)=\{x\in X:d(x,x_0)\le r\}\text{ and }S_X(x_0,r)=\{x\in X:d(x,x_0)=r\}$$

the open and closed balls, and the sphere of radius $r>0$ and centre in $x_0\in X$, respectively. We write $B_X=B_X(0,1)$, $\overline{B}_X=\overline{B}_X(0,1)$ and $S_X=S_X(0,1)$, for short. A sequence $(b_n)_{n=0}^\infty$ in a complex (real) linear metric space X is called a Schauder basis, if, for every $x\in X$, there exists a unique sequence $(\lambda_n)_{n=0}^\infty$ of scalars such that $x=\sum_{n=0}^\infty \lambda_n b_n$.

Let X and Y be normed spaces, then $\mathcal{B}(X,Y)$ denotes the space of all bounded linear operators $L:X\to Y$ which is a normed space with the operator norm defined by $\|L\|=\sup\{\|L(x)\|:x\in S_X\}$ for all $L\in\mathcal{B}(X,Y)$, and a Banach space if Y is a Banach space; we write $\mathcal{B}(X)=\mathcal{B}(X,X)$, for short. Also $X^*=\mathcal{B}(X,\mathbb{C})$ denotes the Banach space of all continuous linear functionals with the norm $\|f\|=\sup\{|f(x)|:x\in S_X\}$ for all $f\in\mathcal{B}(X,\mathbb{C})$.

Let ω be the set of all complex sequences $x=(x_k)_{k=0}^\infty$, and ℓ_∞, c, c_0 and ϕ, denote the sets of all bounded, convergent, null and finite sequences, respectively; we also write $\ell_p=\{x\in\omega:\sum_{k=0}^\infty|x_k|^p<\infty\}$ for $1\le p<\infty$, and bs and cs for the sets of all bounded and convergent series. It is well known that ω is a complete linear metric space with its metric d defined by

$$d(x,y)=\sum_{k=0}^\infty \frac{1}{2^k}\cdot\frac{|x_k-y_k|}{1+|x_k-y_k|}\text{ for all }x=(x_k)_{k=0}^\infty,\,y=(y_k)_{k=0}^\infty\in\omega,$$

the so–called Fréchet combination of its coordinates, and that convergence in ω is equivalent to coordinatewise convergence. Also ℓ_∞, c, c_0, ℓ_p, bs and cs are Banach spaces with their natural norms defined by $\|x\|_\infty=\sup_k|x_k|$ for ℓ_∞, c_0 and c, $\|x\|_p=(\sum_{k=0}^\infty|x_k|^p)^{1/p}$ for ℓ_p, and $\|x\|_{bs}=\sup_n|\sum_{k=0}^n x_k|$ for bs and cs.

Let e and $e^{(n)}$ $(n=0,1,\dots)$ be the sequences with $e_k=1$ for all k, and $e_n^{(n)}=1$ and $e_k^{(n)}=0$ $(k\ne n)$.

For any subset X of ω, the set $X^\beta=\{a\in\omega:a\cdot x=(a_kx_k)_{k=0}^\infty\in cs\}$ is called the *β–dual of X*.

Let $A=(a_{nk})_{n,k=0}^\infty$ be an infinite matrix of complex numbers, X and Y be subsets of ω and $x\in\omega$. We write $A_n=(a_{nk})_{k=0}^\infty$ for the sequence in the n–th row of A, $A_nx=\sum_{k=0}^\infty a_{nk}x_k$, $Ax=(A_nx)_{n=0}^\infty$ (provided all the series A_nx converge), and $X_A=\{x\in\omega:Ax\in X\}$ for the *matrix domain of A in X*. Also (X,Y) is the class of all matrices A such that $X\subset Y_A$; so $A\in(X,Y)$ if and only if $A_n\in X^\beta$ for all n and $Ax\in Y$ for all $x\in X$.

An infinite matrix $T=(t_{nk})_{n,k=0}^\infty$ is said to be a triangle, if $t_{nk}=0$ for $k>n$ and $t_{nn}\ne 0$ $(n=0,1,\dots)$.

A Fréchet sequence space, that is, a complete linear (locally convex) metric space (X,d_X) is said to be an *FKspace* if its metric d_X is stronger than the metric $d|_X$ of ω on X. Thus an *FK* space is a Fréchet sequence space with continuous *coordinates* $P_n:X\to\mathbb{C}$ defined by $P_n(x)=x_n$ $(n=0,1,\dots)$ for all $x=(x_k)_{k=0}^\infty\in X$. A *BK space* is an FK space which is a Banach space. An *FK* or *BK* space $X\supset\phi$ is said to have *AK* if $x^{[m]}=\sum_{k=0}^m x_ke^{(k)}\to x$ $(m\to\infty)$ for every sequence $x=(x_k)_{k=0}^\infty\in X$; $x^{[m]}$ is called the *m–section* of the sequence $x=(x_k)_{k=0}^\infty$.

The following results are known.

Proposition 2.1. *(a) Let (X,d_X) be a Fréchet space, (Y,d_Y) be an FK space, $f:X\to Y$ be a linear map and $d|_Y$ denote the restriction of d on Y. Then $f:(X,d_X)\to(Y,d|_Y)$ is continuous if and only if $f:(X,d_X)\to(Y,d_Y)$ is continuous* ([66, Theorem 4.2.2] or [54, Theorem 1.14])*; in particular, f is continuous if and only if each map $P_n\circ f:X\to\mathbb{C}$ $(n\in\mathbb{N})$ is continuous.*
(b) Let X and Y be FK spaces and $X\subset Y$. Then the metric d_X on X is stronger than the metric $d_Y|_X$ of Y on X. The metrics are equivalent if and only if X is a closed subspace of Y. In particular, the metric of an FK space is unique, this means there is at most one way to make a linear subspace of ω into an FK space ([66, Theorem 4.2.4] or [54, Theorem 1.21]).
(c) Any matrix map between FK spaces is continuous ([66, Theorem 4.2.8] or [54, Theorem 1.17]).

The following useful result holds for bounded linear operators between BK spaces; the first part is a rephrasing of Proposition 2.1 (c).

Proposition 2.2. *Let X and Y be BK spaces.*
(a) Then we have $(X, Y) \subset \mathcal{B}(X, Y)$, that is, every $A \in (X, Y)$ defines a linear operator $L_A \in \mathcal{B}(X, Y)$, where $L_A(x) = Ax$ $(x \in X)$.
(b) If X has AK then we have $\mathcal{B}(X, Y) \subset (X, Y)$, that is, every $L \in \mathcal{B}(X, Y)$ is given by a matrix $A \in (X, Y)$ such that $Ax = L(x)$ for all $x \in X$ ([34, Theorem 1.9]).

We list a few useful known facts.

Example 2.3. *(a) The space ω is an FK space with its natural metric d; ℓ_∞, c, c_0, ℓ_p $(1 \le p < \infty)$, bs and cs are BK spaces with their natural norms; c is a closed subspace of ℓ_∞, c_0 is a closed subspace of c, and cs is a closed subspace of bs; if $1 \le p < p' \le \infty$, then the BK norm on ℓ_p is strictly stronger than that of $\ell_{p'}$ on ℓ_p.*
(b) The spaces ω, c_0, ℓ_p $(1 \le p < \infty)$ and cs have AK; every sequence $x = (x_k)_{k=0}^\infty \in c$ has a unique representation

$$x = \xi e + \sum_{k=0}^{\infty}(x_k - \xi)e^{(k)}, \text{ where } \xi = \lim_{k\to\infty} x_k;$$

the spaces ℓ_∞ and bs have no Schauder basis.
(c) The set ϕ has no Féchet topology ([66, 4.0.5]).

We refer the interested reader to the monographs [8, 12, 22, 60, 68] for the theory of summability methods, to [8, 14, 35, 62, 66, 68] and the survey article [64] for the theory of sequence spaces and matrix transformations, and [54, 65, 66] for the theory of FK and BK spaces and its applications. Many more references can be found in the mentioned monographs and survey article.

3. The Spaces of $[C_1]^p$ Summable and Bounded Sequences

Here we study the sets of sequences that are strongly summable $[C_1]^p$ or strongly bounded $[C_1]^p$ for $p \ge 1$.

Throughout the remainder of the paper, let $1 \le p < \infty$ unless explicitly stated otherwise, and q denote the conjugate number of p, that is, $q = \infty$ for $p = 1$ and $q = p/(p-1)$ for $1 < p < \infty$.

Following Maddox [40], we use the notation w^p for the set of all complex sequences that are strongly summable $[C_1]^p$. We also define the sets

$$w_0^p = \left\{x \in \omega : \lim_{n\to\infty} \sigma_n(|x|^p) = 0\right\} \text{ and } w_\infty^p = \left\{x \in \omega : \sup_n \sigma_n(|x|^p) = 0\right\}$$

of all complex sequences that are strongly summable $[C_1]^p$ to zero and strongly bounded $[C_1]^p$; if $p = 1$, then we omit the index p, that is, we write $w_0 = w_0^1$, $w = w^1$ and $w_\infty = w_\infty^1$, for short. We observe that $w^p = w_0^p \oplus e = \{x \in \omega : x - \xi e \in w_0^p \text{ for some } \xi \in \mathbb{C}\}$. If $x \in w^p$, then the unique complex number ξ for which $x - \xi e \in w_0^p$ is referred to as the *strong C_1 limit of index p* or the *w^p limit of the sequence x*. Furthermore, we obviously have $w_0^p \subset w^p \subset w_\infty^p$ and an application of Hölder's inequality yields $w_0^p \supset w_0^{p'}$, $w^p \supset w^{p'}$ and $w_\infty^p \supset w_\infty^{p'}$ for $1 \le p \le p'$.

We denote the dyadic blocks by $I_0 = \{0, 1\}$ and $I_\nu = \{k \in \mathbb{N} : 2^\nu \le k \le 2^{\nu+1} - 1\}$ for $\nu = 1, 2, \dots$, and write $\sum_\nu = \sum_{k\in I_\nu}$ and $\max_\nu = \max_{k\in I_\nu}$ for $\nu = 0, 1, \dots$.

The basic topological properties are stated in the following result; they are analogous to the corresponding ones for c_0, c and ℓ_∞ stated in Example 2.3 (a) and (b).

Theorem 3.1. ([40], [54, Proposition 3.44]) *Let $1 \leq p < \infty$. Then w_0^p, w^p and w_∞^p are BK spaces with respect to the (equivalent) block and sectional norms $\|\cdot\|_b$ and $\|\cdot\|_s$ defined by*

$$\|x\|_b = \sup_{\nu \in \mathbb{N}_0} \left(\frac{1}{2^\nu} \sum\nolimits_\nu |x_k|^p \right)^{1/p} \text{ and } \|x\|_s = \sup_{n \in \mathbb{N}_0} \left(\frac{1}{n+1} \sum_{k=0}^{n} |x_k|^p \right)^{1/p};$$

w_0^p is a closed subspace of w^p and w^p is a closed subspace of w_∞^p; w_0^p has AK and every sequence $x = (x_k)_{k=0}^\infty \in w^p$ has a unique representation

$$x = \xi e + \sum_{k=0}^{\infty} (x_k - \xi) e^{(k)}, \text{ where } \xi \text{ is the } w^p \text{ limit of } x;$$

finally, w_∞^p has no Schauder basis.

From now on, we will always assume that w_0^p, w^p and w_∞^p are endowed with the block norm $\|\cdot\|_{w_\infty^p} = \|\cdot\|_b$ unless explicitly stated otherwise.

Next we give the β–duals of w_0^p, w^p and w_∞^p, and the continuous duals of w_0^p and w^p. We put

$$\mathcal{M}_p = \left\{ a \in \omega : \|a\|_{\mathcal{M}_p} < \infty \right\}, \text{ where } \|a\|_{\mathcal{M}_p} = \begin{cases} \sum_{\nu=0}^{\infty} 2^\nu \cdot \max_\nu |a_k| & (p = 1) \\ \sum_{\nu=0}^{\infty} 2^{\nu/p} \left(\sum_\nu |a_k|^q \right)^{1/q} & (1 < p < \infty). \end{cases}$$

Furthermore, given any $a \in \omega$ and any normed sequence space X, we write

$$\|a\|_X^* = \sup \left\{ \left| \sum_{k=0}^{\infty} a_k x_k \right| : x \in S_X \right\}$$

provided the expression on the right hand side is defined and finite which is the case whenever X is a *BK* space and $a \in X^\beta$ ([66, Theorem 7.2.9]).

Theorem 3.2. ([40], [54, Proposition 3.47] for (a)–(d)) *We have*

(a) $(w_0^p)^\beta = (w^p)^\beta = (w_\infty^p)^\beta = \mathcal{M}_p$; *(b)* $(w_0^p)^*$ *is norm isomorphic to* $(\mathcal{M}_p, \|\cdot\|_{\mathcal{M}_p})$;

(c) $f \in (w^p)^*$ *if and only if there exist $b \in \mathbb{C}$ and a sequence $a = (a_k)_{k=0}^\infty \in \mathcal{M}_p$ such that*

$$f(x) = \xi b + \sum_{k=0}^{\infty} a_k x_k \text{ for all } x \in w^p, \text{ where } \xi \text{ is the } w^p \text{ limit of } x,\ a = (f(e^{(n)}))_{n=1}^\infty \text{ and } b = f(e) - \sum_{n=1}^{\infty} a_n;$$

moreover $\|f\| = |b| + \|a\|_{\mathcal{M}_p}$ for all $f \in (w^p)^*$;

(d) $\|a\|_{w_\infty^p}^* = \|a\|_{\mathcal{M}_p}$ *for all* $a \in (w_\infty^p)^\beta$; *(e)* $(\mathcal{M}_p, \|\cdot\|_{\mathcal{M}_p})$ *is a BK space with AK* ([56, Theorem 5.7]);

(f) w_∞^p *is β–perfect, that is, $(w_\infty^p)^{\beta\beta} = ((w_\infty^p)^\beta)^\beta = (\mathcal{M}_p)^\beta = w_\infty^p$, and the continuous dual $(\mathcal{M}_p)^*$ of $\mathcal{M}_p$ is norm isomorphic to w_∞^p* ([56, Theorem 5.8]).

First we give the characterizations of the classes (X, Y) for $X = w_0^p, w^p, w_\infty^p$ and $Y = \ell_\infty, c_0, c, \ell_1, w_0, w, w_\infty$ in the known cases. We assume that the spaces w_0, w and w_∞ are endowed with the sectional norm $\|\cdot\|_s$. Let $A = (a_{nk})_{n,k=0}^\infty$ be an infinite matrix, $m \in \mathbb{N}_0$ and N_m be a subset of the set $\{0, 1, \dots, m\}$. Then we write $\max_{N_m}$ for the maximum taken over all subsets $N_m \subset \{0, 1, \dots, m\}$ and S^{N_m} for the matrix with the rows

$$S_m^{N_m}(A) = \frac{1}{m+1} \sum_{n \in N_m} A_n.$$

Theorem 3.3. ([7, Theorem 2.4] for **1.–7.**; **8.** follows from [33, Theorem 4.4 **3.**] and Theorem 3.2 (a) and (d); [2, Proposition 1] for **9. –13.**) *The necessary and sufficient conditions for of $A \in (X,Y)$ when $X = w_0^p, w^p, w_\infty^p$ and $Y = \ell_\infty, c_0, c, \ell_1$ can be read from the following table*

From / *To*	w_∞^p	w_0^p	w^p
ℓ_∞	**1.**	**1.**	**1.**
c_0	**2.**	**3.**	**4.**
c	**5.**	**6.**	**7.**
ℓ_1	**8.**	**8.**	**8.**
w_∞	**9.**	**9.**	**9.**
w_0	**unknown**	**10.**	**11.**
w	**unknown**	**12.**	**13.**

where

1. [3.3](1.1) $\|A\|_{(w_\infty^p,\infty)} = \sup_n \|A_n\|_{\mathcal{M}_p} < \infty$ **2.** [3.3](2.1) $\lim_{n\to\infty} \|A_n\|_{\mathcal{M}_p} = 0$

3. [3.3](1.1) *and* [3.3](3.1), *where* [3.3](3.1) $\lim_{n\to\infty} a_{nk} = 0$ *for all* k

4. [3.3](1.1), [3.3](3.1) *and* [3.3](4.1), *where* [3.3](4.1) $\lim_{n\to\infty} \sum_{k=0}^\infty a_{nk} = 0$

5. [3.3](5.1), [3.3](5.2) *and* [3.3](5.3), *where*

[3.3](5.1) $\alpha_k = \lim_{n\to\infty} a_{nk}$ *exists for all* k

[3.3](5.2) $(\alpha_k)_{k=0}^\infty, A_n \in \mathcal{M}_p$ *for all* n

[3.3](5.3) $\lim_{n\to\infty} \|A_n - (\alpha_k)_{k=0}^\infty\|_{\mathcal{M}_p} = 0$

6. [3.3](1.1) *and* [3.3](5.1)

7. [3.3](1.1), [3.3](5.1) *and* [3.3](7.1), *where* [3.3](7.1) $\alpha = \lim_{n\to\infty} \sum_{k=0}^\infty a_{nk}$ *exists*

8. [3.3](8.1) $\|A\|_{(w_\infty^p,1)} = \sup_{\substack{N \subset \mathbb{N}_0 \\ N \text{ finite}}} \left\| \sum_{n\in N} A_n \right\|_{\mathcal{M}_p} < \infty.$

9. [3.3](9.1) $\|A\|_{(w_\infty^p,w_\infty)} = \sup_m \left(\frac{1}{m+1} \max_{N_m} \left\| \sum_{n\in N_m} A_n \right\|_{\mathcal{M}_p} \right) < \infty$

10. [3.3](9.1) *and* [3.3](10.1), *where* [3.3](10.1) $\lim_{m\to\infty} \frac{1}{m+1} \sum_{n=0}^m |a_{nk}| = 0$ *for each* k

11. [3.3](9.1), [3.3](10.1) *and* [3.3](11.1), *where* [3.3](11.1) $\lim_{m\to\infty} \frac{1}{m+1} \sum_{n=0}^m \left| \sum_{k=0}^\infty a_{nk} \right| = 0$

12. [3.3](9.1) *and* [3.3](12.1), *where*

[3.3](12.1) *for each* k *there is* $\alpha_k \in \mathbb{C}$ *such that* $\lim_{m\to\infty} \frac{1}{m+1} \sum_{n=0}^m |a_{nk} - \alpha_k| = 0$

13. [3.3](9.1), [3.3](12.1) *and* [3.3](13.1), *where*

[3.3](13.1) $\lim_{m\to\infty} \frac{1}{m+1} \sum_{n=0}^m \left| \sum_{k=0}^\infty a_{nk} - \alpha \right| = 0$ *for some* $\alpha \in \mathbb{C}$.

Furthermore, if $A \in (X, \ell_1)$, then $\|A\|_{(w_\infty^p,1)} \le \|L_A\| \le 4 \cdot \|A\|_{(w_\infty^p,1)}$, and if $A \in (X, Y)$ in **1.–7.**, *then $\|L_A\| = \|A\|_{(w_\infty^p,\infty)}$. Also, if $A \in (X, Y)$ in* **9.–13.**, *then $\|A\|_{(w_\infty^p,w_\infty)} \le \|L_A\| \le 4 \cdot \|A\|_{(w_\infty^p,w_\infty)}$.*
We remark that the conditions for $A \in (w_\infty^p, c_0)$ and $A \in (w_\infty^p, c)$ can be replaced by the conditions [3.3](2.1′) *and* [3.3](3.1) *in* **2.** *and* [3.3](2.1′) *and* [3.3](5.1) *in* **5.**, *where* [3.3](2.1′) $\|A_n\|_{\mathcal{M}_p}$ *converges uniformly in n.*

Now we state the *dual result* of Theorem 3.2 **1–8**. Again, we assume that the spaces w_0, w and w_∞ are endowed with the sectional norm $\|\cdot\|_s$.

Theorem 3.4. ([17, Theorem 2.5]) *We write*

$$\|A\|_{(p,w_\infty)} = \sup_m \left(\max_{N_m} \left\|S_m^{N_m}(A)\right\|_q\right) = \begin{cases} \sup_m \left(\max_{N_m} \left(\sup_k \left|\frac{1}{m+1} \sum_{n\in N_m} a_{nk}\right|\right)\right) & (p = 1) \\ \sup_m \left(\max_{N_m} \left(\sum_{k=0}^{\infty} \left|\frac{1}{m+1} \sum_{n\in N_m} a_{nk}\right|^q\right)^{1/q}\right) & (1 < p \le \infty). \end{cases}$$

The necessary and sufficient conditions for $A \in (X, Y)$ when $X \in \{\ell_p, \ell_\infty, c_0, c\}$ and $Y \in \{w_0, w, w_\infty\}$ can be read from the following table

From / To	ℓ_p ($1 \le p < \infty$)	ℓ_∞	c_0	c
w_∞	**1.**	**2.**	**2.**	**2.**
w_0	**3.**	**4.**	**5.**	**6.**
w	**7.**	**8.**	**9.**	**10.**

where

1. [3.4](1.1) $\|A\|_{(p,w_\infty)} < \infty$ **2.** [3.4](2.1) $\|A\|_{(\infty,w_\infty)} < \infty$ **3.** [3.4](2.1) *and* [3.3](10.1)

4. [3.4](4.1) $\lim_{m\to\infty} \left(\max_{N_m} \left(\frac{1}{m+1} \sum_{k=0}^{\infty} \left|\sum_{n\in N_m} a_{nk}\right|\right)\right) = 0$ **5.** [3.4](2.1) *and* [3.3](10.1)

6. [3.4](2.1), [3.3](10.1) *and* [3.3](11.1) **7.** [3.4](1.1) *and* [3.3](12.1)

8. [3.3](12.1), [3.4](8.2) *and* [3.4](8.3), *where*

[3.4](8.2) $(\alpha_k)_{k=0}^\infty \in \ell_1$ *and* $A_n \in \ell_1$ *for all n*

[3.4](8.3) $\lim_{m\to\infty} \left(\max_{N_m} \left(\frac{1}{m+1} \sum_{k=0}^{\infty} \left|\sum_{n\in N_m} a_{nk} - \alpha_k\right|\right)\right) = 0$

9. [3.4](2.1) *and* [3.3](12.1) **10.** [3.4](2.1), [3.3](12.1) *and* [3.3](13.1).

If $A \in (X, Y)$ in the cases above, then we have ([53, Corollary 1, (2.8)]) $\|A\|_{(p,w_\infty)} \le \|L_A\| \le 4 \cdot \|A\|_{(p,w_\infty)}$.

Theorem 3.5. ([18, Theorem 2.5]) *Let $1 < p < \infty$. We write*

$$\|A\|_{(\mathcal{M}_p,\ell_1)} = \sup_{\substack{N\subset\mathbb{N}_0 \\ N \text{ finite}}} \left\|\sum_{n\in N} (a_{kn})_{k=0}^\infty\right\|_{w_\infty^p} = \sup_{\substack{N\subset\mathbb{N}_0 \\ N \text{ finite}}} \left[\sup_\nu \left(\frac{1}{2^\nu} \sum\nolimits_\nu \left|\sum_{n\in N} a_{kn}\right|^p\right)^{1/p}\right]$$

and

$$\|A\|_{(\mathcal{M}_p,\ell_\infty)} = \sup_n \left\|(a_{kn})_{k=0}^\infty\right\|_{w_\infty^p} = \sup_n \left[\sup_\nu \left(\frac{1}{2^\nu} \sum\nolimits_\nu |a_{kn}|^p\right)^{1/p}\right].$$

Then the necessary and sufficient conditions for $A \in (X, Y)$ can be read from the following table:

To \ From	ℓ_∞	c_0	c	ℓ_1
w_∞^p	**1.**	**1.**	**1.**	**2.**
w_0^p	**unknown**	**3.**	**4.**	**5.**
w^p	**unknown**	**6.**	**7.**	**8.**

where

1. [3.5](1.1) $\|A\|_{(\mathcal{M}_p,\ell_1)} < \infty$ **2.** [3.5](2.1) $\|A\|_{(\mathcal{M}_p,\ell_\infty)} < \infty$

3. [3.5](2.1) *and* [3.5](3.1), *where* [3.5](3.1) $\lim_{m\to\infty} \frac{1}{m+1} \sum_{n=0}^{m} |a_{nk}|^p = 0$ *for each* k

4. [3.5](1.1), [3.5](3.1) *and* [3.5](4.1) *where* [3.5](4.1) $\lim_{m\to\infty} \frac{1}{m+1} \sum_{n=0}^{m} \left|\sum_{k=0}^{\infty} a_{nk}\right|^p = 0$

5. [3.5](2.1) *and* [3.5](3.1)

6. [3.5](1.1) *and* [3.5](6.1) *where* [3.5](6.1) $\left\{\begin{array}{c} \text{for each } k \text{ there is } \alpha_k \in \mathbb{C} \text{ such that} \\ \lim\limits_{m\to\infty} \frac{1}{m+1} \sum\limits_{n=0}^{m} |a_{nk} - \alpha_k|^p = 0 \end{array}\right\}$

7. [3.5](1.1) [3.5](6.1) *and* [3.5](7.1) *where*

$$\text{[3.5](7.1) there exists } \alpha_k \in \mathbb{C} \text{ such that } \lim_{m\to\infty} \left(\frac{1}{m+1} \sum_{n=0}^{m} \left| \sum_{k=0}^{\infty} a_{nk} - \alpha_k \right|^p \right) = 0$$

8. [3.5](2.1) *and* [3.5](6.1).

Furthermore, if $A \in (\ell_1, Y)$ *then* $\|L_A\| = \|A\|_{(\mathcal{M}_p,\ell_1)}$ *and if* $A \in (X, Y)$ *in the remaining known cases above then* $\|A\|_{(\mathcal{M}_p,\ell_\infty)} \le \|L_A\| \le 4 \cdot \|A\|_{(\mathcal{M}_p,\ell_\infty)}$.

The reader is also referred to further results on matrix transformations on spaces related to summability $[C_1]^p$ in [11, 38, 57], for a study of the Banach algebras (w, w) and (w_∞, w_∞) and applications to sequence spaces equations and the solvability of infinite system of linear equations in [42, 43, 48]. A great number of additional references can be found in those papers.

4. Spaces of Λ–Strongly Convergent and Bounded Sequences

In this section, we study the spaces of Λ–*strong null sequences* $c_0^p(\Lambda)$, Λ–*strongly convergent* and *bounded sequences* $c(\Lambda)$ and $c_\infty^p(\Lambda)$, of index p for *exponentially bounded sequences* Λ. It will turn out that the case of $p = 1$ is essentially different from that of $1 < p < \infty$. Many of the results of this section for $p = 1$ can be found in [49].

The spaces $c_0^p(\Lambda)$, $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$ were defined and studied for exponentially bounded sequences Λ in [32, 45, 46, 49, 52, 54] for $p = 1$, and in [32, 47, 58] for $1 < p < \infty$. We remark that the extension of the spaces $\tilde{c}_0^p(\mu)$, $\tilde{c}^p(\mu)$ and $\tilde{c}_0^p(\mu)$ to the case of $0 < p < 1$ was studied in [31]. The concept of *exponentially bounded sequences* introduced in [45] plays an important role. A nondecreasing sequence $\Lambda = (\lambda_n)_{n=0}^\infty$ of positive reals is said to be *exponentially bounded* [45] if there exists an integer $m \ge 2$ such that, for each $\nu \in \mathbb{N}_0$, there is at least one λ_n in the interval $[m^\nu, m^{\nu+1})$. The following result is a useful characterization of exponentially bounded sequences.

Lemma 4.1. ([45, Lemma 1]) *A nondecreasing sequence* Λ *of positive reals is exponentially bounded if and only if the following condition holds:*

(I) *There are real numbers s and t with* $0 < s \le t < 1$ *such that for some subsequence* $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$

$$s \le \frac{\lambda_{n(\nu)}}{\lambda_{n(\nu+1)}} \le t \text{ for all } \nu = 0, 1, \ldots .$$

If Λ is an exponentially bounded sequence, then we can always determine a subsequence $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$ which satisfies the condition in (I); such a subsequence will be referred to as an *associated subsequence*.

Throughout, let μ be a nondecreasing sequence of real numbers tending to infinity, Λ be an exponentially bounded sequence and $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$ be an associated subsequence. If $(n(\nu))_{\nu=0}^{\infty}$ with $n(0) = 0$ is a strictly increasing sequence of nonnegative integers and $I(n(\nu))$ denotes the set of all integers k with $n(\nu) \le k \le n(\nu)+1)-1$, then $\sum_{I(n(\nu))}$ and $\max_{I(n(\nu))}$ denote the sum and maximum over all $k \in I(n(\nu))$. We define the sets

$$\tilde{c}_0^p(\mu) = \left\{ x \in \omega : \lim_{n\to\infty} \left(\frac{1}{\mu_n^p} \sum_{k=0}^{n} \left| \Delta_k(\mu \cdot x) \right|^p \right) = 0 \right\},\ \tilde{c}_\infty^p(\mu) = \left\{ x \in \omega : \sup_n \left(\frac{1}{\mu_n^p} \sum_{k=0}^{n} \left| \Delta_k(\mu \cdot x) \right|^p \right) < \infty \right\},$$

$$c_0^p(\Lambda) = \left\{ x \in \omega : \lim_{\nu\to\infty} \left(\frac{1}{\lambda_{n(\nu+1)}^p} \textstyle\sum_{I(n(\nu))} |\Delta_k(\Lambda \cdot x)|^p \right) = 0 \right\},$$

$$c_\infty^p(\Lambda) = \left\{ x \in \omega : \sup_{\nu} \left(\frac{1}{\lambda_{n(\nu+1)}^p} \textstyle\sum_{I(n(\nu))} |\Delta_k(\Lambda \cdot x)|^p \right) < \infty \right\},$$

$\tilde{c}^p(\mu) = \{x \in \omega : x - \xi \cdot e \in \tilde{c}_0^p(\mu) \text{ for some } \xi \in \mathbb{C}\}$, and $c^p(\Lambda) = \{x \in \omega : x - \xi \cdot e \in c_0^p(\Lambda) \text{ for some } \xi \in \mathbb{C}\}$. If $p = 1$, then we omit the index p, that is, we write $\tilde{c}_0(\mu) = \tilde{c}_0^1(\mu)$ etc., for short.

Remark 4.2. *If* $p = 1$ *and* $\mu_{n+1} = n+1$, *then we obviously have* $\tilde{c}(\mu) = [c]_1$ ([39]).

If $x \in \tilde{c}^p(\mu)$ then the number $\xi \in \mathbb{C}$ with $x - \xi \cdot e \in c_0^p(\Lambda)$ is called Λ*–strong limit (of index p)* or $c^p(\Lambda)$ *limit of the sequence* x. The $c^p(\Lambda)$ limit of a sequence $x \in \tilde{c}^p(\mu)$ is unique ([45, Lemma 2]) for $p = 1$, and [47, Theorem 2]) for $1 < p < \infty$ if and only if $\overline{\lim_{n\to\infty}} \mu_{n+1}/\mu_n > 1$.

Our spaces have the following fundamental topological properties similar to those of c_0, c and ℓ_∞, and to those of w_0^p, w^p and w_∞^p in Theorem 3.1.

First we consider the case $p = 1$.

Theorem 4.3. ([45, Theorem 2] and [49, Theorem 2.2]) *Let* $\Lambda = (\lambda_n)_{n=0}^{\infty}$ *be an exponentially bounded sequence and* $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$ *be an associated subsequence.*
Then $c_0(\Lambda) = \tilde{c}_0(\Lambda)$, $c(\Lambda) = \tilde{c}(\Lambda)$ *and* $c_\infty(\Lambda) = \tilde{c}_\infty(\Lambda)$, *and the block and sectional norms* $\|\cdot\|_b$ *and* $\|\cdot\|_s$ *defined by*

$$\|x\|_b = \sup_{\nu} \frac{1}{\lambda_{n(\nu+1)}} \textstyle\sum_{I(n(\nu))} |\Delta_k(\Lambda \cdot x)| \text{ and } \|x\|_s = \sup_n \frac{1}{\lambda_n} \displaystyle\sum_{k=0}^{n} |\Delta_k(\Lambda \cdot x)|$$

are equivalent on $c_\infty(\Lambda)$, $c(\Lambda)$ *and* $c_\infty(\Lambda)$, *more precisely*

$$\|x\|_b \le \|x\|_s \le K(s,t) \cdot \|x\|_b, \text{ where } K(s,t) = (s(1-t))^{-1} \text{ with } s, t \in (0,1) \text{ from condition (I);} \tag{4}$$

Each of the spaces $c_0(\Lambda)$, $c(\Lambda)$ *and* $c_\infty(\Lambda)$ *is a BK space,* $c_0(\Lambda)$ *is a closed subspace of* $c(\Lambda)$, $c(\Lambda)$ *is a closed subspace of* $c_\infty(\Lambda)$; $c_0(\Lambda)$ *has AK and every sequence* $x = (x_k)_{k=1}^{\infty} \in c(\Lambda)$ *has a unique representation*

$$x = \xi \cdot e + \sum_{k=1}^{\infty} (x_k - \xi) e^{(k)} \text{ where } \xi \text{ is the } c(\Lambda) \text{ limit;}$$

finally $c_\infty(\Lambda)$ *has no Schauder basis.*

Now we consider the case $1 < p < \infty$.

Theorem 4.4. *Let $1 < p < \infty$, and $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be an exponentially bounded sequence and $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$ be an associated subsequence.*
Then $c_0^p(\Lambda) = \tilde{c}_0^p(\Lambda)$, $c^p(\Lambda) = \tilde{c}^p(\Lambda)$ and $c_\infty^p(\Lambda) = \tilde{c}_\infty^p(\Lambda)$, and the block and sectional norms $\|\cdot\|_{p,b}$ and $\|\cdot\|_{p,s}$ defined by

$$\|x\|_{p,b} = \sup_\nu \left(\frac{1}{\lambda_{n(\nu+1)}^p} \sum\nolimits_{I(n(\nu))} |\Delta_k(\Lambda\cdot x)|^p\right)^{1/p} \text{ and } \|x\|_{p,s} = \sup_n \left(\frac{1}{\lambda_n^p}\sum_{k=1}^n |\Delta_k(\Lambda\cdot x)|^p\right)^{1/p}$$

are equivalent on $c_\infty^p(\Lambda)$, $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$. Each of the spaces $c_0^p(\Lambda)$, $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$ is a BK space, $c_0^p(\Lambda)$ is a closed subspace of $c^p(\Lambda)$, $c^p(\Lambda)$ is a closed subspace of $c_\infty^p(\Lambda)$. Let $(c^{(k)})_{k=0}^{\infty}$ be the sequence with $c_j^{(k)} = 0$ for $j \le k-1$ and $c_j^{(k)} = 1/\lambda_j$ for $j \ge k$ $(k = 0,1,\dots)$. Then $(c^{(k)})_{k=0}^{\infty}$ is a Schauder basis for $c_0^p(\Lambda)$ ([58, Proposition 2.1 (c)]).
Let $c^{(-1)} = (c_k(-1))_{k=0}^{\infty}$ be the sequence with $c_k^{(-1)} = \sum_{j=0}^k 1/\lambda_j$ $(j = 0,1,\dots)$. Then $(c^{(k)})_{k=-1}^{\infty}$ is a Schauder basis for $c^p(\Lambda)$. ([34, Corollary 2.5])
The space $c_\infty^p(\Lambda)$ has no Schauder basis ([34, Remark 2.3] and Theorem 3.1).

Throughout, we will always assume that the spaces $c_0^p(\Lambda)$, $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$ have the block norm, unless explicitly stated otherwise.

Example 4.5. *([49, Example 2.3]) (a) If $\lambda_n = 2^{n-1}$ $(n = 0,1,\cdots)$, then we may choose the sequence $(\lambda_n)_{n=0}^{\infty}$ itself as an associated subsequence, $\lambda_n/\lambda_{n+1} = 1/2$ $(n = 0,1,\cdots)$ and we obtain, for instance,*

$$c_0(\Lambda) = \left\{x \in \omega : \frac{1}{\lambda_{n+1}}|\lambda_{n-1}x_{n-1} - \lambda_n x_n| \to 0 \ (n \to \infty)\right\}.$$

(b) Let $\alpha > 0$, $\lambda_0 = \lambda_1 = 1$ and $\lambda_{n+1} = n^\alpha$ for $n = 1,2,\cdots$. Then we may choose $(\lambda_{2^\nu})_{\nu=0}^{\infty}$ as an associated subsequence, $\lambda_{2^\nu}/\lambda_{2^{\nu+1}} = 2^{-\alpha}$ for $\nu = 0,1,\cdots$ and we obtain, for instance,

$$c_0(\Lambda) = \left\{x \in \omega : \frac{1}{(2^{\nu+1})^\alpha}\sum_{k=2^\nu}^{2^{\nu+1}-1} |\lambda_{k-1}x_{k-1} - \lambda_k x_k| \to 0 \ (\nu \to \infty)\right\}.$$

If $\alpha = 1$ then the sets $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$ reduce to the sets $[c_0]_1$, $[c]_1$ and $[c_\infty]_1$ introduced and studied by Hyslop, Kuttner and Thorpe [28, 39].
(c) Let $\alpha > 0$, $\lambda_0 = \lambda_1 = 1$ and $\lambda_{n+1} = (\log_2 n)^\alpha$ for $n \ge 1$. Then we may choose $(\lambda_{2^{(2^\nu)}})_{\nu=0}^{\infty}$ as an associated subsequence.

Now we give the duals of the sets $c_0^p(\Lambda)$, $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$. Again, we consider the the case $p = 1$ first. We put

$$C(\Lambda) = \left\{a \in \omega : \|a\|_{C(\Lambda)} < \infty\right\}, \text{ where } \|a\|_{C(\Lambda)} = \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max\nolimits_{I(n(\nu))}\left|\sum_{j=k}^{\infty}\frac{a_j}{\lambda_j}\right|.$$

Theorem 4.6. ([46, Theorem 2] for (a)–(d) and [46, Theorem 3] for (e) and (f)) *We have*

(a) $(c_0(\Lambda))^\beta = (c(\Lambda))^\beta = (c_\infty(\Lambda))^\beta = C(\Lambda)$; *(b) $(c_0(\Lambda))^*$ is norm isomorphic to $(C(\Lambda), \|\cdot\|_{C(\Lambda)})$;*

(c) $f \in (c(\Lambda))^*$ if and only if there exist $b \in \mathbb{C}$ and a sequence $a \in C(\Lambda)$ such that

$$f(x) = b\xi + \sum_{n=0}^{\infty} a_n x_n \text{ for all } x \in c(\Lambda), \text{ where } \xi \text{ is the } c(\Lambda) \text{ limit of } x,\ a = (f(e^{(n)}))_{n=0}^{\infty} \text{ and } b = f(e) - \sum_{n=0}^{\infty} a_n;$$

furthermore, we have $|b| + \|a\|_{C(\Lambda)} \le \|f\| \le K(s,t)\cdot\left(|b| + \|a\|_{C(\Lambda)}\right)$ for all $f \in (c(\Lambda))^*$.

(d) $\|a\|^*_{c_\infty(\Lambda)} = \|a\|_{C(\Lambda)}$ *for all* $a \in (c_\infty(\Lambda))^\beta$; *(e)* $(C(\Lambda), \|\cdot\|_{C(\Lambda)})$ *is a BK space with AK;*

(f) $c_\infty(\Lambda)$ *is* β*–perfect, that is,* $(c_\infty(\Lambda))^{\beta\beta} = ((c_\infty(\Lambda))^\beta)^\beta = (C(\Lambda))^\beta = c_\infty(\Lambda)$*, and the continuous dual* $(C(\Lambda))^*$ *of* $C(\Lambda)$ *is norm isomorphic to* $c_\infty(\Lambda)$*.*

Now we consider the case $1 < p < \infty$. We put

$$C_p(\Lambda) = \left\{a \in \omega : \|a\|_{C_p(\Lambda)} < \infty\right\}, \text{ where } \|a\|_{C_p(\Lambda)} = \sum_{\nu=0}^\infty \lambda_{k(\nu+1)} \left(\sum_{I(n(\nu))} \left|\sum_{j=k}^\infty \frac{a_j}{\lambda_j}\right|^q\right)^{1/q}.$$

For each $n \in \mathbb{N}_0$, let $\nu(n)$ be the uniquely defined integer such that $n \in I_{\nu(n)}$. Then we define the sequence $d = (d_n)_{n=0}^\infty$ by

$$d_n = \sum_{\nu=0}^{\nu(n)-1} \lambda_{k(\nu+1)} \left(k(\nu+1) - k(\nu)\right)^{1/q} + \lambda_{k(n(\nu)+1)}(n + 1 - k(\nu(n)))^{1/q} \text{ for } n = 0, 1, \ldots.$$

Theorem 4.7. ([47, Theorem 3]) *Let* $1 < p < \infty$*. Then we have*

(a) $(c_0^p(\Lambda))^\beta = (c^p(\Lambda))^\beta$ *and* $a \in (c_0^p(\Lambda))^\beta$ *if and only if*

$$a \in C_p(\Lambda) \text{ and } \sup_n \left(d_n \left|\sum_{k=n}^\infty \frac{a_k}{\lambda_k}\right|\right) < \infty; \tag{5}$$

(b) $a \in (c_\infty^p(\Lambda))^\beta$ *if and only if the condition the first condition in (5) is satisfied and*

$$\lim_{n\to\infty} \left(d_n \left|\sum_{k=n}^\infty \frac{a_k}{\lambda_k}\right|\right) = 0.$$

(c) Let $X = c_0^p(\Lambda)$, $X = c^p(\Lambda)$ *or* $X = c_\infty^p(\Lambda)$*. If* $a \in X^\beta$*, then*

$$\sum_{k=0}^\infty a_k x_k = \sum_{k=0}^\infty \Delta_k(\Lambda x) \sum_{j=k}^\infty \frac{a_j}{\lambda_j} \text{ for all } x \in X \text{ and } \|a\|^*_X = \|a\|_{C_p(\Lambda)} \text{ for all } a \in X^\beta.$$

Now we study matrix transformations. First we give the complete list of characterizations of the classes (X, Y) for $X = c_0(\Lambda), c(\Lambda), c_\infty(\Lambda)$ and $Y = c_0, c, \ell_\infty, \ell_1$.

Theorem 4.8. ([49, Theorem 4.1])
The necessary and sufficient conditions for $A \in (X, Y)$ *when* $X \in \{c_0(\Lambda), c(\Lambda), c_\infty(\Lambda)\}$ *and* $Y \in \{\ell_\infty, c, c_0, \ell_1\}$ *can be read from the following table*

From / *To*	$c_\infty(\Lambda)$	$c_0(\Lambda)$	$c(\Lambda)$
ℓ_∞	**1.**	**1.**	**1.**
c_0	**2.**	**3.**	**4.**
c	**5.**	**6.**	**7.**
ℓ_1	**8.**	**8.**	**8.**

where

1. [4.8](1.1) $\|A\|_{(c_\infty(\Lambda),\infty)} = \sup_n \|A_n\|_{C(\Lambda)} < \infty$ **2.** [4.8](2.1) $\lim_{n\to\infty} \|A_n\|_{C(\Lambda)} = 0$

3. [4.8](1.1) *and* [3.3](3.1) **4.** [4.8](1.1), [3.3](3.1) *and* [3.3](4.1)

5. [3.3](5.1), [4.8](5.2) *and* [4.8](5.3), *where*

[4.8](5.2) $(\alpha_k)_{k=1}^\infty, A_n \in C(\Lambda)$ *for all n*

[4.8](5.3) $\lim_{n\to\infty} \|A_n - (\alpha_k)_{k=1}^\infty\|_{C(\Lambda)} = 0$

6. [4.8](1.1) *and* [3.3](5.1) **7.** [4.8](1.1), [3.3](5.1) *and* [3.3](7.1)

8. [4.8](8.1), *where* [4.8](8.1) $\|A\|_{(c_\infty(\Lambda),1)} = \sup_{\substack{N\subset\mathbb{N}_0 \\ N \text{ finite}}} \left\|\sum_{n\in N} A_n\right\|_{C(\Lambda)} < \infty.$

Furthermore, if $A \in (X, \ell_1)$, *then* $\|A\|_{(c_\infty(\Lambda),1)} \le \|L_A\| \le 4\cdot\|A\|_{(c_\infty(\Lambda),1)}$, *and if* $A \in (X,Y)$ *in the remaining cases above, then* $\|L_A\| = \|A\|_{(c_\infty(\Lambda),\infty)}$.
We remark that the conditions for $A \in (c_\infty(\Lambda), c_0)$ *and* $A \in (c_\infty(\Lambda), c)$ *can be replaced by the conditions* [4.8](2.1′) *and* [3.3](3.1) *in* **2.** *and* [4.8](2.1′) *and* [3.3](5.1) *in* **5.**, *where* [4.8](2.1′) $\|A_n\|_{C(\Lambda)}$ *converges uniformly in n.*

Now we state the *dual result* of Theorem 4.8. We assume that the spaces $\tilde{c}_0(\mu)$, $\tilde{c}(\mu)$ and $\tilde{c}_\infty(\mu)$ are endowed with the sectional norm $\|\cdot\|_s$.

Theorem 4.9. *We write*

$$\|A\|_{(p,\tilde{c}_\infty(\mu))} = \begin{cases} \sup_m \left(\max_{N_m}\left(\sup_k \left|\frac{1}{\mu_m}\sum_{n\in N_m}(\mu_n a_{nk} - \mu_{n-1}a_{n-1,k})\right|\right)\right) & (p=1) \\ \sup_m \left(\max_{N_m}\left(\sum_{k=0}^\infty \left|\frac{1}{\mu_m}\sum_{n\in N_m}(\mu_n a_{nk} - \mu_{n-1}a_{n-1,k})\right|^q\right)^{1/q}\right) & (1 < p \le \infty) \end{cases}$$

and

$$\|A\|_{(c_\infty(\Lambda),\tilde{c}_\infty(\mu))} = \sup_m \left[\max_{N_m} \sum_{\nu=0}^\infty \lambda_{k(\nu+1)} \left(\max_\nu \left|\frac{1}{\mu_m}\sum_{n\in N_m}\left(\mu_n \sum_{j=k}^\infty \frac{a_{nj}}{\lambda_j} - \mu_{n-1}\sum_{j=k}^\infty \frac{a_{n-1,j}}{\lambda_j}\right)\right|\right)\right]$$

The necessary and sufficient conditions for $A \in (X,Y)$ *when* $X \in \{\ell_p, \ell_\infty, c_0, c, c_\infty(\Lambda), c_0(\Lambda), c(\Lambda)\}$ *and* $Y \in \{\tilde{c}_\infty(\mu), \tilde{c}_0(\mu), \tilde{c}(\mu)\}$ *can be read from the following table*

From / To	ℓ_p $(1 \le p < \infty)$	ℓ_∞	c_0	c	$c_\infty(\Lambda)$	$c_0(\Lambda)$	$c(\Lambda)$
$\tilde{c}_\infty(\mu)$	**1.**	**2.**	**2.**	**2.**	**9.**	**9.**	**9.**
$\tilde{c}_0(\mu)$	**3.**	**unknown**	**4.**	**5.**	**unkown**	**10.**	**11.**
$\tilde{c}(\mu)$	**6.**	**unknown**	**7.**	**8.**	**unknown**	**12.**	**13.**

where

1. [4.9](1.1) $\|A\|_{(p,\tilde{c}_\infty(\mu))} < \infty$ **2.** [4.9](2.1) $\|A\|_{(\infty,\tilde{c}_\infty(\mu))} < \infty$

3. [4.9](1.1) *and* [4.9](3.1), *where* [4.9](3.1) $\lim_{m\to\infty}\left(\frac{1}{\mu_m}\sum_{n=0}^m |\mu_n a_{nk} - \mu_{n-1}a_{n-1,k}|\right) = 0$ *for each k*

4. [4.9](2.1) *and* [4.9](3.1)

5. [4.9](2.1), [4.9](3.1) *and* [4.9](5.1), *where*

$$[4.9](5.1)\ \lim_{m\to\infty}\left(\frac{1}{\mu_n}\sum_{n=0}^{m}\left|\sum_{k=0}^{\infty}(\mu_n a_{nk}-\mu_{-1}a_{n-1,k})\right|\right)=0$$

6. [4.9](1.1) *and* [4.9](6.1) *where*

$$[4.9](6.1)\ \begin{cases} \text{for each } k \text{ there is } \alpha_k\in\mathbb{C} \text{ such that}\\ \lim_{m\to\infty}\left(\frac{1}{\mu_m}\sum_{n=0}^{m}|\mu_n(a_{nk}-\alpha_k)-\mu_{n-1}(a_{n-1,k}-\alpha_k)|\right)=0\end{cases}$$

7. [4.9](2.1) *and* [4.9](6.1)

8. [4.9](2.1), [4.9](6.1) *and* [4.9](8.1) *where*

$$[4.9](8.1)\ \begin{cases}\lim_{m\to\infty}\left(\frac{1}{\mu_m}\sum_{n=0}^{m}\left|\sum_{k=0}^{\infty}\mu_n(a_{nk}-\alpha)-\mu_{n-1}(a_{n-1,k}-\alpha)\right|\right)=0\\ \text{for some } \alpha\in\mathbb{C}\end{cases}$$

9. [4.9](9.1) $\|A\|_{(c_\infty(\Lambda),\tilde{c}_\infty(\mu))}<\infty$ **10.** [4.9](9.1) *and* [4.9](3.1)

11. [4.9](9.1), [4.9](3.1) *and* [4.9](5.1) **12.** [4.9](9.1) *and* [4.9](6.1)

13. [4.9](6.1), [4.9](6.1) *and* [4.9](8.1).

If $A\in(X,Y)$ in the cases above, then we have ([53, Corollary 1, (2.8)]) $\|A\|_{(p,w_\infty)}\le\|L_A\|\le 4\cdot\|A\|_{(p,w_\infty)}$.

Proof. Part **1.** is [52, Remark 1 (a)], **2.** follows by [53, Corollary 1 (b)] and the well–known fact that $\|\cdot\|_X^*=\|\cdot\|_1$ for $X\in\ell_\infty,c_0,c$. Since $\tilde{c}_0(\mu)$ and $\tilde{c}_\infty(\mu)$ are closed subspaces of $\tilde{c}(\mu)$ by Theorem 4.4, **3.** and **6.**, and **4.** and **7.** follow by [66, 8.3.6] from **1.** and **2.**, respectively, and finally **5.** and **8.** follow by [66, 8.3.7] from **4.** and **7.**. The characterizations of $(c_\infty(\Lambda),\tilde{c}_\infty(\mu))$ and $(c_0(\Lambda),\tilde{c}_\infty(\mu))$ are given in [54, Remark 3.50 (b)] and [32, Theorem 3.4 **10.**], and $(c_\infty(\Lambda),\tilde{c}_\infty(\mu))=(c_0(\Lambda),\tilde{c}_\infty(\mu))$. Since $c_0(\Lambda)\subset c(\Lambda)\subset c_\infty(\Lambda)$, we also have $(c(\Lambda),\tilde{c}_\infty(\mu))=(c_0(\Lambda),\tilde{c}_\infty(\mu))$. Hence we have established **9.** Furthermore, **10.** and **12.** are [32, Theorem 3.4 **11.** and **12.**], and finally [66, 8.3.7] and **10.** and **12.** yield **11.** and **13.**. □

Finally, we give the known characterizations of the classes (X,Y) when $X=c_\infty^p(\Lambda),c_0^p(\Lambda)$, $c^p(\Lambda)$ and $Y=\ell_\infty,\ell_1,c,c_0,\tilde{c}_\infty(\mu),\tilde{c}_0(\mu),\tilde{c}_\infty(\mu)$.

Theorem 4.10. *Let $1<p<\infty$. We write*

$$\|A\|_{(c_\infty^p(\Lambda),\tilde{c}_\infty(\mu))}=\sup_m\left[\max_{N_m}\sum_{\nu=0}^{\infty}\lambda_{k(\nu+1)}\left(\sum\nolimits_\nu\left|\frac{1}{\mu_m}\sum_{n\in N_m}\left(\mu_n\sum_{j=k}^{\infty}\frac{a_{nj}}{\lambda_j}-\mu_{n-1}\sum_{j=k}^{\infty}\frac{a_{n-1,j}}{\lambda_j}\right)\right|^q\right)^{1/q}\right]$$

Then the necessary and sufficient conditions for $A\in(X,Y)$ when $X=c_\infty^p(\Lambda),c_0^p(\Lambda),c^p(\Lambda)$ and $Y=\ell_\infty,\ell_1,c,c_0,\tilde{c}_\infty(\mu)$, $\tilde{c}_0(\mu),\tilde{c}_\infty(\mu)$. can be read from the following table

To \ *From*	$c_\infty^p(\Lambda)$	$c_0^p(\Lambda)$	$c^p(\Lambda)$
ℓ_∞	**1.**	**2.**	**3.**
c_0	**unknown**	**4.**	**5.**
c	**unknown**	**6.**	**7.**
ℓ_1	**8.**	**9.**	**10.**
$\tilde{c}_\infty(\mu)$	**11.**	**12.**	**13.**
$\tilde{c}_0(\mu)$	**unknown**	**14.**	**15.**
$\tilde{c}(\mu)$	**unknown**	**16.**	**17.**

where

1. [4.10](1.1) *and* [4.10](1.2), *where*

$$[4.10](1.1)\ \|A\|_{(c_\infty^p(\Lambda),\infty)} = \sup_n \|A_n\|_{C_p(\Lambda)} < \infty$$

$$[4.10](1.2)\ \lim_{m\to\infty} d_m \left|\sum_{k=m}^{\infty} \frac{a_{nk}}{\lambda_k}\right| = 0 \text{ for each } n$$

2. [4.10](1.1) *and* [4.10](2.1), *where* [4.10](2.1) $\sup_m d_m \left|\sum_{k=m}^{\infty} \frac{a_{nk}}{\lambda_k}\right| < \infty$ *for each n*

3. [4.10](1.1), [4.10](2.1) *and* [4.10](3.1), *where* [4.10](3.1) $\sup_n \left|\sum_{k=0}^{\infty} a_{nk}\right| < \infty$

4. [4.10](1.1), [4.10](2.1) *and* [4.10](4.1), *where* [4.10](4.1) $\lim_{n\to\infty} \sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j} = 0$ *for each k*

5. [4.10](1.1), [4.10](2.1), [4.10](4.1) *and* [4.8](5.1) *where* [4.10](5.1), $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 0$

6. [4.10](1.1), [4.10](2.1) *and* [4.10](6.1), *where* [4.10](6.1) $\lim_{n\to\infty} \sum_{j=k}^{\infty} \frac{a_{nj}}{\lambda_j} = \alpha_k$ *exists for each k*

7. [4.10](1.1), [4.10](2.1), [4.10](6.1) *and* [4.10](7.1), *where* [4.10](7.1) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \alpha$ *exists*

8. [4.10](8.1) *and* [4.10](1.2), *where* [4.10](8.1) $\sup_{\substack{N\subset\mathbb{N}_0 \\ N \text{ finite}}} \left\|\sum_{n\in N} A_n\right\|_{C_p(\Lambda)} < \infty$

9. [4.10](8.1) *and* [4.10](2.1)

10. [4.10](8.1), [4.10](2.1) *and* [4.10](10.1), *where* [4.10](10.1) $\sum_{n=0}^{\infty} \left|\sum_{k=0}^{\infty} a_{nk}\right| < \infty$

11. [4.10](11.1) *and* [4.10](1.2), *where* [4.10](11.1) $\|A\|_{(c_\infty^p(\Lambda),\bar{c}_\infty(\mu))} < \infty$

12. [4.10](11.1) *and* [4.10](2.1)

13. [4.10](11.1) [4.10](2.1) *and* [4.10](13.1), *where*

$$[4.10](13.1)\ \sup_m \frac{1}{\mu_m} \sum_{n=0}^{m} \left|\mu_n \sum_{k=0}^{\infty} a_{nk} - \mu_{n-1} \sum_{k=0}^{\infty} a_{n-1,k}\right| < \infty$$

14. [4.10](11.1) [4.10](2.1) *and* [4.10](14.1), *where*

$$[4.10](14.1)\ \lim_{m\to\infty} \frac{1}{\mu_m} \sum_{n=0}^{m} \left|\sum_{j=k}^{\infty} \left(\mu_n \frac{a_{nj}}{\lambda_j} - \mu_{n-1} \frac{a_{n-1,k}}{\lambda_j}\right)\right| = 0 \text{ for each } k$$

15. [4.10](11.1) [4.10](2.1) [4.10](14.1) *and* [4.10](15.1), *where*

$$[4.10](13.1)\ \lim_{m\to\infty} \frac{1}{\mu_m} \sum_{n=0}^{m} \left|\mu_n \sum_{k=0}^{\infty} a_{nk} - \mu_{n-1} \sum_{k=0}^{\infty} a_{n-1,k}\right| = 0$$

16. [4.10](11.1) [4.10](2.1) *and* [4.10](16.1), *where*

$$[4.10](16.1)\ \lim_{m\to\infty} \frac{1}{\mu_m} \sum_{n=0}^{m} \left|\sum_{j=k}^{\infty} \left(\mu_n \frac{a_{nj}}{\lambda_j} - \mu_{n-1} \frac{a_{n-1,k}}{\lambda_j}\right)\right| = \beta_k \text{ for each } k$$

17. [4.10](11.1) [4.10](2.1) [4.10](16.1) *and* [4.10](17.1), *where*

$$[4.10](17.1)\ \lim_{m\to\infty}\frac{1}{\mu_m}\sum_{n=0}^{m}\left|\mu_n\left(\sum_{k=0}^{\infty}a_{nk}-\beta\right)-\mu_{n-1}\left(\sum_{k=0}^{\infty}a_{n-1,k}-\beta\right)\right|=0 \text{ for some } \beta\in\mathbb{C}.$$

Furthermore, if $A\in(X,Y)$ in **1.** *–***7.**, *then* $\|L_A\|=\|A\|_{(c^p_\infty(\Lambda),\infty)}$, *and if $A\in(X,Y)$ in* **8.** *–***10.** *or in* **11.** *–***17.** , *then* $\|A\|_{(c^p_\infty(\Lambda),1)}\le\|L_A\|\le 4\cdot\|A\|_{(c^p_\infty(\Lambda),1)}$ *or* $\|A\|_{(c^p_\infty(\Lambda),\tilde{c}_\infty(\mu))}\le\|L_A\|\le 4\cdot\|A\|_{(c^p_\infty(\Lambda),\tilde{c}_\infty(\mu))}$.

Proof. Parts **2.**, **4.**, **6.**, **12.**, **14.** and **16.** are [32, Theorem 3.4 **1.–6.**], **1.** and **3.** are [47, Corollary 1 (a) and (b)], and **11.** and **13.** are [47, Corollary 2 (a) and (b)]. Furthermore, **8.** and **9.** are obtained as follows. By [44, Satz 1], we have for arbitrary BK spaces X

$$A\in(X,\ell_1) \text{ if and only if } \|A\|_{(X,1)}=\sup_{\substack{N\subset\mathbb{N}_0\\ N \text{ finite}}}\left\|\sum_{n\in N}A_n\right\|^*_X<\infty.$$

If $X=c^p_\infty(\Lambda)$ or $X=c^p_0(\Lambda)$ then we must use $a\in(c^p_\infty(\Lambda))^\beta$ or $a\in(c^p_0(\Lambda))^\beta$, and $\|\cdot\|^*_{c^p_\infty(\Lambda)}=\|\cdot\|_{C_p(\Lambda)}$ in (a) and (b) of Theorem 4.6. Finally, **5.**, **7.**, **10.**, **15.** and **17.** follow by [54, 8.3.7] from **4.**, **6.**, **9.**, **14.** and **16.**. □

Additional results on matrix transformations involving the spaces $c^p_0(\Lambda)$, $c^p(\Lambda)$ and $c^p_\infty(\Lambda)$ can be found in [15–18, 39]. Also there is a study on the Banach algebras $(c(\Lambda),c(\Lambda))$ and $(c_\infty(\Lambda),c_\infty(\Lambda))$ in [19], and applications to the solvability of infinite systems of linear equations and sequence spaces equations can be found in [41, 42].

5. Strong Cesàro Summability, Convergence and Boundedness

Throughout this section, let $\alpha\ge 0$ and $1\le p<\infty$. In the case of $\alpha=0$, the indices start from 1 with the convention that every term with an index less than 1 is equal to zero.

We study the sets of all sequences that are strongly summable C_α to zero, strongly summable and bounded C_α, strongly convergent to zero, and strongly convergent and bounded, with index p, defined by

$$[C_\alpha]^p_0=[C_1,C_{\alpha-1}]^p_0=\left\{x\in\omega:\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^{n}\left|\sigma^{\alpha-1}_k(x)\right|^p=0\right\},$$
$$[C_\alpha]^p=[C_1,C_{\alpha-1}]^p=\left\{x\in\omega:x-\xi e\in[C_\alpha]^p_0 \text{ for some } \xi\in\mathbb{C}\right\},$$
$$[C_\alpha]^p_\infty=[C_1,C_{\alpha-1}]^p_\infty=\left\{x\in\omega:\sup_n\frac{1}{n+1}\sum_{k=0}^{n}\left|\sigma^{\alpha-1}_k(x)\right|^p<\infty\right\},$$
$$[C_0]^p_0=\left\{x\in\omega:\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|kx_k-(k-1)x_{k-1}|^p=0\right\},$$
$$[C_0]^p=\left\{x\in\omega:x-\xi e\in[C_0]^p_0 \text{ for some } \xi\in\mathbb{C}\right\}$$

and

$$[C_0]^p_\infty=\left\{x\in\omega:\sup_n\frac{1}{n}\sum_{k=1}^{n}|kx_k-(k-1)x_{k-1}|^p<\infty\right\}$$

Remark 5.1. *Since*

$$\sigma^{\alpha-1}_k(e)=\frac{1}{A^{\alpha-1}_k}\sum_{j=0}^{k}A^{\alpha-2}_{k-j}=1 \text{ and } k-(k-1)=1 \text{ for all } k,$$

the definition of $[C_\alpha]^p$ *is the same as that in (1), and* $[C_0]^p = [H_0]^p$ *in (3).*
In the special case of $\alpha = 1$*, we obtain* $[C_1]_0^p = w_0^p$, $[C_1]^p = w^p$ *and* $[C_1]_\infty^p = w_\infty^p$.
If we write $C_\alpha = (c_{nk}^\alpha)_{n,k=0}^\infty$ *for* $\alpha > -1$ *and* $C_{-1} = (c_{nk}^{-1})_{n,k=0}^\infty$ *for the triangles with*

$$c_{nk}^\alpha = \frac{A_{n-k}^{\alpha-1}}{A_n^\alpha} \text{ for } 0 \le k \le n \text{ and } c_{nk}^{-1} = \begin{cases} n & \text{for } k = n \\ -(n-1) & \text{for } k = n-1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } n,$$

then we obviously have $[C_\alpha]_0^p = (w_0^p)_{C_{\alpha-1}}$, $[C_\alpha]^p = (w^p)_{C_{\alpha-1}}$ *and* $[C_\alpha]_\infty^p = (w_\infty^p)_{C_{\alpha-1}}$ *for all* $\alpha \ge 0$.
For each $x \in [C_\alpha]^p$*, the complex number* ξ *with* $x - \xi e \in [C_\alpha]_0^p$ *is unique;* ξ *is referred to as the* $[C_\alpha]^p$ *limit of the sequence* x.

We frequently use the well–known properties of the Cesàro coefficients A_n^α and the fact that the inverses $S^{\alpha-1} = (s_{nk}^{\alpha-1})_{n,k=0}^\infty$ of the matrices $C_{\alpha-1}$ are the triangles given by

$$s_{nk}^{\alpha-1} = A_{n-k}^{-\alpha} A_k^{\alpha-1} \text{ for } \alpha > 0 \text{ and } s_{nk}^{-1} = \frac{1}{n} \text{ for } \alpha = 0.$$

Finally, we write $R^{\alpha-1} = (r_{nk}^{\alpha-1})$ for the transpose of the matrix $S^{\alpha-1}$ ($\alpha \ge 0$), that is,

$$r_{nk}^{\alpha-1} = \begin{cases} 0 & & \text{for } k \le n-1 \\ A_{k-n}^{-\alpha} A_n^{\alpha-1} & \text{for } \alpha > 0 & \multirow{2}{*}{for $k \ge n$} \\ \dfrac{1}{k} & \text{for } \alpha = 0 & \end{cases} \quad \text{for all } n;$$

we also write $\sigma_n^{-1}(x) = nx_n - (n-1)x_{n-1}$ for all n and all $x \in \omega$.

First we establish some important topological properties of the spaces $[C_\alpha]_0^p$, $[C_\alpha]^p$ and $[C_\alpha]_\infty^p$ for $\alpha \ge 0$ and $p \ge 1$. We write $\sum_0 = \sum_{k=0}^1$, $\max_0 = \max_{0 \le k \le 1}$, and $\sum_\nu = \sum_{k=2^\nu}^{2^{\nu+1}-1}$ and $\max_\nu = \max_{2^\nu \le k \le 2^{\nu+1}-1}$ for $\nu \ge 1$.

Proposition 5.2. ([51, Proposition 2.1] for $\alpha > 0$, and Theorem 3.1, and [66, Theorem 4.3.12] and [34, Corollary 2.5] for $\alpha = 0$) *Let* $\alpha \ge 0$ *and* $p \ge 1$.

(a) *The sets* $[C_\alpha]_0^p$, $[C_\alpha]^p$ *and* $[C_\alpha]_\infty^p$ *are BK spaces with respect to*

$$\|x\|_{[C_\alpha]_\infty^p} = \begin{cases} \sup_\nu \left(\frac{1}{2^\nu} \sum_\nu |\sigma_k^{\alpha-1}(x)|^p \right)^{1/p} = \sup_\nu \left(\frac{1}{2^\nu} \sum_\nu \left| \frac{1}{A_k^{\alpha-1}} \sum_{j=0}^k A_{k-j}^{\alpha-2} x_k \right|^p \right)^{1/p} & \text{for } \alpha > 0 \\ \sup_\nu \left(\frac{1}{2^\nu} \sum_\nu |kx_k - (k-1)x_{k-1}|^p \right)^{1/p} & \text{for } \alpha = 0; \end{cases} \tag{6}$$

$[C_\alpha]_0^p$ *is a closed subspace of* $[C_\alpha]^p$ *and* $[C_\alpha]^p$ *is a closed subspace of* $[C_\alpha]_\infty^p$.

(b) *For each* n*, define the sequence* $c^{(\alpha,n)} = (c_k^{(\alpha,n)})_{k=0}^\infty$ *by*

$$c_k^{(\alpha,n)} = \begin{cases} 0 & & \text{for } 0 \le k \le n-1 \\ A_{k-n}^{-\alpha} A_n^{\alpha-1} & \text{for } \alpha > 0 & \text{for } k \ge n \\ \dfrac{1}{n} & \text{for } \alpha = 0. & \end{cases}$$

Then every sequence $x = (x_k)_{k=0}^\infty \in [C_\alpha]_0^p$ *has a unique representation* $x = \sum_n \sigma_n^{\alpha-1}(x) c^{(n)}$.
Every sequence $x = (x_k)_{k=0}^\infty \in [C_\alpha]^p$ *has a unique representation*

$$x = \xi \cdot e + \sum_n \sigma_n^{\alpha-1}(x - \xi \cdot e) c^{(\alpha,n)}, \text{ where } \xi \text{ is the } [C_\alpha]^p \text{ limit of the sequence } x. \tag{7}$$

Remark 5.3. *(a) Since w_∞^p has no Schauder basis by Theorem 3.1, $[C_\alpha]_\infty^p$ has no Schauder basis by [34, Remark 2.4].*

(b) We have $[C_\alpha]^p = (w_0^p \oplus e)_{C_{\alpha-1}}$ by definition, and so it follows from [34, Corollary 2.5 (c)] that every sequence $x \in [C_\alpha]^p$ has a unique representation

$$x = \xi \cdot c^{(\alpha,-1)} + \sum_n \left(\sigma_n^{\alpha-1}(x) - \xi \cdot e\right) c^{(\alpha,n)} \tag{8}$$

where the sequences $c^{(\alpha,n)}$ for all n are defined as in Proposition 5.2 and the sequence $c^{(\alpha,-1)} = (c_k^{(\alpha,-1)})_{k=0}^\infty$ is given by

$$c_k^{(\alpha,-1)} = \begin{cases} \sum\limits_{j=0}^{n} A_{k-j}^{-\alpha} A_j^{\alpha-1} = A_k^0 = 1 & \text{for } \alpha > 0 \\ \dfrac{1}{k} \sum\limits_{j=1}^{k} 1 = 1 & \text{for } \alpha = 0 \end{cases} \quad \text{for } n = 0, 1, \ldots.$$

hence $c^{(\alpha,-1)} = e$ Since $\sigma_n^{\alpha-1}(x) - \xi = \sigma_n^{\alpha-1}(x - \xi \cdot e)$ for all n, the representations in (8) and (7) are identical.

Now we determine the β duals of the spaces $[C_\alpha]_0^p$, $[C_\alpha]^p$ and $[C_\alpha]_\infty^p$. First we consider the case $\alpha > 0$.

Theorem 5.4. ([51, Theorem 3.3] for $\alpha > 0$; the case $\alpha = 0$ follows from [7, Lemma 3.1]) *Let $p \geq 1$ and $\alpha \geq 0$. We define the triangle $W^{(a;\alpha-1)} = (w_{mk}^{(a;\alpha-1)})$ for $a \in \omega$ by*

$$w_{mk}^{(a;\alpha-1)} = \begin{cases} \sum\limits_{j=m}^{\infty} A_{j-k}^{-\alpha} A_k^{\alpha-1} a_j & \text{for } \alpha > 0 \\ \sum\limits_{j=m}^{\infty} \dfrac{a_j}{j} & \text{for } \alpha = 0 \end{cases} \quad \text{for } k \leq m.$$

Then we have
(a) $a \in ([C_\alpha]_0^p)^\beta$ if and only if $a \in (\mathcal{M}_p)_{R^{\alpha-1}}$ and $W^{(a;\alpha-1)} \in (w_0^p, \ell_\infty)$;
(b) $a \in ([C_\alpha]^p)^\beta$ if and only if $a \in (\mathcal{M}_p)_{R^{\alpha-1}}$ and $W^{(a;\alpha-1)} \in (w^p, c)$;
(c) $a \in ([C_\alpha]_\infty^p)^\beta$ if and only if $a \in (\mathcal{M}_p)_{R^{\alpha-1}}$ and $W^{(a;\alpha-1)} \in (w_\infty^p, c_0)$.
(d) If $X = w_0^p$ or $X = w_\infty^p$ and $a \in (X_{C_{\alpha-1}})^\beta$, then

$$\sum_{k=0}^\infty a_k z_k = \sum_{k=0}^\infty (R_k^{\alpha-1} a)(\sigma_k^{\alpha-1}(x)) \text{ for all } x \in X_{C_{\alpha-1}}; \text{ also } \|a\|_{X_{C_{\alpha-1}}}^* = \|R_k^{\alpha-1} a\|_{\mathcal{M}_p}.$$

If $a \in ([C_\alpha]^p)^\beta$ then

$$\sum_{k=0}^\infty a_k z_k = \sum_{k=0}^\infty (R_k^{\alpha-1} a)(\sigma_k^{\alpha-1}(x)) - \xi\rho \text{ where } \xi \in \mathbb{C} \text{ is the } [C_\alpha]^p \text{ limit of } x \text{ and } \rho = \lim_{m\to\infty} W_m^{(a;\alpha)} e;$$

also $\|a\|_{[C_\alpha]^p}^* = |\rho| + \|R^{\alpha-1} a\|_{\mathcal{M}_p}$ for all $a \in ([C_\alpha]^p)^\beta$.

Remark 5.5. *Writing $W = W^{(a;\alpha-1)}$ and $R = R^{\alpha-1}$, for short, we have*

$$R_k a = \begin{cases} \sum\limits_{j=k}^{\infty} A_{j-k}^{-\alpha} A_k^{\alpha-1} a_j & \text{for } \alpha > 0 \\ \sum\limits_{j=k}^{\infty} \dfrac{a_j}{j} & \text{for } \alpha = 0 \end{cases} \quad \text{for all } k,$$

hence $a \in (\mathcal{M}_p)_R$ if and only if $\|R\|_{w^p_\infty} < \infty$. Furthermore we have by Theorem 3.3 **1.** *$W \in (w^p_0, \ell_\infty)$ if and only if*

$$\sup_m \|W_m\|_{\mathcal{M}_p} < \infty. \tag{9}$$

We have by Theorem 3.3 **7.** *that $W \in (w^p, c)$ if and only if (9) holds,*

$$\rho = \lim_{m\to\infty} W_m e = \lim_{m\to\infty} \sum_{k=0}^{m} w_{mk} = \begin{cases} \lim\limits_{m\to\infty} \sum\limits_{k=0}^{m} \sum\limits_{j=m}^{\infty} A^{-\alpha}_{j-k} A^{\alpha-1}_k a_j & \text{for } \alpha > 0 \\ \lim\limits_{m\to\infty} \sum\limits_{k=0}^{m} \sum\limits_{j=m}^{\infty} \dfrac{a_j}{j} & \text{for } \alpha = 0 \end{cases} \tag{10}$$

exists, and $\lim_{m\to\infty} w_{mk} = \gamma_k$ exists for each k, which is redundant since the convergence of $R_k a$ for each k implies $\lim_{m\to\infty} w_{mk} = \gamma_k = 0$ for each k.
Finally, it follows from Theorem 3.3 **2.** *that $W \in (w^p_\infty, c_0)$ if and only if $\lim_{m\to\infty} \|W_m\|_{\mathcal{M}_p} = 0$.*

Finally, we give the characterizations of the classes (X, Y) for $X = [C_\alpha]^p_\infty, [C_\alpha]^p_0, [C_\alpha]^p$ and $Y = \ell_\infty, c_0, c$.
Let $A = (a_{nk})_{n,k=0}^\infty$ be an infinite matrix. We define the matrix $\hat{A}^{\alpha-1} = (\hat{a}^{\alpha-1}_{nk})_{n,k=0}^\infty$ by $\hat{a}^{\alpha-1}_{nk} = R^{\alpha-1}_k A_n$

Theorem 5.6. ([51, Theorem 4.2] for $\alpha > 0$, [7, Theorem 4.2] for $\alpha = 0$) *The necessary and sufficient conditions for the entries of $A \in (X, Y)$ for $X \in \{[C_\alpha]^p_\infty, [C_\alpha]^p_0, [C_\alpha]^p\}$ and $Y = \{\ell_\infty, c_0, c\}$ can be read from the following table*

To \ *From*	$[C_\alpha]^p_\infty$	$[C_\alpha]^p_0$	$[C_\alpha]^p$
ℓ_∞	**1.**	**2.**	**3.**
c_0	**4.**	**5.**	**6.**
c	**7.**	**8.**	**9.**

where

1. [5.6](1.1) *and* [5.6](1.2).*where*

[5.6](1.1) $\|A\|_{([C_\alpha]^p_\infty,\infty)} = \sup_n \|\hat{A}^{\alpha-1}_n\|_{\mathcal{M}_p} < \infty$

[5.6](1.2) $\lim_{m\to\infty} \|W^{(A_n;\alpha-1)}_m\|_{\mathcal{M}_p} = 0$ *for all n*

2. [5.6](1.1) *and* [5.6](2.1) $\sup_m \|W^{(A_n;\alpha-1)}_m\|_{\mathcal{M}_p} < \infty$ *for all n*

3. [5.6](1.1), [5.6](2.1) [5.6](3.1) *and* [5.6](3.2), *where*

[5.6](3.1) $\rho^{(n)} = \lim_{m\to\infty} W^{(A_n;\alpha-1)}_m e$ *exists for each n*

[5.6](3.2) $\sup_n \left|\hat{A}^{\alpha-1}_n e - \rho^{(n)}\right| < \infty$

4. [5.6](1.2) *and* [5.6](4.1) $\lim_{n\to\infty} \|\hat{A}^{\alpha-1}_n\|_{\mathcal{M}_p} = 0$

5. [5.6](1.2), [5.6](2.1) *and* [5.6](5.1) $\lim_{n\to\infty} \hat{a}^{\alpha-1}_{nk} = 0$ *for all k*

6. [5.6](1.1), [5.6](2.1), [5.6](3.1), [5.6](5.1) *and* [5.6](6.1), *where*

[5.6](6.1) $\lim_{n\to\infty} \left(\hat{A}^{\alpha-1}_n e - \rho^{(n)}\right) = 0$

7. [5.6](1.1), [5.6](7.1), [5.6](7.2) *and* [5.6](7.3), *where*

[5.6](7.1) $\hat{\alpha}_k = \lim_{n\to\infty} \hat{a}^{\alpha-1}_{nk}$ *exists for all k*

[5.6](7.2) $(\hat{\alpha}_k), \hat{A}^{\alpha-1}_n \in \mathcal{M}_p$ *for all n*

[5.6](7.3) $\lim_{n\to\infty} \|\hat{A}_n^{\alpha-1} - (\hat{\alpha}_k)\|_{\mathcal{M}_p} = 0$

8. [5.6](1.1), [5.6](2.1) *and* [5.6](7.1)

9. [5.6](1.1), [5.6](2.1), [5.6](3.1), [5.6](7.1) *and* [5.6](9.1), *where*

[5.6](9.1) $\lim_{n\to\infty} \left(\hat{A}_n^{\alpha-1} e - \rho^{(n)}\right) = \beta$ *exists.*

Furthermore, if $A \in ([C_\alpha]_0^p, Y)$ *or* $A \in ([C_\alpha]_\infty^p, Y)$*, then we have* $\|L_A\| = \|A\|_{([C_\alpha]_\infty^p,\infty)}$*, and if* $A \in ([C_\alpha]^p, Y)$*, then* $\|L_A\| = \|A\|_{([C_\alpha]^p,\infty)} = \sup_n \left(|\rho^n| + \|A\|_{([C_\alpha]_\infty^p,\infty)}\right)$.

Remark 5.7. *We note that by (10) and the definition of the matrices* $\hat{A}^{\alpha-1}$ *and* $W^{(A_n;\alpha-1)}$*, we have*

$$\hat{A}_n^{\alpha-1} e - \rho^{(n)} = \begin{cases} \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} A_{j-k}^{-\alpha} A_k^{\alpha-1} a_{nj} - \lim_{m\to\infty} \sum_{k=0}^{m} \sum_{j=m}^{\infty} A_{j-k}^{-\alpha} A_k^{\alpha-1} a_{nj} & \text{for } \alpha > 0 \\ \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \frac{a_{nj}}{j} - \lim_{m\to\infty} \sum_{k=0}^{m} \sum_{j=m}^{\infty} \frac{a_{nj}}{j} & \text{for } \alpha = 0 \end{cases}$$

$$\hat{\alpha}_k = \begin{cases} \lim_{n\to\infty} \sum_{j=k}^{\infty} A_{j-k}^{-\alpha} A_k^{\alpha-1} a_{nj} \text{ for each } k & \text{for } \alpha > 0 \\ \lim_{n\to\infty} \sum_{j=k}^{\infty} \frac{a_{nj}}{j} \text{ for each } k & \text{for } \alpha = 0 \end{cases}$$

and, for all n,

$$\left\|\hat{A}_n^{\alpha-1} - (\hat{\alpha}_k)\right\|_{\mathcal{M}_p} = \begin{cases} \sum_{\nu=0}^{\infty} 2^\nu \max_\nu \left|\hat{a}_{nk}^{\alpha-1} - \hat{\alpha}_k\right| & (p = 1) \\ \sum_{\nu=0}^{\infty} 2^{\nu/p} \left(\sum_\nu \left|\hat{a}_{nk}^{\alpha-1} - \hat{\alpha}_k\right|^q\right)^{1/q} & \left(p > 1; q = \frac{p}{p-1}\right). \end{cases}$$

6. Compact Operators

In this section, we give the characterizations of some classes of compact operators between the spaces of the previous sections. We achieve this by using the *Hausdorff measure of noncompactness*. Although a different approach for the characterization of compact matrix operators between certain sequence spaces can be found, for instance, in [63], we chose our approach because it seems to be more elementary and suitable for our purpose. Measures of noncompactness are also widely used in fixed point theory.

The first *measure of noncompactness*, the function α, was defined and studied by *Kuratowski* [36] in 1930. Later in 1955, *Darbo* [13] was the first who continued to use the function α. He proved that if T is a continuous self-mapping of a nonempty, bounded, closed and convex subset C of a Banach space X such that $\alpha(T(Q)) \leq k\alpha(Q)$ for all $Q \subset C$, where $k \in (0, 1)$ is a constant, then T has at least one fixed point in the set C. Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem and it includes the existence part of Banach's fixed point theorem.

Other measures were introduced by *Goldenstein, Gohberg and Markus* (the *ball* or *Hausdorff measure of noncompactness*) [20] in 1957 (which was later studied by *Goldenstein and Markus* [21] in 1968), *Istrăţesku* [29] in 1972 and others. Apparently Goldenstein, Gohberg and Markus were unaware of the work of Kuratowski and Darbo. It is surprising that Darbo's theorem was almost never noticed and applied, not until in the 1970's mathematicians working in operator theory, functional analysis and differential equations began to apply Darbo's theorem and develop the theory connected with measures of noncompactness.

These measures and their applications are discussed for example in the monographs [1, 3–5, 27, 30, 61], and a large number of research articles.

We recall that if X and Y are Banach spaces and $L : X \to Y$ is a linear operator, then L is said to be *compact*, if its domain is all of X and, for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y. We write $\mathcal{C}(X, Y)$ for the set of compact operators in $\mathcal{B}(X, Y)$.

Let (X, d) be a complete metric space and $\mathcal{M}_X$ denote the class of all bounded subsets of X. Then the *Hausdorff measure of noncompactness of the set* $Q \in \mathcal{M}_X$ is given by

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ can be covered by finitely many open balls of radius } < \varepsilon\};$$

the function $\chi : \mathcal{M}_X \to [0, \infty)$ is called the *Hausdorff measure of noncompactness*.

Let X and Y be Banach spaces and χ_1 and χ_2 be measures of noncompactness on X and Y. Then the operator $L : X \to Y$ is called (χ_1, χ_2)*–bounded* if $L(Q) \in \mathcal{M}_Y$ for every $Q \in \mathcal{M}_X$ and there exists a constant $C > 0$ such that

$$\chi_2(L(Q)) \leq C \cdot \chi_1(Q) \text{ for all } Q \in \mathcal{M}_X; \tag{11}$$

if L is (χ_1, χ)–bounded then the number $\|L\|_{(\chi_1,\chi_2)} = \inf\{C > 0 : (11) \text{ holds}\}$ is called the (χ_1, χ_2)*–measure of noncompactness of* L ([54, Definition 2.24]); we also write $\|L\|_\chi = \|L\|_{(\chi_1,\chi_2)}$, for short, and call $\|L\|_\chi$ the *Hausdorff measure of noncompactness of* L.

If X and Y are Banach spaces and $L \in \mathcal{B}(X, Y)$ then the following vitally important facts are well known:

$$\|L\|_\chi = \chi(L(S_X)) \quad \text{([54, Theorem 2.25])} \tag{12}$$

and

$$L \text{ is compact if and only if } \|L\|_\chi = 0 \quad \text{([54, Corollary 2.26 (2.58)])}. \tag{13}$$

We also need the following results.

Theorem 6.1 (Goldenstein, Gohberg, Markus). ([54, Theorem 2.23]) *Let X be a Banach space with a Schauder basis $(b_n)_{n=1}^\infty$, $Q \in \mathcal{M}_X$, $\mathcal{P}_n : X \to X$ be the projector onto the linear span of $b_1, b_2, \dots, b_n$ and $\mathcal{R}_n = I - \mathcal{P}_n$ where I is the identity on X. Then we have*

$$\frac{1}{a} \limsup_{n\to\infty} \left(\sup_{x\in Q} \|\mathcal{R}_n(x)\| \right) \leq \chi(Q) \leq \inf_n \left(\sup_{x\in Q} \|\mathcal{R}_n(x)\| \right) \leq \limsup_{n\to\infty} \left(\sup_{x\in Q} \|\mathcal{R}_n(x)\| \right), \tag{14}$$

where $a = \limsup_{n\to\infty} \|\mathcal{R}_n\|$ is the basis constant.

We give some prelimimary well–known examples of the Hausdorff measure of noncompactness of bounded subsets of ℓ_p for $1 \leq p < \infty$ and c_0, and of the measure of noncompactness of operators $L \in \mathcal{B}(\ell_1)$.

Example 6.2. ([54, Theorem 2.15 and Corollary 2.29]) *(a) We have*

$$\chi(Q) = \begin{cases} \lim_{n\to\infty} \left(\sup_{x\in Q} \left(\sup_{k\geq n} |x_k| \right) \right) & \text{for } Q \in \mathcal{M}_{c_0} \\ \lim_{n\to\infty} \left(\sup_{x\in Q} \left(\sum_{k=n}^\infty |x_k|^p \right)^{1/p} \right) & \text{for } Q \in \mathcal{M}_{\ell_p}. \end{cases}$$

(b) Let $L \in \mathcal{B}(\ell_1)$. Since ℓ_1 is a BK space with AK by Example 2.3, L is given by a matrix $A \in (\ell_1, \ell_1)$ by Proposition 2.2. Then L is compact if and only if

$$\|L\|_\chi = \lim_{m\to\infty} \left(\sup_k \sum_{n=m}^\infty |a_{nk}| \right) = 0.$$

First we give a few identities or estimates for the Hausdorff measure of noncompactness of bounded linear operators on the spaces w_0^p and w^p. We need the representations of bounded linear operators on w^p.

If X and Y are Banach sequence spaces and $L \in \mathcal{B}(X, Y)$, then we write $L_n = P_n \circ L$ for all n.

Theorem 6.3. ([6, Theorem 3.9]) *We have $L \in \mathcal{B}(w^p, c)$ if and only if there exist a sequence $b \in c$ and a matrix $A = (a_{nk})_{n=0,k=0}^{\infty} \in (w_0^p, c)$ such that*

$$L(x) = b\xi + Ax \text{ for all } x \in w^p, \tag{15}$$

where $\xi \in \mathbb{C}$ is the strong w^p limit of x,

$$a_{nk} = L_n(e^{(k)}) \text{ and } b_n = L_n(e) - \sum_{k=0}^{\infty} a_{nk} \text{ for all } n \text{ and } k. \tag{16}$$

Moreover, if $L \in \mathcal{B}(w^p, c)$, then we have

$$\frac{1}{2} \cdot \limsup_{n\to\infty} \left(\left| b_n - \beta + \sum_{k=0}^{\infty} \alpha_k \right| + \left\| \tilde{A}_n \right\|_{\mathcal{M}_p} \right) \leq \|L\|_{\chi} \leq \limsup_{n\to\infty} \left(\left| b_n - \beta + \sum_{k=0}^{\infty} \alpha_k \right| + \left\| \tilde{A}_n \right\|_{\mathcal{M}_p} \right), \tag{17}$$

where $\tilde{A} = (\tilde{a}_{nk})_{n,k=0}^{\infty}$ is the matrix with $\tilde{a}_{nk} = a_{nk} - \alpha_k$ for all n and k, and

$$\alpha_k = \lim_{n\to\infty} a_{nk} \text{ for all } k \geq 0 \text{ and } \beta = \lim_{n\to\infty} \left(b_n + \sum_{k=0}^{\infty} a_{nk} \right). \tag{18}$$

We also have

$$\eta = \lim_{n\to\infty} L_n(x) = \xi\beta + \sum_{k=0}^{\infty} \alpha_k(x_k - \xi) = \left(\beta - \sum_{k=0}^{\infty} \alpha_k \right) \xi + \sum_{k=0}^{\infty} \alpha_k x_k \text{ for all } x \in w^p. \tag{19}$$

Now we give the estimates for $\|L\|_{\chi}$ when $L \in \mathcal{B}(X, Y)$ for $X = w_0^p, w^p$ and $Y = c_0, c$.

Theorem 6.4. *We use the notations of Theorem 6.4. Let X be any of the spaces w_0^p and w^p, and $Y = c_0$ or $Y = c$. Then identities or estimates for $\|L\|_{\chi}$ when $L \in \mathcal{B}(X, Y)$ can be read from the following table*

From / *To*	w_0^p	w^p
c_0	**1.**	**2.**
c	**3.**	**4.**

where

1. [6.4](1.1) $\|L_A\|_{\chi} = \lim_{r\to\infty} \left(\sup_{n \geq r} \|A_n\|_{\mathcal{M}_p} \right)$ ([6, Corollary 3.8])

2. [6.4](2.1) $\|L_A\|_{\chi} = \limsup_{n\to\infty} \left(|b_n| + \|A_n\|_{\mathcal{M}_p} \right)$ ([6, Corollary 3.11])

3. [6.4](3.1) $\left\{ \frac{1}{2} \cdot \lim_{r\to\infty} \left(\sup_{n\geq r} \left\| \tilde{A}_n \right\|_{\mathcal{M}_p} \right) \leq \|L_A\|_{\chi} \leq \lim_{r\to\infty} \left(\sup_{n\geq r} \left\| \tilde{A}_n \right\|_{\mathcal{M}_p} \right) \right.$ ([6, Corollary 3.6])

4. [6.4](4.1), *where [6.4](4.1) is (17).*

We obtain as an immediate consequence of (13) and Theorem 6.3 the following characterization of compact operators

Corollary 6.5. ([6, Corollary 3.13]) *Let X be any of the spaces w_0^p and w^p, and $Y = c_0$ or $Y = c$. Then the necessary and sufficient conditions for $L \in \mathcal{C}(X, Y)$ and be read from the following table*

From / To	w_0^p	w^p
c_0	**1.**	**2.**
c	**3.**	**4.**

where

1. [6.5](1.1) $\lim_{r\to\infty}\left(\sup_{n\ge r}\|A_n\|_{\mathcal{M}_p}\right)=0$ **2.** [6.5](2.1) $\lim_{n\to\infty}\left(|b_n|+\|A_n\|_{\mathcal{M}_p}\right)=0$

3. [6.5](3.1) $\lim_{r\to\infty}\left(\sup_{n\ge r}\left\|\tilde{A}_n\right\|_{\mathcal{M}_p}\right)=0$ **4.** [6.4](4.1) $\lim_{n\to\infty}\left(\left|b_n-\beta+\sum_{k=0}^{\infty}\alpha_k\right|+\left\|\tilde{A}_n\right\|_{\mathcal{M}_p}\right)=0.$

Remark 6.6. *Putting $b_n=0$ for all n in* **2.** *and* **4.** *of Theorem 6.4 and Corollary 6.5 we obtain*

(a) if $A\in(w^p,c_0)$ then ([6, Corollary 3.12])

$$\|L_A\|_{\mathcal{X}}=\lim_{n\to\infty}\|A_n\|_{\mathcal{M}_p} \text{ and } L_A\in\mathcal{C}(w^p,c_0) \text{ if and only if } \lim_{n\to\infty}\|A_n\|_{\mathcal{M}_p}=0 \tag{20}$$

(b) if $A\in(w^p,c)$ then ([6, Corollary 3.10])

$$\frac{1}{2}\limsup_{n\to\infty}\left(\left|\sum_{k=0}^{\infty}\alpha_k-\alpha\right|+\|\tilde{A}_n\|_{\mathcal{M}_p}\right)\le\|L_A\|_{\mathcal{X}}\le\limsup_{n\to\infty}\left(\left|\sum_{k=0}^{\infty}\alpha_k-\alpha\right|+\|\tilde{A}_n\|_{\mathcal{M}_p}\right),$$
$$\text{with } \alpha_k \text{ from (18) and } \alpha=\lim_{n\to\infty}\sum_{k=0}^{\infty}a_{nk}; \tag{21}$$

also $L_A\in\mathcal{C}(w^p,c)$ if and only if

$$\lim_{n\to\infty}\left(\left|\sum_{k=0}^{\infty}\alpha_k-\alpha\right|+\|\tilde{A}_n\|_{\mathcal{M}_p}\right)=0.$$

Now we characterize the compact matrix operators in (w_∞^p,c_0) and (w_∞^p,c).

Corollary 6.7. ([6, Corollary 3.15]) *(a) Let $A\in(w_\infty^p,c_0)$. Then we have $L_A\in\mathcal{C}(w_\infty^p,c_0)$ if and only if* **1.** *in Corollary* 6.5 *holds.*
(b) Let $A\in(w_\infty^p,c)$. Then we have $L_A\in\mathcal{C}(w_\infty^p,c_0)$ if and only if **3.** *in Corollary* 6.5 *holds.*

We say that an operator $L\in\mathcal{B}(w^p,c)$ is w^p–regular, if $\lim_{n\to\infty}L_n(x)=\xi$ for all $x\in w^p$, where ξ is the w^p limit of x. A matrix $A\in(w^p,c)$ is said to be w^p–regular, if the operator L_A is w^p–regular.

Now we give a characterization of compact w^p–regular operators.

Corollary 6.8. ([6, Corollary 3.16]) *Let $L\in\mathcal{B}(w^p,c)$ be w^p–regular. Then we have $L\in\mathcal{C}(w^p,c)$ if and only if*

$$\lim_{n\to\infty}\left(|b_n-1|+\|A_n\|_{\mathcal{M}_p}\right)=0. \tag{22}$$

The next result is similar to a well–known result by Cohen and Dunford [10] which states that a *regular matrix*, that is, a matrix $A\in(c,c)$ with $\lim_n A_nx=\lim_n x_n$ for all $x\in c$, cannot be a compact matrix operator.

Remark 6.9. ([6, Remark 3.17]) *If A is a w^p–regular matrix then L_A cannot be compact, since, with $b_n=0$ for all n, we have $1+\|A_n\|_{\mathcal{M}_p}\ge 1\ne 0$ for all n, and so* (22) *in Corollary* 6.8 *cannot hold.*

Now we give a result for the representations of $L \in \mathcal{B}(w^p, w)$ and $L \in \mathcal{B}(w^p, w_0)$, similar to Theorem 6.3, and for the operator norm $\|L\|$.

Theorem 6.10. ([2, Theorem 2 and Corollary 7]) *We use the notations of Theorem 6.3, and assume that w and w_0 have the sectional norm.*

(a) We have $L \in \mathcal{B}(w^p, w)$ if and only if there exists a matrix $A \in (w_0^p, w)$ and a sequence $b \in w_\infty$ with

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{n=0}^{m} |b_n + A_n e - \tilde{\beta}| = 0 \text{ for some } \tilde{\beta} \in \mathbb{C} \tag{23}$$

such that (15) holds; moreover, we have, writing $S^N(b) = \sum_{n\in N} b_n$ and $S^N(A) = \sum_{n\in N} A_n$ for any finite subset N of $\mathbb{N} \cup \{0\}$,

$$\sup_m \nu^{N_m}(b, A) \le \|L\| \le 4 \cdot \nu^{N_m}(b, A), \text{ where } \nu^{N_m}(b, A) = \frac{1}{m+1} \max_{N_m} \left(\left|S^{N_m}(b)\right| + \left\|S^{N_m}(A_n)\right\|_{\mathcal{M}_p} \right). \tag{24}$$

Also, if ξ is the w^p limit of $x \in w^p$, then the w limit η of $y = Ax$ is given by ([2, Remark 3])

$$\eta = \tilde{\beta}\xi + \sum_{k=0}^{\infty} \alpha_k (x_k - \xi). \tag{25}$$

(b) We have $L \in \mathcal{B}(w^p, w_0)$ if and only if there exists a matrix $A \in (w_0^p, w_0)$ and a sequence $b \in w_\infty$ with

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{n=0}^{m} |b_n + A_n e| = 0 \tag{26}$$

such that (15) holds; moreover, we have (24).

Next we give the estimates for $\|L\|_\chi$ when $L \in \mathcal{B}(X, Y)$ for $X = w_0^p, w^p$ and $Y = (w_0, w)$.

Theorem 6.11. ([2, Corollary 7] for $X = w_0^p$ and [2, Theorem 6] for $X = w^p$) *Let $m, r \in \mathbb{N} \cup \{0\}$ and $m > r$. Then we write $N(m, r)$ for any subset of the set $\{r+1, r+2, \dots, m\}$. We also use the notations of Theorem 6.10, write $\nu^{N(m,r)}(A) = \nu^{N(m,r)}(0, A)$, for short, and assume that w and w_0 have the sectional norm.*
Let X be any of the spaces w_0^p and w^p, and $Y = c_0$ or $Y = c$. Then estimates for $\|L\|_\chi$ when $L \in \mathcal{B}(X, Y)$ can be read from the following table

To \ From	w_0^p	w^p
w_0	**1.**	**2.**
w	**3.**	**4.**

where

1. [6.11](1.1) $\lim_{r\to\infty} \left(\sup_m \left(\frac{1}{m+1} \nu^{N(m,r)}(A) \right) \right) \le \|L\|_\chi \le 4 \cdot \lim_{r\to\infty} \left(\sup_m \left(\frac{1}{m+1} \nu^{N(m,r)}(A) \right) \right)$

2. [6.11](2.1) $\lim_{r\to\infty} \left(\sup_m \left(\frac{1}{m+1} \nu^{N(m,r)}(b, A) \right) \right) \le \|L\|_\chi \le 4 \cdot \lim_{r\to\infty} \left(\sup_m \left(\frac{1}{m+1} \nu^{N(m,r)}(b, A) \right) \right)$

3. [6.11](3.1) $\lim_{r\to\infty} \left(\sup_m \left(\frac{1}{m+1} \nu^{N(m,r)}(\tilde{A}) \right) \right) \le \|L\|_\chi \le 4 \cdot \lim_{r\to\infty} \left(\sup_m \left(\frac{1}{m+1} \nu^{N(m,r)}(\tilde{A}) \right) \right)$

4. [6.11](4.1), $\begin{cases} \lim_{r\to\infty} \left(\sup_m \left(\frac{1}{m+1} \nu^{N(m,r)}(\gamma, \hat{A}) \right) \right) \le \|L\|_\chi \le 4 \cdot \lim_{r\to\infty} \left(\sup_m \left(\frac{1}{m+1} \nu^{N(m,r)}(\gamma, \hat{A}) \right) \right), \\ \textit{where } \gamma = (\gamma_n) \textit{ is the sequence with } \gamma_n = b_n - \tilde{\beta} + \sum_{k=0}^{\infty} \alpha_k \textit{ for all } n. \end{cases}$

Next we give the characterizations of the the classes $\mathcal{C}(X,Y)$ when $X = w_0^p, w^p$ and $Y = w_0, w$.

Corollary 6.12. ([2, Corollary 8]) *Let $L \in \mathcal{B}(X,Y)$. Then the necessary and sufficient conditions for $L \in \mathcal{C}(X,Y)$ when $X = w_0^p, w^p$ and $Y = w_0, w$ can be read from the table*

From / *To*	w_0^p	w^p
w_0	**1.**	**2.**
w	**3.**	**4.**

where

1. [6.12](1.1) $\lim_{r\to\infty}\left(\sup_m\left(\frac{1}{m+1}\nu^{N(m,r)}(A)\right)\right) = 0;$ **2.** [6.12](2.1) $\lim_{r\to\infty}\left(\sup_m\left(\frac{1}{m+1}\nu^{N(m,r)}(b,A)\right)\right) = 0;$

3. [6.12](3.1) $\lim_{r\to\infty}\left(\sup_m\left(\frac{1}{m+1}\nu^{N(m,r)}(\hat{A})\right)\right) = 0;$ **4.** [6.12](4.1) $\lim_{r\to\infty}\left(\sup_m\left(\frac{1}{m+1}\nu^{N(m,r)}(\gamma,\hat{A})\right)\right) = 0.$

Now we study compact operators on $c_0^p(\Lambda)$, $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$. The results are similar to those for w_0^p, w^p and w_∞^p.

We start with the case $p = 1$. The first results concern the representations of bounded linear operators from $c(\Lambda)$ into c and c_0, similar to those in Theorem 6.3.

Theorem 6.13. ([49, Theorem 5.9] for (a)–(d), and [49, Theorem 5.10] for (e)) *We have*
(a) $L \in \mathcal{B}(c(\Lambda), c)$ if and only if there exists a matrix $A \in (c_0(\Lambda), c)$ and a sequence $b \in \ell_\infty$ for which the limit β in (18) exists such that

$$L(x) = b \cdot \xi + Ax \text{ for all } x \in c(\Lambda) \tag{27}$$

holds, where the entries of A and the terms of b are defined as in (16), and ξ is the $c(\Lambda)$ limit of the sequence x;
(b) $L \in \mathcal{B}(c(\Lambda), c_0)$ if and only if there exists a matrix $A \in (c_0(\Lambda), c_0)$ and a sequence $b \in \ell_\infty$ with $\beta = 0$ such that (27) holds.
(c) If $L \in \mathcal{B}(c(\Lambda), Y)$ for $Y = c, c_0$ then we have

$$\sup_n\left(|b_n| + \|A_n\|_{C(\Lambda)}\right) \le \|L\| \le K(s,t)\cdot\sup_n\left(|b_n| + \|A_n\|_{C(\Lambda)}\right) \text{ with } K(s,t) \text{ from (4).} \tag{28}$$

(d) Let $L \in \mathcal{B}(c(\Lambda), c)$, $x \in c(\Lambda)$ and ξ be the $c(\Lambda)$ limit of the sequence x. Then we have

$$\eta = \lim_{n\to\infty}(L_n(x)) = \beta\cdot\xi + \sum_{k=0}^\infty \alpha_k(x_k - \xi) = \left(\beta - \sum_{k=0}^\infty \alpha_k\right)\xi + \sum_{k=0}^\infty \alpha_k x_k \text{ with } \alpha_k\ (k = 0, 1, \dots) \text{ from (18).} \tag{29}$$

(e) Finally, if $L \in \mathcal{B}(c(\Lambda), c)$, then we have

$$\frac{1}{2}\cdot\limsup_{n\to\infty}\left(\left|b_n - \beta + \sum_{k=0}^\infty \alpha_k\right| + \left\|\tilde{A}_n\right\|_{C(\Lambda)}\right) \le \|L\|_\chi \le K(s,t)\cdot\limsup_{n\to\infty}\left(\left|b_n - \beta + \sum_{k=0}^\infty \alpha_k\right| + \left\|\tilde{A}_n\right\|_{C(\Lambda)}\right). \tag{30}$$

Now we give the estimates for $\|L\|_\chi$ when $L \in \mathcal{B}(X,Y)$ for $X = c_0(\Lambda), c(\Lambda)$ and $Y = c_0, c$.

Theorem 6.14. *We use the notations of Theorem 6.13. Let X be any of the spaces $c_0(\Lambda)$ and $c(\Lambda)$, and $Y = c_0$ or $Y = c$. Then identities or estimates for $\|L\|_\chi$ when $L \in \mathcal{B}(X,Y)$ can be read from the following table*

From / *To*	$c_0(\Lambda)$	$c(\Lambda)$
c_0	**1.**	**2.**
c	**3.**	**4.**

where

1. [6.14](1.1) $\|L_A\|_\chi = \lim_{r\to\infty}\left(\sup_{n\geq r}\|A_n\|_{C(\Lambda)}\right)$ ([49, Corollary 5.14 (b)])

2. [6.14](2.1), $\begin{cases} \limsup_{n\to\infty}\left(|b_n| + \|A_n\|_{C(\Lambda)}\right) \leq \|L_A\|_\chi \leq K(s,t)\limsup_{n\to\infty}\left(|b_n| + \|A_n\|_{C(\Lambda)}\right) \\ \text{([49, Corollary 5.12])}\end{cases}$

3. [6.14](3.1) $\begin{cases} \frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{n\geq r}\left\|\tilde{A}_n\right\|_{C(\Lambda)}\right) \leq \|L_A\|_\chi \leq \lim_{r\to\infty}\left(\sup_{n\geq r}\left\|\tilde{A}_n\right\|_{C(\Lambda)}\right) \\ \text{([49, Corollary 5.14 (a)])}\end{cases}$

4. [6.14](4.1), *where [6.14](4.1) is (30).*

We obtain as an immediate consequence of (13) and Theorem 6.14 the following characterization of compact operators

Corollary 6.15. ([49, Corollary 5.15]) *Let X be any of the spaces $c_0(\Lambda)$ and $c(\Lambda)$, and $Y = c_0$ or $Y = c$. Then the necessary and sufficient conditions for $L \in \mathcal{C}(X, Y)$ and be read from the following table*

From / *To*	$c_0(\Lambda)$	$c(\Lambda)$
c_0	**1.**	**2.**
c	**3.**	**4.**

where

1. [6.15](1.1) $\lim_{r\to\infty}\left(\sup_{n\geq r}\|A_n\|_{C(\Lambda)}\right) = 0$ **2.** [6.15](2.1) $\lim_{n\to\infty}\left(|b_n| + \|A_n\|_{C(\Lambda)}\right) = 0$

3. [6.15](3.1) $\lim_{r\to\infty}\left(\sup_{n\geq r}\left\|\tilde{A}_n\right\|_{C(\Lambda)}\right) = 0$ **4.** [6.15](4.1) $\lim_{n\to\infty}\left(\left|b_n - \beta + \sum_{k=0}^\infty \alpha_k\right| + \left\|\tilde{A}_n\right\|_{C(\Lambda)}\right) = 0.$

Remark 6.16. ([49, Corollary 5.16]) *Putting $b_n = 0$ for all n in* **2.** *and* **4.** *of Theorem 6.14 and Corollary 6.15 we obtain*

(a) if $A \in (c(\Lambda), c_0)$ then $\|L_A\|_\chi = \lim_{n\to\infty}\|A_n\|_{C(\Lambda)}$, and $L_A \in \mathcal{C}(c(\Lambda), c_0)$ if and only if $\lim_{n\to\infty}\|A_n\|_{C(\Lambda)} = 0$;

(b) if $A \in (c(\Lambda), c)$ then

$$\frac{1}{2}\limsup_{n\to\infty}\left(\left|\sum_{k=0}^\infty \alpha_k - \alpha\right| + \|\tilde{A}_n\|_{C(\Lambda)}\right) \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\left(\left|\sum_{k=0}^\infty \alpha_k - \alpha\right| + \|\tilde{A}_n\|_{C(\Lambda)}\right),$$

with α_k and α from (18) and (21); (31)

also $L_A \in \mathcal{C}(c(\Lambda), c)$ if and only if

$$\lim_{n\to\infty}\left(\left|\sum_{k=0}^\infty \alpha_k - \alpha\right| + \|\tilde{A}_n\|_{C(\Lambda)}\right) = 0. \tag{32}$$

Now we characterize the compact matrix operators in $(c_\infty(\Lambda), c_0)$ and $(c_\infty(\Lambda), c)$.

Corollary 6.17. ([49, Corollary 5.17]) *(a) Let $A \in (c_\infty(\Lambda), c_0)$. Then we have $L_A \in \mathcal{C}(c_\infty(\Lambda), c_0)$ if and only if* **1.** *in Corollary* 6.15 *holds.*
(b) Let $A \in (c_\infty(\Lambda), c)$. Then we have $L_A \in \mathcal{C}(c_\infty(\Lambda), c_0)$ if and only if **3.** *in Corollary* 6.15 *holds.*

Similarly as before, we say that an operator $L \in \mathcal{B}(c(\Lambda), c)$ is $c(\Lambda)$–regular, if $\lim_{n\to\infty} L_n(x) = \xi$ for all $x \in c(\Lambda)$, where ξ is the $c(\Lambda)$ limit of x. A matrix $A \in (c(\Lambda), c)$ is said to be $c(\Lambda)$–regular, if the operator L_A is $c(\Lambda)$–regular.

Now we give a characterization of compact $c(\Lambda)$–regular operators.

Corollary 6.18. ([49, Corollary 5.18]) *Let $L \in \mathcal{B}(c(\Lambda), c)$ be $c(\Lambda)$–regular. Then we have $L \in \mathcal{C}(c(\Lambda), c)$ if and only if*

$$\lim_{n\to\infty}\left(|b_n - 1| + \|A_n\|_{C(\Lambda)}\right) = 0. \tag{33}$$

Similarly, as in the case of w^p–regularity in Corollary 6.8, it turns out that a $c(\Lambda)$–regular matrix cannot be a compact operator.

Remark 6.19. ([49, Remark 5.19]) *If A is a $c(\Lambda)$–regular matrix then L_A cannot be compact, since, with $b_n = 0$ for all n, we have $1 + \|A_n\|_{C(\Lambda)} \geq 1 \neq 0$ for all n, and so (33) in Corollary 6.18 cannot hold.*

Next we study some more Hausdorff measures of noncompactness of matrix operators on $c_0^p(\Lambda)$ for $1 \leq p < \infty$

Theorem 6.20. *Let Y be any of the spaces c_0, c, $\tilde{c}_0(\mu)$ and $\tilde{c}(\mu)$. Then the identities and estimates for $\|L_A\|_\chi$ when $A \in (c_0^p(\Lambda), Y)$ for $1 < p < \infty$ can be read from the following table*

From / *To*	$c_0^p(\Lambda)$
c_0	**1.**
c	**2.**
$\tilde{c}_0(\mu)$	**3.**
$\tilde{c}(\mu)$	**4.**

where

1. [6.20](1.1) $\|L_A\|_\chi = \lim_{r\to\infty}\left\|A^{>r<}\right\|_{(c_0^p(\Lambda),\infty)}$, *where*

$$\left\|A^{>r<}\right\|_{(c_0^p(\Lambda),\infty)} = \sup_{n\geq r}\sum_{\nu=0}^{\infty}\lambda_{k(\nu+1)}\left|\sum\nolimits_\nu\left(\sum_{j=k}^{\infty}\frac{a_{nj}}{\lambda_j}\right|^q\right)^{1/q} \quad \text{([58, Theorem 5.3 (a)])}$$

2. [6.20](2.1) $\frac{1}{2}\cdot\lim_{r\to\infty}\left\|A^{>r<}\right\|_{(c_0^p(\Lambda),\infty)} \leq \|L_A\|_\chi \leq \lim_{r\to\infty}\left\|A^{>r<}\right\|_{(c_0^p(\Lambda),\infty)}$ ([58, Theorem 5.3 (b)])

3. [6.20](3.1)

$$\begin{cases} \lim_{r\to\infty}\|A^{>r<}\|_{(c_0^p(\Lambda),\tilde{c}_0(\mu))} \leq \|L_A\|_\chi \leq 4\cdot\lim_{r\to\infty}\|A^{>r<}\|_{(c_0^p(\Lambda),\tilde{c}_0(\mu))}, \text{ where} \\ \|A^{>r<}\|_{(c_0(\Lambda),\tilde{c}_0(\mu))} = \sup_{m\geq r}\left(\frac{1}{\mu_m}\max_{N(m,r)}\left\|\sum_{n\in N(m,r)}(\mu_n A_n - \mu_{n-1}A_{n-1})\right\|_{C_p(\Lambda)}\right) \\ \text{and} \\ \left\|\sum_{n\in N(m,r)}(\mu_n A_n - \mu_{n-1}A_{n-1})\right\|_{C_p(\Lambda)} = \sum_{\nu=0}^{\infty}\lambda_{k(\nu+1)}\left(\sum\nolimits_\nu\left|\sum_{n\in N(m,r)}\sum_{j=k}^{\infty}\frac{\mu_n a_{nj} - \mu_{n-1}a_{n-1,j}}{\lambda_j}\right|^q\right)^{1/q} \\ \text{([58, Theorem 5.4 (a)])} \end{cases}$$

4. [6.20](4.1) $\frac{1}{2}\cdot\lim_{r\to\infty}\left\|A^{>r<}\right\|_{(c_0^p(\Lambda),\tilde{c}_0(\mu))} \leq \|L_A\|_\chi \leq \lim_{r\to\infty}\left\|A^{>r<}\right\|_{(c_0^p(\Lambda),\tilde{c}_0(\mu))}$ ([58, Theorem 5.4 (b)]).

Now we establish some identities or estimates for the Hausdorff measures of noncompactness of matrix operators on the matrix domains of arbitrary triangles in w_0^p, w^p and w_∞^p.

It is well known that every triangle T has a unique inverse S, which is also a triangle, and $T(Sx) = S(Tx) = x$ for all $x \in \omega$ ([66, Theorem 1.4.8] and [12, Remark 22 (a)]).

Theorem 6.21. ([51, Theorem 5.3]) *Let X be any of the spaces w_0^p and w^p and $Y = c_0$ or $Y = c$. If $A = (a_{nk})$ is an infinite matrix, then we write $\hat{A} = (\hat{a}_{nk})$ for the matrix with $\hat{a}_{nk} = \sum_{j=k}^{\infty} s_{jk} a_{nk}$ for all n and k. Then identities or estimates for $\|L_A\|_\chi$ when $A \in (X_T, Y)$ can be read from the following table*

To \ From	$(w_0^p)_T$	$(w^p)_T$
c_0	**1.**	**2.**
c	**3.**	**4.**

where

1. [6.21](1.1) $\|L_A\|_\chi = \lim_{r\to\infty}\left(\sup_{n\ge r}\|\hat{A}_n\|_{\mathcal{M}_p}\right)$

2. [6.21](2.1) $\|L_A\|_\chi = \lim_{r\to\infty}\left(\sup_{n\ge r}\left(\|\hat{A}_n\|_{\mathcal{M}_p} + |\rho^{(n)}|\right)\right)$ *where* $\rho^{(n)} = \lim_{m\to\infty} W_m^{(n)} e$ *for all* n

3. [6.21](3.1)
$$\begin{cases} \frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{n\ge r}\left\|\hat{A}_n - (\hat{\alpha}_k)\right\|_{\mathcal{M}_p}\right) \le \|L_A\|_\chi \le \lim_{r\to\infty}\left(\sup_{n\ge r}\left\|\hat{A}_n - (\hat{\alpha}_k)\right\|_{\mathcal{M}_p}\right) \\ \text{where } \hat{\alpha}_k = \lim_{n\to\infty}\hat{a}_{nk} \text{ for each } k \end{cases}$$

4. [6.21](4.1)
$$\begin{cases} \frac{1}{2}\cdot\lim_{r\to\infty}\left(\sup_{n\ge r}\left\|\hat{A}_n - (\hat{\alpha}_k)\right\|_{\mathcal{M}_p} + \left|\sum_{k=0}^{\infty}\hat{\alpha}_k - \hat{\beta} - \rho^{(n)}\right|\right) \le \|L_A\|_\chi \\ \le \lim_{r\to\infty}\left(\sup_{n\ge r}\left\|\hat{A}_n - (\hat{\alpha}_k)\right\|_{\mathcal{M}_p} + \left|\sum_{k=0}^{\infty}\hat{\alpha}_k - \hat{\beta} - \rho^{(n)}\right|\right) \\ \text{where } \hat{\beta} = \lim_{n\to\infty}(\hat{A}_n e - \rho^{(n)}) \end{cases}$$

We obtain the following characterizations of compact operators from Theorem 6.21.

Corollary 6.22. ([51, Corollary 5.4]) *Let X and Y be any of the spaces of Theorem 6.21. Then if $A \in (X_T, Y)$ then the conditions for L_A to be compact can be read from the following table*

To \ From	$(w_0^p)_T$, $(w_\infty^p)_T$	$(w^p)_T$
c_0	**1.**	**2.**
c	**3.**	**4.**

where

1. [6.22](1.1) $\lim_{r\to\infty}\left(\sup_{n\ge r}\|\hat{A}_n\|_{\mathcal{M}_p}\right) = 0$ **2.** [6.22](2.1) $\lim_{r\to\infty}\left(\sup_{n\ge r}\left(\|\hat{A}_n\|_{\mathcal{M}_p} + |\rho^{(n)}|\right)\right) = 0$

3. [6.22](3.1) $\lim_{r\to\infty}\left(\sup_{n\ge r}\left\|\hat{A}_n - (\hat{\alpha}_k)\right\|_{\mathcal{M}_p}\right) = 0$

4. [6.22](4.1) $\lim_{r\to\infty}\left(\sup_{n\ge r}\left\|\hat{A}_n - (\hat{\alpha}_k)\right\|_{\mathcal{M}_p} + \left|\sum_{k=0}^{\infty}\hat{\alpha}_k - \hat{\beta} - \rho^{(n)}\right|\right) = 0.$

Remark 6.23. *The estimates and identities for the Hausdorff measure of noncompactness of L_A when $A \in ([C_\alpha]_0^p, Y)$, $([C_\alpha]^p, Y)$, $([C_\alpha]_\infty^p, Y)$, and the characterizations of the corresponding compact operators are obtained from Theorem 6.21 and Corollary 6.22 with $T = C_{\alpha-1}$.*

Additional related results on compact operators can be found in [15, 16, 50, 54, 55, 57].

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