FILOMAT (Niš) **10** (1996), 99–105

CHARACTERIZATIONS OF SOME KINDS OF REGULAR SEMIGROUPS BY INVERSES OF ELEMENTS

M. K. Sen, H. X. Yang and Y. Q. Guo

Abstract. In this paper, several new characterizations of some kinds of regular semigroups are obtained by an unified way with the help of the mutual relations between inverses of some special kinds of elements, in particular, idempotents.

The defining modes of some kinds of regular semigroups are usually very different. In this short paper, we are going to give them some new characterizations by an unified fashion, that is, by the mutual relations between inverses of some special kinds of elements, in particular, idempotents.

We recall first the definitions of some kinds of regular semigroups.

Let S be a regular semigroup, $E(S)$ the set of idempotents of S. S is

- *orthodox*: $E(S)$ is a band;
- *right* (*left*) *inverse*: each $\mathcal{R}$ ($\mathcal{L}$)-class of S contains only one idempotent;
- *inverse*: S is right inverse and left inverse;
- *groupbound*: every element has a power in a subgroup of S;
- *completely regular*: every element is in a subgroup of S;
- an *orthogroup*: S is orthodox and completely regular, i.e. S is a semilattice of rectangular groups,
- a *left C-semigroup*: S is right inverse and completely regular;
- a *C-semigroup*: $E(S)$ is in the center of S (equivalently, S is inverse and completely regular);
- a *G.C-semigroup*: S is inverse and groupbound.

Received June 26, 1996

1991 *Mathematics Subject Classification*: 20M17.

Project Supported by the NNSF of China and two Foundation of STC and EC of Yunnan Province.

M. K. Sen and H. X. Yang are visiting Professor and visiting Scholar respectively in the Institute of Pure Mathematics, Yunnan University

We shall use the results of [1] without further notice.

By the following theorem we describe a regular semigroup S to be orthodox by $V(a)$ for $a \in S$, where $V(a)$ denotes the set of all inverses of a.

Theorem 1. *The following statements on a regular semigroup S are equivalent:*

(1) *S is orthodox;*
(2) $(\forall e, f \in E(S)$ *with* $e\mathcal{R}f)$ $V(e) = V(f)$;
(3) $(\forall e, f \in E(S)$ *with* $e\mathcal{L}f)$ $V(e) = V(f)$;
(4) $(\forall e, f \in E(S))$ $V(ef) = V(fe)$;
(5) $(\forall a \in S)(\forall x \in S)(\forall n \in \mathbb{Z}^+)$ $x^n \in V(a^n)$.

Proof. (1)⇒(4). It is easy to see that $efe \in V(ef) \cap V(fe)$, and hence $V(ef) = V(fe)$.

(4)⇒(2). If $e, f \in E(S)$, $e\mathcal{R}f$, then $ef = f$, $fe = e$, so $V(e) = V(f)$.

(2)⇒(1). Let $e \in E(S)$ and $a \in V(e)$. We only need to prove $a \in E(S)$.

By the hypothesis, we have $ea\mathcal{R}e$ and $ea \in E(S)$. By $V(e) = V(ea)$, we have $a \in V(ea)$, and hence $ea = (ea)a(ea) = ea(aea) = ea^2$. Therefore, $a = aea = a(ea^2) = (aea)a = a^2$, that is, $a \in E(S)$, so S is orthodox.

Since (3) is dual to (2), it follows (3) ⇔ (1).

(1)⇒(5). It is clear when $n = 1$. Now assume that the conclusion holds for $k \in \mathbb{Z}^+$. We shall show that it is also true for $k + 1$. By the inductive assumption, we have $a^{k+1} = a^k a = (a^k x^k a^k)(axa) = a^k(x^k a^k)(ax)a$. Since $x^k a^k, ax \in E(S)$, $(x^k a^k)(ax) \in E(S)$. Thus $a^{k+1} = a^k(x^k a^k)(ax)a = a^k(x^k a^k)(ax)(x^k a^k)(ax)a = a^{k+1}x^{k+1}a^{k+1}$. Similarly we can get the relation $x^{k+1} = x^{k+1}a^{k+1}x^{k+1}$. So, the conclusion holds for all $n \in \mathbb{Z}^+$.

(5)⇒(1). Let $e \in E(S)$ and $a \in V(e)$. Then by the hypothesis, $e^2 = e^2a^2e^2$, i.e. $e = ea^2e$. Therefore, $a = aea = a(ea^2ea) = (aea)(aea) = a^2$, i.e. $a \in E(S)$. So, S is orthodox. □

Corollary 2. *The following statements on a regular semigroup S are equivalent:*

(1) *S is orthodox;*
(2) $(\forall e, f \in E(S))$ $(\exists n \in Z^+)$ $V((ef)^n) = V((fe)^n)$;
(3) $(\forall e, f \in E(S))$ $V(efe) = V(ef)$;
(3') $(\forall e, f \in E(S))$ $V(efe) = V(fe)$;
(4) $(\forall e, f \in E(S))$ $(\exists n \in Z^+)$ $V((efe)^n) = V((ef)^n)$;
(4') $(\forall e, f \in E(S))$ $(\exists n \in Z^+)$ $V((efe)^n) = V((fe)^n)$.

Proof. (1) ⇒ (2). By Theorem 1 and the facts that $(ef)^n = ef$ and $(fe)^n = fe$, the conclusion follows.

(2) $\Rightarrow$ (1). If $e, f \in E(S)$, $e\mathcal{R}f$, then $ef = f$, $fe = e$. Whence we have $V(e) = V(f)$. Thus the conclusion follows from Theorem 1.

Similarly the statements (3), (3'), (4) and (4') are equivalent to statement (1). $\square$

Remark 1. The equalities in statements (2), (3), (3'), (4) and (4') can be changed to inclusions.

Theorem 3. *Let S be a regular semigroup. Then S is right inverse if and only if*

$$(\forall e, f \in E(S))\ V(ef)e = V(ef).$$

Proof. ($\Rightarrow$). Let $g \in V(ef)$. It sufficies to show $g = ge$. Since S is right inverse, by (2) in Theorem 1, S is orthodox. Therefore $efe\mathcal{R}ef$ for any $e, f \in E(S)$. Thus $efe = ef$ (S is right inverse) and $g(ef) = g(ef)g$. So $g = g(ef)g = g(ef) = g(efe)$, and hence $ge = g$.

($\Leftarrow$). If $e, f \in E(S)$, $e\mathcal{R}f$, then $ef = f$, $fe = e$. By the hypothesis, we have $f = xe$ for some $x \in V(f)$. Therefore $e = fe = (xe)e = xe = f$. So S is right inverse. $\square$

Theorem 4. *Let S be a regular semigroup. Then S is inverse if and only if $eV(ef)e = V(ef)$.*

Proof. ($\Rightarrow$). Immediately.

($\Leftarrow$). If $e, f \in E(S)$, $e\mathcal{R}f$, then $ef = f$, $fe = e$. By the hypothesis, we have $eV(f)e = V(f)$, and hence $f = exe$ for some $x \in V(f)$. Therefore $e = fe = (exe)e = exe = f$.

If $e, f \in E(S)$, $e\mathcal{L}f$, then $ef = e$, $fe = f$. By the hypothesis, we have $eV(e)e = V(e)$, and hence $f = exe$ for some $x \in V(e)$. Therefore $e = ef = e(exe) = exe = f$. So S is inverse. $\square$

Lemma 5 [6]. *Let S be a regular semigroup. Then*

(1) *S is groupbound if and only if*

$$(\forall a \in S)\ (\exists n \in Z^+)\ a^n \in Sa^{2n};$$

(1') *S is groupbound if and only if*

$$(\forall a \in S)\ (\exists n \in Z^+)\ a^n \in a^{2n}S;$$

(2) *S is completely regular if and only if*

$$(\forall a \in S)\ a \in Sa^2;$$

(2') *S is completely regular if and only if*

$$(\forall a \in S)\ a \in a^2S.$$

Theorem 6. *Let S be a regular semigroup. Then S is a groupbound if and only if*

$$(\forall a \in S)\ (\exists n \in Z^+)\ a^n V(a^n) \cap V(a^n) a^n \neq \varnothing.$$

Proof. ($\Rightarrow$). Immediately.

($\Leftarrow$). Let $a \in S$ and $n \in Z^+$, and $a^n x = y a^n$ for some $x, y \in V(a^n)$. Then $a^n = a^n x a^n = y(a^n)^2$. By Lemma 5, S is groupbound. □

Theorem 7. *The following statements on a regular semigroup S are equivalent:*

(1) $(\forall e \in E(S))\ V(eS) \subseteq V(Se)$;
(1') $(\forall e \in E(S))\ V(Se) \subseteq V(eS)$;
(2) $(\forall e \in E(S))\ V(Se) = V(eS)$;
(3) *S is completely regular;*
(4) $(\forall a \in S)\ V(aS) \subseteq V(Sa)$;
(4') $(\forall a \in S)\ V(Sa) \subseteq V(aS)$;
(5) $(\forall a \in S)\ V(Sa) = V(aS)$.

Proof. We only need to show (1)$\Rightarrow$(3) and (3)$\Rightarrow$(5).

(1)$\Rightarrow$(3). Let $a \in S$ and $a' \in V(a)$. By the hypothesis, we have that $V(a') = V(a'aa') \subseteq V(Sa'a)$. Since $a \in V(a')$, $a \in V(xa'a)$ for some $x \in S$. Therefore $a = a(xa'a)a \in Sa^2$. By Lemma 5, S is completely regular.

(3)$\Rightarrow$(5). Since S is completely regular, S is a semilattice of completely simple semigroups S_α. In fact, each S_α is a $\mathcal{D}$-class of S.

Let $a, b \in S$ and $x \in V(ab)$. We shall show that $x \in V(Sa)$.

Assume that ab, ba, $x \in S_\alpha$. Choose an inverse x' of x in the $\mathcal{L}$-class L_{ba}, and hence $x' = yba$ for some $y \in S$. Therefore $x \in V(yba) \subseteq V(Sa)$.

Thus $V(aS) \subseteq V(Sa)$. Similarly $V(Sa) \subseteq V(aS)$, and hence $V(aS) = V(Sa)$, as required. □

Theorem 8. *The following statements on a regular semigroup S are equivalent:*

(1) *S is orthodox and groupbound;*
(2) $(\forall e \in E(S))\ (\forall a \in S)\ (\exists n \in Z^+)\ V((ea)^n) = V((ae)^n)$;
(3) $(\forall e \in E(S))\ (\forall a \in S)\ (\exists n \in Z^+)\ V((eae)^n) = V((ae)^n)$;
(3') $(\forall e \in E(S))\ (\forall a \in S)\ (\exists n \in Z^+)\ V((eae)^n) = V((ea)^n)$.

Proof. (1)$\Rightarrow$(2). Let $E \in E(S)$ and $a \in S$. Then there exists an enough big $N \in Z^+$ such that $(ea)^n$, $(ae)^n$ and $(eae)^n$ are all in some subgroups of S. Denote the group inverse of the element $(ea)^n$ $[(ae)^n,\ (eae)^n]$ by $(ea)^{-n}$

$[(ae)^{-n}, (eae)^{-n}]$. Then

$$\begin{aligned}(ea)^n(eae)^{-n}(ea)^n &= (ea)^n(eae)^n(eae)^{-2n}(ea)^n(ea)^n(ea)^{-n} \\ &= (eae)^n(eae)^n(eae)^{-2n}(eae)^n(ea)^n(ea)^{-n} \\ &= (eae)^n(ea)^n(ea)^{-n} \\ &= (ea)^n(ea)^n(ea)^{-n} \\ &= (ea)^n,\end{aligned}$$

$$\begin{aligned}(eae)^{-n}(ea)^n(eae)^{-n} &= (eae)^{-n}(ea)^n(eae)^n(eae)^{-2n} \\ &= (eae)^{-n}(eae)^n(eae)^n(eae)^{-2n} \\ &= (eae)^{-n},\end{aligned}$$

and hence $(eae)^{-n} \in V((ea)^n)$. Similarly $(eae)^{-n}) \in V((ae)^n)$.

(2)⇒(3). Immediately.

(3)⇒(1). By Corollary 2, S is orthodox. It remains to show that S is groupbound.

Let $a \in S$ and $a' \in V(a)$. By the hypothesis, there exists $n \in Z^+$ such that $V((a'aa')^n) = V((aa'a'aa')^n)$, i.e. $V((a')^n) = V(a(a')^{n+1})$.

By Theorem 1, $a^n \in V((a')^n)$ which gives $a^n \in V(a(a')^{n+1})$. Thus we have $a^n = a^na(a')^{n+1}a^n \in a^{n+1}Sa^n$. Therefore $a^n \in a(a^nSa^n) \subseteq a(a^{n+1}Sa^nSa^n) \subseteq a^{n+2}Sa^n \subseteq \ldots \subseteq a^{2n}Sa^n$. By Lemma 5, S is groupbound.

Since (3') is dual to (3), similarly we have (3')⇒(1). □

Remark 2. The equalities in statements (2), (3) and (3') can be changed to inclusions.

Wang Liming [5] obtained that a regular semigroup S is an orthogroup if and only if $V(ea) = V(ae)$ for all $e \in E(S)$ and $a \in S$.[1]

By the following theorem, we obtain some other sufficient and necessary conditions for a regular semigroup to be an orthogroup.

Theorem 9. *The following statements on a regular semigroup S are equivalent:*

(1) *S is an orthogroup;*
(2) $(\forall e \in E(S))\ (\forall a \in S)\ V(eae) = V(ae)$;
(2') $(\forall e \in E(S))\ (\forall a \in S)\ V(eae) = V(ea)$.

---

[1]In the doctorial thesis of H. X. Yang, he gave a characterization for a regular semigroup S to be a semilattice of rectangular commutative groups: $V(ab) = V(ba)$ for all $a, b \in S$.

Proof. (1)⇒(2). Let $e \in E(S)$ and $a \in S$. Denote the group inverse of eae $[ae]$ by $(eae)^{-1}$ $[(ae)^{-1}]$. Then

$$\begin{aligned}(ae)(eae)^{-1}(ae) &= (ae)^{-1}(ae)(ae)(eae)^{-2}(eae)(ae)\\ &= (ae)^{-1}(ae)(eae)(eae)^{-2}(eae)(eae)\\ &= (ae)^{-1}(ae)(eae)\\ &= (ae)^{-1}(ae)(ae)\\ &= (ae)^{-1},\end{aligned}$$

$$\begin{aligned}(eae)^{-1}(ae)(eae)^{-1} &= (eae)^{-2}(eae)(ae)(eae)^{-1}\\ &= (eae)^{-2}(eae)(eae)(eae)^{-1}\\ &= (eae)^{-1}.\end{aligned}$$

Therefore $V(eae) \cap V(ae) \neq \varnothing$. so $V(eae) = V(ae)$.

(2)⇒(1). By Corollary 2, S is orthodox. By (1') in Theorem 7, S is completely regular. Therefore S is an orthogroup.

Since (2') is dual to (2), similarly we have (2') ⇔ (1). □

Remark 3. The equalities in statements (2) and (2') can be changed to inclusions.

Theorem 10. *Let S be a regular semigroup. Then S is right inverse and groupbound if and only if*

$$(\forall a \in S)\ (\exists n \in Z^+)\ a^nV(a^n) \subseteq V(a^n)a^n.$$

Proof. (⇒). If $a \in S$, then there exists $n \in Z^+$ such that a^n lies in a subgroup of S. Let z be the group inverse of a^n. Then $za^n = a^nz$. If $x \in V(a^n)$, then $a^nx\mathcal{R}a^n\mathcal{R}a^nz$. Since S is right inverse and $a^nx,\ a^nz \in E(S)$, $a^nx = a^nz = za^n$. So $a^nV(a^n) \subseteq V(a^n)a^n$.

(⇐). By Theorem 6, S is groupbound. It remains to show that S is right inverse. If $e, f \in E(S)$, $e\mathcal{R}f$ (clearly $f \in V(e)$), then $f = ef = xe$ for some $x \in V(e)$. Therefore $e = fe = (xe)e = xe = f$. So S is right inverse. □

By using the method in the proof of Theorem 10, we can prove

Theorem 11. *Let S be a regular semigroup. Then S is a left C-semigroup if and only if $aV(a) \subseteq V(a)a$ for every $a \in S$.*

By Theorem 10 and its dual, we have

Theorem 12. *Let S be a regular semigroup. Then S is a $G.C$–semigroup if and only if*

$$(\forall a \in S)\ (\exists n \in Z^+)\ a^n V(a^n) = V(a^n)a^n.$$

By Theorem 11 and its dual, we have

Theorem 13. *Let S be a regular semigroup. Then S is a C–semigroup if and only if*

$$(\forall a \in S)\ aV(a) = V(a)a.$$

References

[1] J. M. Howie, *An Introduction to Semigroup Theory*, Academic Press, London, 1976.

[2] P. Y. Zhu, Y. Q. guo and K. P. Shum, *Characterization and structure of left Clifford semigroups*, Science in China, ser. A **35** (1992), no. 7, 791–805.

[3] Y. Q. Guo, X. M. Ren and K. P. Shum, *Another structure of left C–semigroups*, Advancein Math. (Chinese) **24** (1995), no. 1, 39–43.

[4] M. K. Sen, H. X. Yang and Y. Q. Guo, *G.C–semigroups I* (1995 (preprint)).

[5] L. M. Wang, *Some congruences on regular semigroups and E–solid semigroups*, Algebra Colloq. **2** (1995), no. 3, 231–240.

[6] S. Bogdanović, *Semigroups with a system of subsemigroups* (1985), Novi Sad.

Department of Pure Mathematics, University of Calcutta, 35, Ballygunge circular Road, Calcutta – 700019, India

Department of Mathematics, University of Lanzhou, Lanzhou, Gansu, 730000, China

Institute of Pure Mathematics, University of Yunnan, Kunming, Yunnan, 650091, China