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NOTE ON SEMIDIRECT PRODUCTS AND WREATH PRODUCTS OF P–REGULAR SEMIGROUPS

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Abstract. In this note, we mainly investigate the semidirect products and wreath products of P–regular semigroups. The main results are Theorem 2.4 and Theorem 3.2.

1. Introduction

The first use of semidirect product made in semigroup theory was, so far as I know, by B. H. Neumann [1] in 1960. From then on, W. R. Nico, T. Saito, G. B. Preston and H. W. Zheng [2–5] have investigated some properties and structures on the semidirect and wreath product of semigroups, especially, T. Saito [4] obtained same conditions under which $S \times_\alpha T$ is orthodox. H. W. Zheng [5] investigated semidirect product of P–regular semigrops and imposed many restrictions on S, T, P, α in order to prove that $S \times_\alpha T$ is P–regular. Now, His main result Theorem 1.2 in chap. 3 is corollary of our result (Theorem 2.4). During prof. M. K. Sen Visiting Institute of Mathematics, Yunann University, in sept.–oct., 1995, he suggest me studying the semidirect product of P–regular semigroups. In the present note, we only consider the structures of $S \times_\alpha T$, where $\alpha : S \longrightarrow Aut(T)$. But no answer is obtaained in this note, where $\alpha : S \longrightarrow End(T)$.

Let S, T be a semigroups, $End(T)$ the endomorphisam monoid of T, and write endomorphisam as exponent to the right of arguments. Let $\alpha : S \longrightarrow End(T), s \longrightarrow \alpha(s)$, be given homomorphisam. If $s \in S$ and $t \in T$, write t^s for $t^\alpha(s)$. Then, since $\alpha(s) \in End(T)$, for $s \in S$, $(tu)^s = t^s u^s$ for every $t, u \in T$, and since α is a homomorphisam, $(t^s)^r = t^{sr}$ for every $s, r \in S$.

The semidirect product $S \times_\alpha T$ is the semigroup with elements $\{(s,t) : s \in S, t \in T\}$ and multiplication $(s,t)(r,u) = (sr, t^r u)$.

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Let S be a regular semigroup, E_S the set of idempotents of S and $P \subseteq E_S$. Then $S(P)$ is called a P–regular semigroup [6] if it satisfies the following:

(1) $P^2 \subseteq E_S$;
(2) for $q \in P, qPq \subseteq P$, and
(3) for any $a \in S$, there exists $a^+ \in V(a)$ such that $aP^1a^+ \subseteq P$ and $a^+P^1a \subseteq P$.

In this case, P is called the *characteric set* of S. For $a \in S$, if there exists $a^+ \in V(a)$ such that $aP^1a^+ \subseteq P$ and $a^+P^1a \subseteq P$, then a^+ is called a *P–inverse* of a and $V_P(a)$ denote the set of all *P–inverse* of a. As usual, $V(s)$ denotes the set of inverses of $s \in S$. Further, we say that a semigroup S is *P–regular* if there exists $P \subseteq E_S$ such that (S,P) is *P–regular.*

A *P–regular* semigroup $S(P)$ is *left* [resp. *right*] *P–inverse* if $ege = ge$ $[resp. ege = eg]$ for every $e, g \in P$.

2. Semidirect product

Let $S(P_S), T(P_T)$ be P–regular semigroups, $\alpha : S \longrightarrow End(T)$ a given homomorphisam and $S \times_\alpha T$ the semidirect product of S and T. Let $Sur(T) = \{\beta \in End(T) : \beta \text{ is surjective}\}$, $Inj(T) = \{\beta \in End(T) : \beta \text{ is injective}\}$ and $Aut(T) = Sur(T) \cap Inj(T)$.

Proposition 2.1. *Let S be a regular semigroup. Then the following are equivalent:*

(1) $\alpha(S) \subseteq Sur(T)$;
(2) $\alpha(S) \subseteq Inj(T)$;
(3) $\alpha(S) \subseteq Aut(T)$.

Proof. (1) $\Rightarrow$ (2) Suppose that $e \in E_S$, then $\alpha(e) \in Sur(T)$. For every $t \in T$, there exists $t_1 \in T$ such that $t = {t_1}^e$. So $t^e = ({t_1}^e)^e = {t_1}^{e^2} = {t_1}^e = t$ since $e \in E_S$. Consequently, $\alpha(e) = i_T$ (identily mapping). Let $s_1 \in V(s)$. Then $\alpha(ss_1) = \alpha(s_1s) = i_T$ since $ss_1, s_1s \in E_S$. Thus, $\alpha(s) \in Aut(T)$.

(3) $\Rightarrow$ (2) It is obvious.

(2) $\Rightarrow$ (1) Let $e \in E_S$. Then $t^e = (t^e)^e$ for any $t \in T$. By $\alpha(s) \in Inj(T)$ for ant $s \in S$, we have $t = t^e$, that is, $\alpha(e) = i_T$. Let $s_1 \in V(s)$. Then $\alpha(ss_1) = \alpha(s_1s) = i_T$. Since ss_1 and $s_1s \in E_S, \alpha(s) \in Sur(T)$. □

In the following discussions, suppose that $\alpha : S \longrightarrow Aut(T)$.

Lemma 2.2. *Let $(s,t) \in S \times_\alpha T$. Then $(s,t)^2 = (s,t)$ if and only if $s^2 = s$ and $t^2 = t$.*

Proof. It is obvious. □

Lemma 2.3. *If $S \times_\alpha T$ is a P–regular or left P–inverse or right P–inverse semigroup, then S and T have same property.*

Proof. Suppose that P is the characteric set of $S \times_\alpha T$. Let $P_S = \{p :$ there exist $q \in T$ such that $(p,q) \in P\}$ and $P_T = \{q :$ there exist $p \in S$ such that $(p,q) \in P\}$. For any $t \in T$, let $p \in P_S$. Then every $(p_1,t_1) \in V_P((p,t))$ satisfies $t_1 \in V(t), t_1t, tt_1 \in P_T$. Thus, by Lemma 2.2 and the definition of P–regular semigroups, $S(P_S)$ and $T(P_T)$ are P–regular semigroup and have the same property as $S \times_\alpha T$. □

Notice $\alpha(s_1) = \alpha(s)^{-1}$ for every $s_1 \in V(s)$. So we write $t^{s^{-1}}$ for t^{s_1} and $1^s = 1$ for $1 \in S^1$.

Theorem 2.4. *Let S,T be semigroups and $\alpha : S \longrightarrow Aut(T)$ a given homomorphisam. Suppose that $P \subseteq E(S \times_\alpha T)$. Let $P_S = \{p :$ there exists $q \in T$ such that $(p,q) \in P\}$ and $P_T = \{q :$ there exists $p \in S$ such that $(p,q) \in P\}$. Then the semidirect $S \times_\alpha T(P)$ is a P–regular semigroup if and only if*

(1) *$S(P_S)$ and $T(P_T)$ are P–regular and*
(2) *$(\forall s' \in V_P(S))(\exists\, t' \in V(t))$ such that $t({P_T}')^{s^{-1}}t' \subseteq P_T$ and $(t'{P_T}'t) \subseteq P_T$.*

Proof. " $\Rightarrow$ " Suppose that $S \times_\alpha T(P)$ is a P– regular. Let $P_S = \{p :$ there exists $q \in T$ such that $(p,q) \in P\}$ and $P_T = \{q :$ there exists $p \in S$ such that $(p,q) \in P\}$. By Lemma 2.3, $S(P_S)$ and $T(P_T)$ are a P–regular semigroup. Let $(s_1,t_1) \in V_P(s,t^s)$. Then for any $q \in P_T^1$, there exists $p \in P_S^1$ such that $(p,q) \in P^1$ and so that $(s_1,t_1)(p,q)(s,t^s) = (s_1ps, t_1^sq^st^s) \subseteq P$. Therefore $(t_1{P_T}^1t)^s \subseteq P_T$ and it is clear that $s_1 \in V_P(s)$. Similarly, $t({P_T}^1)^{s^{-1}}t_1 \subseteq P_T$.

" $\Leftarrow$ " Let $P = P_S \times P_T$. Then it is clear that P, $P^2 \subseteq E(S \times T)$ and $(p,q)P(p,q) \subseteq P$ for every $(p,q) \in P$. Suppose that s_1,t_1 satisfy (2), then $(s_1,t_1)P^1(s,t^s) = (s_1,t_1)({P_S}^1 \times {P_T}^1)(s,t^s) = (s_1{P_S}^1s, {t_1}^s({P_T}^1)^st^s) \subseteq P_S \times P_T = P$ and $(s,t^s)P^1(s_1,t_1) = (s,t^s)({P_S}^1 \times {P_T}^1)(s_1,t_1) = (s{P_S}^1s_1, t^{ss_1}{P_T}^1 t_1) = (s{P_S}^1s_1, t{P_T}^1t_1) \subseteq P$ since $ss_1 \in E_S$. Consequently, $S \times_\alpha T(P)$ is a P– regular semigroup. □

Corollary 2.5. *S,T,P,P_S,P_T and α are as in Theorem 2.4. If, for every $q \in P_T, q^s \in P_T$, then $S \times_\alpha T(P)$ is P–regular if and only if $S(P_S)$ and $T(P_T)$ are P–regular semigroups.*

Proof. It is obvious since $q^s \in P_T$ if and only if $q \in P_T$. □

Corollary 2.6. *S,T,P,P_S,P_T and α are as in Theorem 2.4 then $S \times_\alpha T(P)$ is left P–inverse or right P–inverse if and only if S,T satisfy*

(1) *$S(P_S)$ and $T(P_T)$ have the same property, and*

(2) $(\forall\ s' \in V_P(s))(\exists\ t' \in V(t))$ *such that* $t(P_T{}')^{s^{-1}}t' \subseteq P_T$ *and* $(t'P_T{}'t)^s \subseteq P_T$.

Proof. It is clear that we notice Lemma 2.3 and Theorem 2.4. □

3. Wreath products

Let S and T be semigroups. If S acts on a set X from the left, that is $x \in X, s(rx) = (sr)x$ for every $s, r \in S$ and every $x \in X$, then the wreath product SW_XT is the semidirect product $S \times_\alpha T^X$, where $T^X = \{f : f : X \longrightarrow T \text{ is a function}\}$ is the cartesian power of T, that is $(fh)(x) = f(x)h(x)$ for every $f, h \in T_X$ and every $x \in X$, and where the homomorphisam $\alpha : S \longrightarrow End(T^X)$ is defined by $(f^s)(x) = f(sx)$ for every $s \in S$, $f \in T^X$ and $x \in X$.

The standard wreath product SWT is defined by using left regular representation of S on itself for the left S–set S.

If $|T| = 1$, then $SW_XT \cong S$. Thus assume hereafter that $|T| \geq 2$. In the following discussions, suppose that $\alpha(S) \subseteq Aut(T)$.

The following is a rephrasing of Theorem 2.4 for the wreath product: "The wreath product $SW_XT(P)$ is a P–regular semigroup if and only if

(1) $S(P_S)$ and $T^X(P_{T^s})$ are P–regular, and

(2) $(\forall\ s_1 \in V_T(s))(\exists\ t_1 \in V(t)$ for every $t \in T^X)$ $t(P_{T^s}{}^1)^{s^{-1}}t_1 \subseteq P_{T^s}$ and $(t_1 P_{T^s}{}^1 t)^s \subseteq P_{T^s}$."

Lemma 3.1. *T^X is regular if and only if T is regular.*

Proof. Let $c_t \in T^X$ such that $c_t(x) = t$ for any $t \in T$ and $x \in X$. Then it is easy to see that result holds. □

Lemma 3.2. *T^X is P–regular if and only if also is T.*

Proof. " ⇒ " Suppose that T^X is P–regular and its characteric set is P_X. Let $P_T = \{f(x_0) : f \in P_X$ for a given $x_0 \in X\}$. By $c_t, t \in T$, we denote the mapping in T^X such that $c_t(x) = t$ for all $x \in X$. Then P_T, $P_T{}^2 \subseteq E_T$ and $f(x_0)P_T{}^1 f(x_0) \subseteq P_T$. For every $t \in T$, let $f \in V_P(c_t)$, then $fP_X{}^1 c_t \subseteq P_X$ and $c_t P_X{}^1 f \subseteq P_X$. So that $f(x_0) = f(x_0)tf(x_0), t = tf(x_0)t$ and $f(x_0)g(x_0)t \in P_T, tg(x_0)f(x_0) \in P_T$ for every $g \in P_X{}^1$. Namely, $f(x_0) \in V(t), f(x_0)P_T{}^1 t \subseteq P_T$ and $tP_T{}^1 f(x_0) \subseteq P_T$. Consequently, $T(P_T)$ is P–regular.

" ⇐ " Suppose that $T(P_T)$ is P –regular. Let $P = \{c_p : p \in P_T\}$. Then P, $P^2 \subseteq E(T^X)$ and $c_p P c_p \subseteq P$ for every $c_p \in P$. By Lemma 3.1, T^X is regular. Let $f \in T^X$. Then $f(x) \in T$ for every $x \in X$. For a given $g(x) \in V_P(f(x)), g \in V(f), f(x)P_T{}^1 g(x) \subseteq P_T$ and $g(x)P_T{}^1 f(x) \subseteq P_T$. Consequently, $T^X(P)$ is P–regular. □

Theorem 3.3. *Let S, T be semigroups. Then SW_XT is P–regular if and only if S and T are P–regular.*

Proof. Suppose that SW_XT is a P–regular and its characteric set is P. Let $P_S = \{p :$ there exists $q \in T^X$ such that $(p,q) \in P\}$, $P_{T^X} = \{f :$ there exists $p \in S$ such that $(p,f) \in P\}$ and $P_T = \{f(x) :$ for a given $x \in X\}$. Then, by Theorem 2.4, $S(P_S)$ and $T^X(P_{T^X})$ are P–regular and $P_T, {P_T}^2 \subseteq E_T, f(x)P_Tf(x) \subseteq P_T$ for every $f(x) \in P_T$. By the proof of Lemma 3.2, $T(P_T)$ is P–regular. Consequently, S, T are P–rregular.

Conversly, suppose that S, T are P–regular and its characteic set P_S, P_T respectively. Let P_{T^S} be the same $\{c_p : p \in P_T\}$ as in the proving of lemma 3.2. Then T^X (P_{T^S}) is P–regular. Furthemore, ${c_p}^s = c_p$ for every $s \in S$ and $c_p \in P_T$. Then, by Corollary 2.5, SW_XT is P–regular. □

Theorem 3.4. *Let S, T be semigroups. Then SWT is P–regular if and only if S and T are P–regular semigroups.*

Proof. It is a corollary of Theorem 3.3. □

Theorem 3.5. *Let $\alpha : S \longrightarrow End(T^X)$. Then $\alpha(s)$ is injective if and only if $sX = X$.*

Proof. "$\Rightarrow$" Suppose that $sX \neq X$. Let $y \in X/sX$. Then there exists $g \in T^X$ such that $g(sx) = f(sx)$ and $g(y) \neq f(y)$. Thus $g^s = f^s$ and $g \neq f$.

"$\Leftarrow$" It is clear since $f^s(x) = f(sx)$. □

Corollary 3.6. *Let S, T be regular semigroups. Then $\alpha(S) \subseteq Aut(T^X)$ if and only if $sX = X$ for every $s \in S$.*

Corollary 3.7. *Let S, T be regular semigroups. Then $\alpha(S) \subseteq Aut(T^S)$ if and only if S is a right group.*

Proof. By corollary 3.6, $\alpha(S) \subseteq Aut(T^S)$ if and only if $sS = S$ for every $s \in S$ if and only if S is right simple if and only if S is right group (since S is regular, every idempotent of S is primitive). □

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