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COMPLETE SCHEME OF ONE-, TWO- AND THREE-DIMENSIONAL MIHEEV HOMOLOGY GROUPS

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Abstract. After introductory definitions and theorethical basis of Miheev homology, survey of results obtained by generalization of all categories of two- and three-dimensional crystallographic groups to corresponding homology groups, is given, as well as subordination scheme for categories of groups considered.

1. The classical theory of symmetry, completed at the end of 19th century by E.S. Fedorov and A. Schoenflies, was the basis for scientific research, not only for crystallography, but for the most of natural sciences. The need for a more profound knowledge about real crystal and nature surrounding us, resulted in various generalizations of classical symmetry, mostly exploring two possible ways.

The first is represented by Shubnikov antisymmetry and its generalizations – multiple antisymmetry, colored symmetry, colored simple and multiple antisymmetry, cryptosymmetry, P-symmetry, W-symmetry, etc., and could be conditionally called physical. Such generalizations do not change the geometrical nature of symmetry, which is extended by assigning to figure points several symbols denoting some general properties, and by combining geometrical transformations with the property changes. Namely, in the case of antisymmetry and colored symmetries mentioned, the signs "+" and "-", and the indexes $i = 1, 2, \ldots, p$ have a certain non-geometrical meaning regarding the space that figure belongs to [1,2,3,4]. In an additional dimension that symbols could be interpreted geometrically, making possible to describe certain categories of multi-dimensional symmetry groups by one-, two- and three-dimensional symmetry and generalized antisymmetry groups.

The other way of symmetry generalizations is represented by Nalivkin curvilinear symmetry, Miheev crystal homology [11] ("visible symmetry" according to Fedorov, or affine symmetry [12]) and Shubnikov similarity sym-

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metry [13]. They are characterized by the generalized equality and symmetry criteria for figures, resulting from partial or complete leaving of isometrism.

The survey papers on curvilinear symmetry [11] and similarity symmetry [14] could give to the reader a complete and concise information about the subject. A deficiency of such works about affine symmetry is the reason for writing this paper. Its purpose is to consider from a common point of view, all diverse results obtained by extending different categories of two- and three-dimensional symmetry groups to the corresponding homology groups.

The analysis of Miheev homology, the survey of some well known and recent results obtained by generalization of different categories of classical symmetry groups, the construction of a complete scheme of one-, two- and three-dimensional homology groups, and the explanation of new-introduced symbols (different from [11]) for such groups, are the topics of this paper.

2. Concisely, the theoretical basis of Miheev homology theory [11], derived from [15], is the following:

A discrete group H of affine transformations of a finite figure F, that transform F into itself, is called the homology group of F, and each transformation h from H is a homology transformation of F. The homology group of any finite figure is a point group, and its elements are equiaffine transformations.

For a finite figure with the symmetry group S, the results of the action of an affine transformation σ on S are:

1) The elements of symmetry of S (rotation and rotary-reflection axes of order k, inversion center and mirror planes) are transformed into the homology elements of H (perpendicular circular G_k or perpendicular elliptical μ_k, oblique circular ν_k or oblique elliptical λ_k homology axes of order k, perpendicular circular G_{ki} or perpendicular elliptical μ_{ki}, oblique circular ν_{ki} or oblique elliptical λ_{ki} rotary-reflection homology axes of order k, inversion center i, perpendicular m or oblique π reflection planes, respectively);

2) Every s from S is transformed into a homology transformation h of the new figure according to the rule $h = \sigma s \sigma^{-1}$, and the set $\{\sigma s \sigma^{-1}\}$ with σ given, and when s exausts all the discrete group S, makes the discrete group H, i.e. the action of an affine transformation on a discrete symmetry group of a finite figure results in a homology group.

By using the corollary of Maschke theorem [16], it is possible to prove the reverse: every homology group H of a finite figure we could obtain from a certain point symmetry group S, if we transform it by some affine transformation σ, i.e.

$$H = \sigma S \sigma^{-1} \qquad (1).$$

The group S is called the generating group of H.

A homology group is called crystallographic if its generating group is crystallographic. The set of all homology groups with the same generating symmetry group is called a homology class, generated by it. The generating symmetry group always belongs to its homology class.

Two homology groups H and H' are equal if there is such isomorphism between them, that their mutually corresponding transformations are equal, regarding from homology point of view.

According to this:

1) The set of symmetry elements characterizing group S consists from the same number of elements as the set of homology elements characterizing group H, derived from S according to the rule (1);

2) Equal homology groups may appear only in the same class;

3) Symmetry transformations of finite figures are the particular case of homology transformations, and symmetry groups of finite figures are the particular case of homology groups.

In transition from finite figures to space the situation is more complicated, because it is not possible to derive every discrete space group of affine transformations from some space symmetry group by a homogenous deformation; this proves the example of groups interpreting in Euclidian space pseudo-Euclidian Fedorov groups [17].

Plane or space affine transformation groups are a large class. This shows the following example: let us consider any two-dimensional latice with two arbitrary points, chosen in such way that the vector defined by them is the basic vector of this lattice in its dirrection. Then the line defined by these points is for the lattice an oblique reflection axis. Namely, thanks to the high degree of freedom for the choice of the basic parallelogram, it could be always taken that way, that the points mentioned are its toward vertices, and in every parallelogram the homology axis is incident to them. Also, if the first chosen point is fixed, the other can be chosen in an infinite number of ways, only respecting the condition that the corresponding segment cannot contain any other lattice point. This means, through every point of two-dimensional lattice there is an infinite number of oblique reflection axes, and such collection of homology elements it is not possible to obtain by a homogenous deformation of a collection of symmetry elements corresponding to any two-dimensional Fedorov group. Analogously, by using a basic parallelograme of a three-dimensional lattice, it is possible to prove that through every point there is an infinite number of rotation homology axes of order 2,3,4,6, and such collection of homology elements it is not possible to obtain by a homogenous deformation of symmetry elements corresponding to any three-dimensional Fedorov group. It is necessary to introduce additional restrictions for the transformations of regular point systems, in order to explore first even a part of discrete plane or space affine transformation groups.

The set Γ of space affine transformations, obtained from some Fedorov group Φ by an affine transformation σ, according to the rule

$$\Gamma = \sigma\Phi\sigma^{-1} \tag{2}$$

is called a space crystallographic homology group, and each $\gamma \in \Gamma$ (where $\gamma = \sigma f \sigma^{-1}$, $f \in \Phi$) is called a homology transformation of space. From the discreteness and homogeneity of Φ result the same properties of Γ. The set of all space crystallographic homology groups, derivable from a given Fedorov group Φ by all possible affine transformations σ according to the rule (2) is a homology class with the generating group Φ. Each space crystallographic homology group has a unique (to the equality) generating Fedorov group.

Every Fedorov group Φ is resembling to one from 32 crystallographic point symmetry groups G_{30}, and contains three-dimensional translational subgroup T, i.e. Φ is an extension of its translational subgroup by a crystallographic point group W: $\Phi/T \simeq W$ (where W is a group of "rotations", belonging to the transformations of group G [2]). By the action of affine transformation σ to the group Φ according to the rule (2), its translational subgroup is transformed into the translational subgroup of the group G, and the point group W corresponding to Φ is transformed into one of point crystallographic homology groups H, occuring as a group of "rotations", belonging to the transformations of the group Γ. In such a case we will say that the group Γ is resembling to the group H. Therefor, in order to generalize Fedorov groups G_3 to homology groups H_3, we need first to study all point crystallographic homology groups H_{30} resembling to them.

3. The survey of derivation of all categories of finite and infinite two- and three-dimensional crystallographic homology groups will begin with the space groups H_3. To their large derivation proceeded studies, extending and making more exact Miheev affine symmetry theory and his results of the derivation of groups H_{30} [11].

The derivation of point crystallographic homology groups H_{30} is realized in [14,18] by all possible affine deformations of crystallographic point groups G_{30} according to the rule (1), using the distribution of G_{30} into three classes: axial, central and planar [19]. The corresponding homology groups are named in the same way. By using geometrical arguments, it was proved the existence of exactly 215 (and not 218, as in [11]) groups H_{30}, consisting of 70 axial, 70 central and 75 planar groups [14,18]. The list of 215 H_{30} is given and compared with results [11] in the first chapter of the dissertation [15]. The analogous list of 1848 space crystallographic homology groups H_3, consisting of 570 symmorphic, 408 hemisymmorphic and 870 asymmorphic groups, derived by using all affine deformations of Fedorov groups G_3, is given in the second chapter of the same dissertation.

4. The survey of derivation of the finite and infinite plane-linear crystallographic homology groups will begin with band groups. Before solving this problem, let us remain that symmetry groups of finite bands consist from the complete symmetry group of rectangular parallelepiped and all its subgroups. That groups preserve invariant in 3-space a two-dimensional plane and a line with the invariant point, laying in this plane, i.e. they are symmetry groups of the category G_{3210}, so they could be modelled by symmetry and antisymmetry groups of finite frizes G^1_{210}, if the signs "+" and "-" assigned to the points of finite friezes are interpreted as the position of the point with regard to the invariant plane (above-below) [20].

Because by homology and antihomology groups of finite frizes H^1_{210} cannot be modelled all the homology groups of finite bands H_{3210}, the groups mentioned are not sufficient to obtain homology groups having oblique axes or planes, which are in a slanting position with regard to the invariant plane of a finite band. The reasons for this are following: e.g. to the rotation around 2-center in a homology group of finite friezes H_{210} corresponds the rotation around 2-axis perpendicular to the invariant plane in the group of finite homology bands H_{3210}; to a line reflection with invariant homology axis corresponds the plane reflection with invariant homology plane, which is perpendicular to the invariant plane, etc.

All homology groups of finite bands are derived in [20] from 45 their axial homology and antihomology groups given in Table 1. Generating symmetry groups G_{3210} are denoted by International symbols, and the generated homology groups H_{3210} are given by the set of homology elements, obtained from the elements characterizing symmetry group S by the action of the transformation σ, according to the rule (1). In the table, the symmetry groups are denoted by S, and the corresponding homology groups by H ($H = \sigma S \sigma^{-1}$).

Table 1: Point subgroups of infinite band groups

S	$H = \sigma S \sigma^{-1}$
$1; 211; 121; 112$	$1; G_2 11, \lambda_2 11; 1G_2 1, 1\lambda_2 1; 11G_2, 11\lambda_2$
222	$G_2G_2G_2; \lambda_2\lambda_2G_2; \lambda_2G_2\lambda_2; G_2\lambda_2\lambda_2; \lambda_2\lambda_2\lambda_2$
$i; 2/m11; 12/m1;$	$i; G_2/m11, \lambda_2/\pi 11; 1G_2/m1;$
$112/m; mmm$	$1\lambda_2/\pi 1, 11G_2/m, 11\lambda_2/pi; mmm, \pi\pi m, \pi m \pi, m\pi\pi, \pi\pi\pi$
$m11; 1m1; 11m$	$m11, \pi 11; 1m1, 1\pi 1; 11m, 11\pi$
$mm2$	$mmg_2, \pi\pi G_2, \pi m \lambda_2, m\pi\lambda_2, \pi\pi\lambda_2$

$m2m$	$mG_2m, \pi\lambda_2m, \pi G_2\pi, m\lambda_2\pi, \pi\lambda_2\pi$
$2mm$	$G_2mm, G_2\pi\pi, \lambda_2m\pi, \lambda_2\pi m, \lambda_2\pi\pi$

The orientation of homology elements in the notation of groups H_{3210} corresponds to the same rules for symmetry groups G_{3210} in International symbols. In symmetry groups, the line / denotes the perpendicularity of elements adjacent to it, but in homology groups such intersecting elements are not always perpendicular. The perpendicularity holds only for elements G_2 and m.

In the list of 45 groups of the category H_{3210} are included all the homology groups of the category H_{210}, because the groups of one-sided finite bands: 1, $11G_2$, $m11$, $\pi 11$, $1m1$, $1\pi 1$, mmG_2 and $\pi\pi G_2$, are the copy of homology groups of finite bands, if in the groups mentioned G_2 is a 2-rotation center, and m and π are corresponding perpendicular and oblique reflection in homology axis. That way, 5 symmetry groups of finite bands G_{210}: 1, 2, $m1$, $1m$ and mm (the symmetry group of a rectangle and its subgroups), generate 8 homology groups (the homology group of a complete parallelogram of a general form, a rectangle homology group and all their subgroups) [2].

Naturally, the translational subgroup of an infinite band group is one-dimensional, because it preserves invariant a plane in space and a line belonging to the plane, i.e. it is a symmetry group of the category G_{321} [1,2]. Hence, the existing 1-1 correspondence between symmorphic groups of this category and their point subgroups G_{3210}, holds as well for the corresponding homology groups. Therefore, if we write to the every symmetry group symbol S from Table 1 the translational subgroup symbol P on the left, we will obtain 16 symmorphic symmetry groups of bands [1,2]. In transition from symmetry groups S to homology groups $H = \sigma S \sigma^{-1}$, the translational group P is transformed into the translational homology group, so the same procedure results in 45 symmorphic band homology groups [20].

Because every affine transformation of a figure is 1-1 and preserves collinearity, to generating symmorphic, hemisymmorphic and asymmorphic symmetry groups correspond generated symmorphic, hemisymmorphic and asymmorphic homology groups, respectively. This means, that symmmetry elements of generating groups are transformed into the homology elements having the same name [20]. From 15 hemisymmorphic and asymmorphic symmetry groups of bands are derived 57 hemisymmorphic and asymmorphic homology groups of bands [1,2,20].

Their complete list is given in Table 2. The symbols m, π, G_2 and λ_2 are used in the same sense as before. A glide reflection plane is denoted by a, and its transform– a plane of oblique glide reflection by α. In the same way, by G_{2_1} and λ_{2_1} are denoted a perpendicular and oblique axis 2_1, respectively.

This way, from 31 generating symmetry groups of bands G_321 are derived 102 homology groups of bands H_{321}, where in this number are included the generating symmetry groups as well.

Table 2: Band groups

S	$H = \sigma S \sigma^{-1}$
$p1a1; p11a1;$	$p1a1, p1\alpha1; p11a, p11\alpha;$
$p12/a1; p112/a$	$p1g_2/a1, p1\lambda_2/\alpha1; p11G_2/a, p11\lambda_2/\alpha$
$pma2$	$pmaG_2, pm\alpha\lambda_2, p\pi a\lambda_2, p\pi\alpha G_2, p\pi\alpha\lambda_2$
$pm2a$	$pmG_2a, pm\lambda_2\alpha, p\pi\lambda_2a, p\pi G_2\alpha, p\pi\lambda_2\alpha$
$p2_1ma$	$pG_{2_1}am, pG_{2_1}\pi\alpha, p\lambda_{2_1}m\alpha, p\lambda_{2_1}\pi\alpha$
$p2aa$	$pG_2aa, pG_2\alpha\alpha, p\lambda_2a\alpha, p\lambda_2\alpha a, p\lambda_2\alpha\alpha$
$pmma$	$pmma, p\pi\pi a, p\pi m\alpha, pm\pi\alpha, p\pi\pi\alpha$
$pmam$	$pmaa, p\pi\alpha m, p\pi a\pi, pm\alpha\pi, p\pi\alpha\pi$
$pmaa$	$pmaa, p\pi\alpha a, p\pi a\alpha, pm\alpha\alpha, p\pi\alpha\alpha$
$p2_111; p2_1m1$	$pG_{2_1}11, p\lambda_{2_1}11; pG_{2_1}/m11, p\lambda_{2_1}/\pi11$
$p2_122$	$pG_{2_1}G_2G_2, pG_{2_1}\lambda_2\lambda_2, p\lambda_{2_1}G_2\lambda_2, p\lambda_{2_1}\lambda_2G_2, p\lambda_{2_1}\lambda_2\lambda_2$
$p2_1am$	$pG_{2_1}am, p\lambda_{2_1}a\pi, p\lambda_{2_1}\alpha m, pG_{2_1}\alpha\pi, p\lambda_{2_1}\alpha\pi$

Let us notice that from 102 band groups H_{321} is possible to separate all homology groups of infinite friezes H_{21}, because the symmetry groups $p1$, $p112$, $pm11$, $p1m1$, $p1a1$, $pmm2$ and $pma2$ of one-sided bands are the copies [2] of 7 frieze groups G_{21}, so the 12 gomology groups generated by them, without rotation axes and reflection planes in a slanting position with regard to the invariant plane of a band, exaust all the diferent homology groups of infinite friezes H_{21}.

5. By using tablet homology groups H_{320} [21], preserving invariant in 3-space a plane, a line and their intersection point, it is possible to solve two problems: to generalize rod symmetry groups (three-dimensional line groups) G_{31}, the groups without invariant points, having invariant 3-space and a line [1,2] and to generalize layer symmetry groups (three-dimensional plane groups) G_{32}, the groups without invariant points or lines, having invariant 3-space and a plane, in order to derive homology groups of rods H_{31} and layers H_{32}, respectively.

For this, according Section 2 of this work, we need to transform by all possible affine transformations σ according to the rule (2) the rod and layer symmetry groups S. This way, every resembling point symmetry group S' is transformed into the point homology group $H = \sigma S' \sigma^{-1}$, resembling to the corresponding infinite homology group $H = \sigma S \sigma^{-1}$. The point subgroups G_{32}, which are copies of the crystallographic symmetry groups of tablets

G_{320}, coincide to the analogous symmetry groups of finite rods G_{310}, occuring as point subgroups of the groups G_{31}. Hense, the rod H_{31} and layer H_{32} homology groups are resembling to the tablet homology groups H_{320}.

According to this, the groups H_{320} are the basis for the solution of the problems mentioned. All of them are derived in [21] by using 145 tablet axial crystallographic homology and antihomology groups, distributed in subclasses, and listed in Table 3. The tablet symmetry groups G_{320} are denoted by International symbols, and the generated homology groups are denoted by the corresponding set of homology elements characterizing them. If a homology group is characterized by homology axis of order k and by the set of k homology 2-axes of the same kind intersecting it, or by the same main axis and by the set of k homology planes of the same kind, the symbol of such homology group could be reduced to the form, analogous to the International symbol of the generating group (e.g. $G_4 22$ instead $G_4 2222$, $\lambda_4 22$ instead $\lambda_4 2222$, $\lambda_4 \pi\pi$ instead $\lambda_4 \pi\pi\pi\pi$, etc.).

In the list of H_{320} (Table 3) are included 21 crystallographic homology groups of one-sided rosettes H_{20}, derived in [22]. They are connected with the groups H_{320} in the same way, as they are connected the homology groups of finite friezes H_{210} to the homology groups of finite bands H_{3210} [20,21].

Table 3: Catalogue of tablet symmetry groups and their generalizing homology groups

Tablet groups

Singony	S	$H = \sigma S \sigma^{-1}$
T	$1; \bar{1}$	$1; \bar{1}$
	$112; 121$	$11G_2, 11\lambda_2; 1G_2 1, 1\lambda_2 1$
M	$m11; 11m$	$m11, \pi 11; 11m, 11\pi$
	$112/m; 2/m11$	$11G_2/m, 11\lambda_2/\pi; G_2/m11, \lambda_2/\pi 11$
	222	$G_2G_2G_2, \lambda_2G_2\lambda_2, G_2\lambda_2\lambda_2, \lambda_2\lambda_2\lambda_2$
O	$mm2$	$mmG_2, m\pi\lambda_2, \pi\pi G_2, \pi\pi\lambda_2$
	$m2m$	$mG_2m, \pi G_2\pi, m\lambda_2\pi, \pi\lambda_2 m, \pi\lambda_2\pi$
	mmm	$mmm, \pi\pi m, \pi m\pi, \pi\pi\pi$
	$4; \bar{4}$	$G_4, \lambda_4, \mu_4, \nu_4;$ $G_{4i}, \lambda_{4i}, \mu_{4i}, \nu_{4i}$
	422	$G_4G_2G_2, \lambda_4G_2\lambda_2\lambda_2\lambda_2, \lambda_4\lambda_2\lambda_2,$ $\mu_4G_2\lambda_2G_2\lambda_2, \mu_4\lambda_2\lambda_2, \nu_4G_2\lambda_2\lambda_2\lambda_2, \nu_4\lambda_2\lambda_2$
	$\bar{4}m2(=\bar{4}2m)$	$G_{4i}mG_2, \lambda_{4i}m\lambda_2\pi\lambda_2, \lambda_{4i}\pi\lambda_2\pi G_2,$ $\lambda_{4i}\pi\lambda_2, \mu_{4i}m\lambda_2, \mu_{4i}\pi G_2, \mu_{4i}\pi\lambda_2,$ $\nu_{4i}m\lambda_2\pi\lambda_2, \nu_{4i}\pi\lambda_2\pi G_2, \nu_{4i}\pi\lambda_2$
Q	$4mm$	$G_4mm, \lambda_4m\pi\pi\pi, \lambda_4\pi\pi,$

		$\mu_4 m\pi, \mu_4\pi\pi, \nu_4 m\pi\pi\pi, \nu_4\pi pi$
	$4/m$	$G_4/m, \lambda_4/\pi, \mu_4/m, \nu_4/\pi$
	$4/mmm$	$G_4/mmm, \lambda_4/\pi m\pi\pi\pi, \lambda_4/\pi\pi\pi,$
		$\mu_4/mm\pi m\pi, \mu_4/m\pi\pi, \nu_4/\pi m\pi\pi\pi, \nu_4/\pi\pi\pi$
	$3; \overline{3}$	$G_3, \lambda_3, \mu_3, \nu_3;$
		$G_{3i}, \lambda_{3i}, \mu_{3i}, \nu_{3i}$
R	$321(=312)$	$G_3G_2, \lambda_3G_2\lambda_2\lambda_2, \lambda_3\lambda_2,$
		$\mu_3G_2\lambda_2\lambda_2, \mu_3\lambda_2, \nu_3G_2\lambda_2\lambda_2, \nu_3\lambda_2$
	$\overline{3}m1(=\overline{3}1m)$	$G_{3i}m1, \lambda_{3i}m\pi\pi, \lambda_{3i}\pi 1,$
		$\mu_{3i}m\pi\pi, \mu_{3i}\pi 1, \mu_{3i}m\pi\pi,$
		$\mu_{3i}\pi 1, \nu_{3i}m\pi\pi, \nu_{3i}\pi 1$
	$3m1(=31m)$	$G_3m1, \lambda_3m\pi\pi, \lambda_3\pi 1,$
		$\mu_3m\pi\pi, \nu_3m\pi\pi, \nu_3\pi 1, \mu_3\pi 1$
	6	$G_6, \lambda_6, \mu_6, \nu_6$
H	622	$G_6G_2G_2, \lambda_6G_2\lambda_2\lambda_2\lambda_2\lambda_2\lambda_2, \lambda_6\lambda_2\lambda_2,$
		$\mu_6G_2\lambda_2\lambda_2G_2\lambda_2\lambda_2, \mu_6\lambda_2\lambda_2, \nu_6G_2\lambda_2\lambda_2\lambda_2\lambda_2\lambda_2, \nu_6\lambda_2\lambda_2$
	$\overline{6}$	$G_{6i}, \lambda_{6i}, \mu_{6i}, \nu_{6i}$
	$\overline{6}2m(=\overline{6}m2)$	$G_3/mmmm, \lambda_3/\pi m\pi\pi\lambda_2\lambda_2\lambda_2, \lambda_3/\pi\pi\pi\pi G_2\lambda_2\lambda_2,$
		$\lambda_3/\pi\pi\pi\pi\lambda_2\lambda_2\lambda_2, \mu_3/\pi m\pi\pi G_2\lambda_2\lambda_2, \mu_3/m\pi\pi\pi\lambda_2\lambda_2\lambda_2,$
		$\nu_3/\pi m\pi\pi\lambda_2\lambda_2\lambda_2,$
		$\nu_3/\pi\pi\pi\pi G_2\lambda_2\lambda_2, \nu_3/\pi\pi\pi\pi\lambda_2\lambda_2\lambda_2$
	$6/m$	$G_6/m, \lambda_6/\pi, \mu_6/m, \nu_6/\pi$
	$6/mmm$	$G_6/mmm, \lambda_6/\pi m\pi\pi\pi\pi\pi, \lambda_6\pi\pi\pi,$
		$\mu_6/m\pi\pi, \mu_6/mm\pi\pi m\pi\pi, \mu_6/\pi m\pi\pi\pi\pi\pi, \nu_6/\pi\pi\pi$
	$6mm$	$G_6mm, \lambda_6m\pi\pi\pi\pi\pi, \lambda 6\pi\pi,$
		$\mu_6m\pi\pi m\pi\pi, \mu_6\pi\pi, \nu_6m\pi\pi\pi\pi\pi, \nu_6\pi\pi$

Because the translational subgroups of groups G_{31} and G_{321} are one-dimensional, the derivation of symmorphic rod homology groups is analogous to the derivation of the band homology groups. If in the symbols of groups from Table 3 on the left we write the symbol P, denoting a translational subgroup, from the second coloumn we will obtain 31 symmorphic symmetry group of rods [1,2], and from the third coloumn 145 symmorphic homology groups of rods, derived from the generating symmetry groups according to the rule (2). E.g. the rod symmetry group $pmm2$ generates 4 homology groups of rods: $pmmG_2$, $pm\pi\lambda_2$, $p\pi\pi G_2$, $p\pi\pi\lambda_2$.

Each of 13 hemisimorphic symmetry groups of rods: pc, $p2/c11$, $pcc2$, $p2cm$, $p3c$, $p4cc$, $p6cc$, $pccm$, $p\overline{4}2c$, $p\overline{6}c2$, $p4/mcc$, $p6/mcc$, $p\overline{3}c$ [1,2], generates the same nubber of groups as the corresponding generating symmorphic group. E.g. the group $pcc2$ generates 4 hemisymmorphic homology groups of rods: $pccG_2$, $pc\gamma\lambda_2$, $p\gamma\gamma\lambda_2$, $p\gamma\gamma G_2$. In their symbols, by c is denoted a plane of perpendicular glide reflection, and by γ an oblique glide reflection plane.

That samples show how to obtain, by using "rotation" groups given in

Table 3, belonging to the transformations of 13 hemisymmorphic groups mentioned, the complete list of all generated 78 hemisymmorphic homology groups of rods.

Each of asymmorphic symmetry groups of rods $p2_1$ and $p112_1/m$ generate two homology groups: pG_{2_1} and $p\lambda_{2_1}$, $p11G_{2_1}/m$ and $p11\lambda_{2_1}/\pi$, respectively. Each of 13 groups $p3_k$ $(k = 1,2)$, $p4_r$ $(r = 1,2,3)$, $p6_q$ $(q = 1,2,3,4,5)$, $p222_1$, $p4_2/m$ and $p6_3/m$ generates 4 homology groups, and each of 10 groups $p3_k2$ $(k = 1,2)$, $p4_r22$ $(r = 1,2,3)$ and $p6_q22$ $(q = 1,2,3,4,5)$ generates 5 homology groups, each of the groups $p6_3mc$ and $p6_3/mcm$ generates 9, and each of groups $p4_2mc$ and $p4/mcm$ generates 10 homology groups.

Hence, 31 asymmorphic symmetry group of rods generate 174 homology grups. As the final result, we may conclude that they are 397 crystallographic homology groups of rods H_{31}, where in this number are included 75 generating crystallographic symmetry groups of rods G_{31}.

The derivation of layer homology groups H_{32} is more complicated then the derivation of rod groups, because the translational subgroup of layer groups is not one-dimensional as the translational subgroup of rod groups, but two-dimensional [1,2]; this results in more diverse posibilities for different positions of translational vectors from the translational subgroup, with regard to the other homology elements.

The property mentioned is essential, because implyes that two layer homology groups are equal, if there is such isomorphism between them, that their mutually corresponding transformations are equal from the point of view of homology, and that the mutually corresponding homology elements are equally placed with regard to the translation vectors of their translational subgroups [15,20].

If in Table 3 to the each of 36 groups in the second coloumn and to generated groups in the third coloumn we write on the left the symbol P, and to the symbols of 7 groups of monoclinic and orthogonal syngony: 121, $m11$, $2/m11$, 222, $mm2$, $m2m$ and mmm in the same coloumns symbol C, we will obtain 43 symmorphic symmetry groups of layers [1,2] and 223 symmorphic homology groups of layers generated by them.

In transition from the symmorphic to hemisymmorphic groups, it is possible to conclude that 16 hemisymmorphic symmetry groups of layers [1,2] generate 80 hemisymmorphic homology groups. Analogously, in transition from the symmorphic layer groups to asymmorphic, 21 asymmorphic symmetry group of layers [1,2] generate 111 homology groups of layers.

The 80 symmetry groups of layers (43 symmorphic + 16 hemisymmorphic + 21 asymmorphic) generate 414 homology groups of layers (223 symmorphic + 80 hemisymmorphic + 111 asymmorphic).

As a reliabile orientation for tabulating groups in question, we will show the number of homology groups generated by each symmetry group of layers. Trying to be concise, instead of complete symbols of generating layer groups, we are giving only their ordering numbers, according to the table P3 [2], where the groups G_{32} are given together with their corresponding plane antisymmetry groups G_2^1.

The layer groups $P1$ and $P\bar{1}$ (as well as the analogous rod groups) do not generate nontrivial homology groups. Each layer groups, denoted in Table P3 [2] by the ordering number: a) 3,4,...,18 generate two homology groups each ($16 \times 2 = 32$ homology groups); b) 19, 20, 22, 23, 33, 34, 37, 39, 41, 44, 46, 47,..., 52, 65, 73, 74, 75 generate 4 homology groups each ($22 \times 4 = 88$ homology groups); c) 21, 24, 25,...,32, 35, 36, 38, 40, 42, 43, 45 generate 5 homology groups each ($17 \times 5 = 85$ homology groups); d) 67, 68,...,72 generate 7 groups each ($6 \times 7 = 42$ homology groups); e) 76, 77,...,80 generate 9 grpups each ($5 \times 9 = 45$ homology groups); f) 53, 54,...,64 generate 10 groups each ($12 \times 10 = 120$ homology groups).

Their distribution according to the classes and families it is possible to follow according to Table 4, where for example, are given certain generating symmorphic and not-symmorphic groups with the common "rotational" subgroup belonging to their transformations, and their generated homology groups with the "rotational" subgroups belonging to the same class. The symbols G_k, λ_k, μ_k, ν_k, m and π have the same meaning, as in Tables 1,2,3. By the symbols a, b, n in Table 4 are denoted (according to International symbol rules) planes of perpendicular glide reflection, with different translation vectors, and by symbols α, β, γ, in the third coloumn are denoted the corresponding planes of oblique glide reflection.

Table 4: Layer groups

S	$H = \sigma S \sigma^{-1}$
$Pm2m$	$PmG_2m, Pm\lambda_2\pi, P\pi G_2\pi, P\pi\lambda_2\pi, P\pi\lambda_2 m$
$Pm2a$	$PmG_2a, Pm\lambda_2\alpha, P\pi G_2\alpha, P\pi\lambda_2\alpha, P\pi\lambda_2 a$
$Pb2b$	$PbG_2b, Pb\lambda_2\beta, P\beta G_2\beta, P\beta\lambda_2\beta, P\beta\lambda_2 b$
$Pm2_1b$	$PmG_{2_1}b, Pm\lambda_{2_1}\beta, P\pi G_{2_1}\beta, P\pi\lambda_{2_1}\beta, P\pi\lambda_{2_1}b$
$Pb2_1m$	$PbG_{2_1}m, Pb\lambda_{2_1}\pi, P\beta\lambda_{2_1}m, P\beta G_{2_1}\pi, P\beta\lambda_{2_1}\pi$
$Pb2_1a$	$PbG_{2_1}a, P\beta\lambda_{2_1}a, Pb\lambda_{2_1}\alpha, P\beta G_{2_1}\alpha, P\beta\lambda_{2_1}\alpha$
$Pm2_1n$	$PmG_{2_1}n, Pm\lambda_{2_1}\nu, P\pi\lambda_{2_1}n, P\pi G_{2_1}\nu, P\pi\lambda_{2_1}\nu$
$Cm2m$	$CmG_2m, Cm\lambda_2\pi, C\pi\lambda_2 m, C\pi G_2\pi, P\pi\lambda_2\pi$
$Cm2a$	$CmG_2a, Cm\lambda_2\alpha, C\pi\lambda_2 a, C\pi G_2\alpha, C\pi\lambda_2\alpha$

From the given layer groups H_{32} it is possible to separate all 39 crystallographic infinite two-dimensional homology groups H_2, derived in [23], because they are connected with layer groups H_{32} in the same way as the frieze groups H_{21} with band groups H_{321}. By the two-dimensional homology and antihomology groups H_2^1 discussed in [23] are modelled not all, but only 189 layer homology groups, without homology oblique elements in a slanting position with regard to the invariant plane of layer.

The number of non-isomorphic groups in each of the categories mentioned, according to Section 2 of this work, will be the same as the number of non-isomorphic group in the corresponding generating categories.

6. By the derivation of rod H_{31} and layer homology groups H_{32} it is completed the study of all categories of two-dimensional and three-dimensional crystallographic homology groups. The one-dimensional homology groups of line segments H_{10} and lines H_1 are identical with one-dimensional symmetry groups G_{10} and G_1, respectively.

As we can conclude from the works [11,15,18,20,21,22,32] and from this work, the numbers of the homology groups obtained in the corresponding categories are: 2(2) H_{10} ($= G_{10}$); 2(2) H_1 ($= G_1$); 21(9) H_{20}; 39(17) H_2; 8(3) H_{210}; 12(4) H_{21}; 215(18) H_{30}; 1848(219) H_3, 145(14) H_{310} ($=H_{320}$); 397(36) H_{31}; 414(34) H_{32}; 45(4) H_{3210}; 102(6) H_{321}, where the numbers of non-isomorphic groups are given in parentheses.

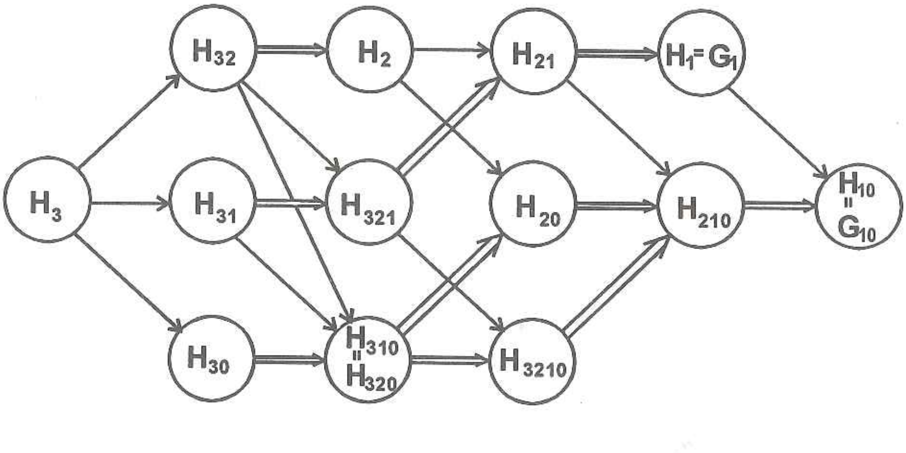

Figure 1: Subordination scheme for all categories of one-, two- and three-dimensional homology groups.

Figure 1 (constructed in the same manner as Fig. 15 [3]), represents the subordination scheme for all categories of one-, two- and three-dimensional homology groups. The simple arrows denote that groups of other category are subgroups of groups belonging to first, and double arrows indicate that the other category is included in first.

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