FILOMAT (Niš) 10 (1996), 33–48

PRODUCT OF TURÁN QUADRATURES FOR SPHERE, CONE, CYLINDER AND TORUS

Miodrag M. Spalević

Abstract. We continue the construction of a cubature formulas for approximate calculations of multiple integrals, which is starting in [8], for regions: the Sphere, the Cone, the Cylinder, the Torus, by using combinations, or products, of the generalized Turán quadratures (see [4],[5],[7]). Here $s \in N \cup \{0\}$. The particular case $s = 0$ is given in [1],[12]. Some numerical examples are included.

1. Introduction

The purpose of our consideration is a construction of cubature formulas, for approximate calculation of multiple integrals, by using the generalized Turán quadrature formulas, i.e., their products. A way for determining the nodes and the coefficients of the generalized Gauss–Turán quadrature formulas

$$\int_a^b p_1(t)g(t)\,dt \cong \sum_{i=0}^{2s}\sum_{\nu=1}^{m} A_{i,\nu}g^{(i)}(\tau_\nu), \tag{1.1}$$

$$\int_a^b p_1(t)g(t)\,dt \cong \sum_{k=0}^{p}\left[\alpha_k g^{(k)}(a)+\beta_k g^{(k)}(b)\right]+\sum_{i=0}^{2s}\sum_{\nu=1}^{m} A^L_{i,\nu}g^{(i)}(\tau_\nu), \tag{1.2}$$

is given in the papers [4], [5], [7]. The degree of exactness of the formula (1.1) is $2(s+1)m-1$. Formula (1.2) is the generalized formula of Lobatto type. The degree of exactness of such formula is $2(s+1)m+2p+1$. $t \mapsto p_1(t)$ is a nonnegative weight function for an interval $[a,b] \subset R$.

Received March 24, 1996; Revised December 7, 1996

1991 *Mathematics Subject Classification*: 03F65, 13A99.

Key words and phrases: "product" cubatures, Turán quadratures, formulas of the type, a polynomial of degree.

An iterative process for computing the coefficients of s–orthogonal polynomials (whose zeros are the nodes of the formula (1.1)) in a special case, when the interval $[a, b]$ is symmetric with respect to the origin and the weight function is an even function, was proposed by Vincenti [13]. He applied his process to the Legendre case. When n and s increase, the process becomes numerically unstable.

In [4] (see also [2]) is given a stable method for numerically constructing s–orthogonal polynomials and their zeros. In [5] was given a numerical procedure for finding the coefficients in (1.1). Some alternative methods were proposed in [11] (see also [10]) and in [2].

A way for determining the nodes and the coefficients of the formulas (1.2) was given in [7].

By using the results performed in [4],[5],[7] we can computing a multiple integrals for some regions of integration by using a products of formulas (1.1) or (1.2). As basis for our construction we use the results from [1], [12]. We shall make a generalization of "product" formulas obtained using combinations of Gauss quadrature formulas, by replacing these to the generalized Turán quadrature formulas. "Product" formulas in [1], [12] are the special case ($s = 0$) of the "product" formulas which shall be construct here ($s \in N \cup \{0\}$). Our formulas contain the values of integrand in nodes and the values of a partial derivatives of order not exceeding $2sn$.

So, because of a much number of pieces of information in nodes, we reduce the number of nodes of cubature formulas and we hold their degree of exactness.

Hence, we cosider the cubature formula

(1.3)
$$\int_{R_n} w(x_1, \ldots, x_n) f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n \cong \sum_{i=1}^{N} \sum_{j \in J} B_{ij} f^{(j)}(\nu_{i,1}, \ldots, \nu_{i,n})$$

which can be construct by using the combinations, or products, formulas for regions of dimensions $< n$.

$w(x_1, \ldots, x_n)$ is a nonnegative weight function which is defined on region $R_n \subset E_n$, where E_n is n–dimensional Euclidean space. B_{ij} are coefficients of cubature formula (1.3) and

$$f^{(j)}(\nu_{i,1}, \ldots, \nu_{i,n}) = \frac{\partial^{|j|} f(\nu_{i,1}, \ldots, \nu_{i,n})}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}}$$

are a partial derivatives of the integrand f in nodes $\nu_i \equiv (\nu_{i,1}, \ldots, \nu_{i,n})$, for $i = 1, \ldots, N$. $j \in J$ is the multiindex with the nonnegative integer components $j = (j_1, \ldots, j_n)$, $|j| = j_1 + \cdots + j_n$. In the general case: $J = J(i)$;

$j_p \in J_{i_p}$; $i = 1,\dots,N$, $p = 1,\dots,n$, and $J_{i_p} = \{0,1,\dots,2s_{i_p}\}$, $s_{i_p} \in \{s_1, s_2, \dots, s_m\}$; $s_l \in N \cup \{0\}$, $l = 1,\dots,m$. This case can be consider by using the results from [3],[6]. We shall assume, not reducing generality, that

$$J_1 = J_2 = \ \dots\ = J_n = \{0,1,2,\ \dots\ ,2s\}, \quad s \in N \cup \{0\}.$$

Hence, for multiindex j holds $j = (j_1,\dots,j_n) \in J = J_1 \times \dots \times J_n = J_1^n$.

We cannot construct "product" cubature formulas (1.3) for an arbitrary region $R_n \subset E_n$, but we can to construct such formulas for the "nice" regions which are often encountered. These include the n–cube, the n–sphere, the n–simplex, and various cones, pyramids, prisms, cylinders, and so forth.

In most cases n one–dimensional formulas (1.1) or (1.2), each of degree of exactness d, are combined to give a formula (1.3) of degree d for R_n. The regions we consider do not exhaust the regions for which product formulas (1.3) can be constructed, the number of such regions is large. Here, we shall consider the formulas (1.3) for regions which are considered in [1] and [12]: the Sphere, the Cone, the Cylinder, the Torus (see and [8], [9]).

For regions such as the n–cube, n–sphere, n–simplex one can construct product formulas (1.1) or (1.2) each of which use, say, m–points, $(2s+1)$–values of derivatives of integrand in node and have degree d (see and [8], [9]).

In all the cases in which partial derivatives of integrand are relative simply computing, "product" cubatures (1.3) are of particular importance for approximate calculation of such integrals.

2. The n–Sphere

Let R_n be a bounded n–dimensional region which contains the origin θ and let Y_n be the surface of R_n. We assume that R_n is starlike with respect to θ. By this we mean that each ray which begins at θ intersects Y_n in exactly one point. We assume we are given a formula of the type (1.3) of degree d for Y_n:

$$\int_{Y_n} u(x_1,\dots,x_n)\, d\sigma \cong \sum_{j=1}^{N_0} \sum_{k\in K} B_j^{(k)} u^{(k)}(\mu_{j,1},\dots,\mu_{j,n}). \tag{2.1}$$

(We do not assume that the points in this formula lie on Y_n, but it is more desirable if they do.)

Let r be a real number > 0 and define $rY_n = \{rz |\ z \in Y_n\}$. Then

$$\int_{rY_n} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}\, d\sigma = \int_{Y_n} r^{n-1}(rx_1)^{\alpha_1} \dots (rx_n)^{\alpha_n}\, d\sigma$$

$$= r^{n-1+|\alpha|} \int_{Y_n} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}\, d\sigma,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is multiindex and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Therefore

$$
\begin{aligned}
\int_{R_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n}\, dx_1 \ldots dx_n &= \int_0^1 \left[\int_{rY_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n}\, d\sigma \right] dr \\
&= \int_0^1 r^{n-1+|\alpha|}\, dr \int_{Y_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n}\, d\sigma.
\end{aligned} \tag{2.2}
$$

Let

$$
\int_0^1 r^{n-1} h(r)\, dr \cong \sum_{\nu=1}^{m} \sum_{i=0}^{2s} A_{i,\nu} h^{(i)}(r_\nu) \tag{2.3}
$$

be a formula of the type (1.1) of degree of exactness d. As an immediate consequence of (2.2), by using the formula (2.1), we can construct the formula (1.3) of degree d for region R_n

$$
\begin{aligned}
&\int_{R_n} f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \\
&\cong \sum_{\nu=1}^{m} \sum_{j=1}^{N_0} \sum_{l=(i,k)\in I\times K} A_{i,\nu} B_j^{(k)} f^{(l)}(r_\nu \mu_{j,1}, \ldots, r_\nu \mu_{j,n}),
\end{aligned} \tag{2.4}
$$

where

$$
f^{(l)}(r_\nu \mu_{j,1}, \ldots, r_\nu \mu_{j,n}) = \frac{\partial^{i+|k|} g(r_\nu, \mu_{j,1}, \ldots, \mu_{j,n})}{\partial r^i \partial x_1^{k_1} \ldots \partial x_n^{k_n}}
$$

and $g(r, x_1, \ldots, x_n) = f(rx_1, \ldots, rx_n)$. Multiindex l is defined as the pair $(i,k) = (i, k_1, \ldots, k_n)$, where $i \in I = \{0, \ldots, 2s\}$, the components of $k = (k_1, \ldots, k_n)$, $k \in K$, are nonnegative integers from $\{0, \ldots, 2s\}$, and $|k| = k_1 + k_2 + \cdots + k_n$.

It is easy to modify this result to construct integration formulas (1.3) for the shell

$$
R_n^{\text{shell}} \equiv \{rz \mid z \in Y_n,\ 0 < \varrho_1 \le r \le \varrho_2 < \infty\} \tag{2.5}
$$

by replacing the formula (2.3) with

$$
\int_{\varrho_1}^{\varrho_2} r^{n-1} h(r)\, dr \cong \sum_{\nu=1}^{m} \sum_{i=0}^{2s} A_{i,\nu} h^{(i)}(r_\nu) \tag{2.6}
$$

in (2.4). Then (2.4) is the formula of degree d for R_n^{shell}. We are especially interested in constructing formulas for the n–spherical shell S_n^{shell}

$$\varrho_1^2 \le x_1^2 + x_2^2 + \cdots + x_n^2 \le \varrho_2^2, \quad 0 \le \varrho_1 < \varrho_2 < \infty, \tag{2.7}$$

and the n–cubical shell C_n^{shell}

$$\varrho_1 \le |x_i| \le \varrho_2, \quad i = 1, \dots, n, \quad 0 \le \varrho_1 < \varrho_2 < \infty, \tag{2.8}$$

from formulas for their surface of the type (1.3).

In [8] we are constructed the cubature formula of the type (1.3) for the surface of the unit of n–sphere U_n in n–dimensional Euclidean space. By using the procedure of obtaining of the cubature (2.4), and the corresponding formula from [8] and (2.3) we can construct the cubature formula (1.3) for the approximation of integral over solid of n–sphere S_n

(2.9)

$$\int_{S_n} f(x_1, \dots, x_n)\, dx_1 \cdots dx_n$$

$$\cong \frac{\pi}{(s+1)m} \sum_{j=1}^{2m(s+1)} \sum_{\nu,\nu_1,\dots,\nu_{n-2}=1}^{m} \sum_{i,i_1,\dots,i_{n-2}=0}^{2s} A_{i,\nu} \times$$

$$\times A_{i_1,\nu_1}^{(n-3)} \dots A_{i_{n-2},\nu_{n-2}}^{(0)} F^{(i,i_1,\dots,i_{n-2})}\left(r_\nu, \varphi_1^{(\nu_1)}, \dots, \varphi_{n-2}^{(\nu_{n-2})}, \frac{\pi j}{(s+1)m}\right),$$

where

$$F^{(i,i_1,\dots,i_{n-2})} = \frac{\partial^{|p|} F}{\partial r^i \partial(\cos\varphi_1)^{i_1} \dots \partial(\cos\varphi_{n-2})^{i_{n-2}}},$$

$$p = (i, i_1, \dots, i_{n-2}), \qquad |p| = i + \sum_{k=1}^{n-2} i_k.$$

For the computing an integrals over the n–spherical shell, we use the formula (2.9) which is obtained by using the corresponding formula from [8] and (2.6).

Now, we shall construct the formula of the type (1.3) for the unit n–sphere S_n, which has some other form of the formula (2.9) (the both formulas belong to so–called group "spherical product" formulas (see [12])). Hence, our purposse is the constructing of the formula (1.3) for the unit n–sphere defined by

$$S_n = \{x = (x_1, \dots, x_n) |\ x_1^2 + x_2^2 + \cdots + x_n^2 \le 1\}.$$

To do this we need to transform a monomial integral

$$\int_{S_n} x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}\, dx_1 \cdots dx_n, \tag{2.10}$$

so that it separates into the product of n–single integrals. This happens in n–dimensional spherical coordinates.

There are many ways to define n–dimensional spherical coordinates, one of these we have used in the construction of the formula (2.9). Here, we use the definition (see [12])

$$\begin{aligned} x_1 &= r\cos\theta_{n-1}\cos\theta_{n-2}\dots\cos\theta_2\cos\theta_1, \\ x_2 &= r\cos\theta_{n-1}\cos\theta_{n-2}\dots\cos\theta_2\sin\theta_1, \\ x_3 &= r\cos\theta_{n-1}\cos\theta_{n-2}\dots\sin\theta_2, \\ &\;\vdots \\ x_{n-1} &= r\cos\theta_{n-1}\sin\theta_{n-2}, \\ x_n &= r\sin\theta_{n-1}. \end{aligned} \tag{2.11}$$

Thus we use the n new variables which satisfy $\theta_1,\dots,\theta_{n-1} \in [-\pi/2,\pi/2]$, and
$r \in [-1,1]$.

The Jacobian of transformation (2.11) is

$$J = r^{n-1}(\cos\theta_{n-1})^{n-2}(\cos\theta_{n-2})^{n-3}\dots(\cos\theta_3)^2(\cos\theta_2),$$

and therefore (2.10) transforms into the product of the following integrals:

$$\int_{-\pi/2}^{\pi/2} (\cos\theta_1)^{\alpha_1}(\sin\theta_1)^{\alpha_2}\, d\theta_1, \tag{2.12.1}$$

$$\int_{-\pi/2}^{\pi/2} (\cos\theta_2)(\cos\theta_2)^{\beta_2}(\sin\theta_2)^{\alpha_3}\, d\theta_2, \tag{2.12.2}$$

$$\vdots$$

$$\int_{-\pi/2}^{\pi/2} (\cos\theta_{n-1})^{n-2}(\cos\theta_{n-1})^{\beta_{n-1}}(\sin\theta_{n-1})^{\alpha_n}\, d\theta_{n-1}, \tag{2.12.n-1}$$

$$\text{(2.12.n)} \qquad \int_{-1}^{1} |r|^{n-1} r^{\beta_n}\, dr,$$

$$\beta_k = \alpha_1 + \cdots + \alpha_k, \quad k = 2, \ldots, n.$$

Now we discuss the one–variable formulas needed for these single integrals. We assume that these are to be the Turán quadratures (1.1).

Consider the transformation

$$\text{(2.13)} \qquad \begin{aligned} y_k &= \sin\theta_k, & 1 - y_k^2 &= (\cos\theta_k)^2, \\ dy_k &= (\cos\theta_k)\, d\theta_k, & d\theta_k &= (1 - y_k^2)^{-1/2}\, dy_k. \end{aligned}$$

For $k = 1, \ldots, n-1$ we consider the formulas of the type (1.1) of degree of exactness d,

$$\text{(2.14.k)} \qquad \int_{-1}^{1} (1 - y_k^2)^{(k-2)/2} g(y_k)\, dy_k \cong \sum_{i_k=0}^{2s} \sum_{\nu_k=1}^{m} A_{i_k,\nu_k}^{(k)} g^{(i_k)}(y_{k,\nu_k}).$$

The formulas (2.14.k), $k = 1, \ldots, n-1$, are Gauss–Jacobi–Turán quadrature formulas. For $k = n$ we consider the formula of the type (1.1)

$$\text{(2.14.n)} \qquad \int_{-1}^{1} |r|^{n-1} h(r)\, dr \cong \sum_{i=0}^{2s} \sum_{\nu=1}^{m} A_{i,\nu}^{(n)} h^{(i)}(r_\nu),$$

of degree of exactness d, too.

Now, we can write the formula of the type (1.3) for S_n in the form

(2.15)

$$\begin{aligned} &\int_{S_n} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n \\ &\cong \sum_{\nu,\nu_1,\ldots,\nu_{n-1}=1}^{m} \; \sum_{i,i_1,\ldots,i_{n-1}=0}^{2s} A_{i,\nu}^{(n)} A_{i_1,\nu_1}^{(1)} \ldots A_{i_{n-1},\nu_{n-1}}^{(n-1)} \times \\ &\times f^{(i,i_1,\ldots,i_{n-1})}\left(r_\nu \sqrt{1 - y_{n-1,\nu_{n-1}}^2} \cdots \sqrt{1 - y_{1,\nu_1}^2}, \ldots, r_\nu y_{n-1,\nu_{n-1}}\right), \end{aligned}$$

where

$$\begin{aligned} &f^{(i,i_1,\ldots,i_{n-1})}\left(r_\nu \sqrt{1 - y_{n-1,\nu_{n-1}}^2} \cdots \sqrt{1 - y_{1,\nu_1}^2}, \ldots, r_\nu y_{n-1,\nu_{n-1}}\right) = \\ &\qquad = \frac{\partial^{i+i_1+\cdots+i_n} F\left(r_\nu, y_{1,\nu_1}, \ldots, y_{n-1,\nu_{n-1}}\right)}{\partial r^i \partial y_1^{i_1} \ldots \partial y_{n-1}^{i_{n-1}}} \end{aligned}$$

and $F(r, y_1, \ldots, y_{n-1}) = f(r\sqrt{1-y_{n-1}^2}\cdots\sqrt{1-y_1^2}, \ldots, ry_{n-1})$. The number of nodes of the formula (2.15) is $2m^{n-1}$ for m even, and $m^n - m^{n-1} + 1$ for m odd (see [12]).

It is easy to generalize formula (2.15) slightly and obtain a formula of degree $2(s+1)m-1$ for S_n with weight function $w(x_1, \ldots, x_n) = (x_1^2 + \cdots + x_n^2)^{\beta/2}$, $\beta > -n$. Everything we have said about the construction of the "spherical product" formula remains the same except that in place (2.14.n) we must use a formula

$$\int_{-1}^{1} |r|^{n-1+\beta} h(r)\, dr \cong \sum_{i=0}^{2s} \sum_{\nu=1}^{m} A_{i,\nu}^{(n)} h^{(i)}(r_\nu).$$

3. n–Dimensional Cones

Let R_{n-1} be a region in $(n-1)$–dimensional Euclidean space in the variables $u_1, \ldots, u_{n-1}$. Let R_n be defined by

$$R_n = \{(x_1, \ldots, x_n) |\ x_i = u_i(1-\lambda),\ i = 1, \ldots, n-1,\\ x_n = \lambda,\ \lambda \in [0,1],\ (u_1, \ldots, u_{n-1}) \in R_{n-1}\}. \tag{3.1}$$

R_n is called a cone with base R_{n-1}. The vertex of R_n is the point $(0, \ldots, 0, 1)$.

As examples of cones we have the following (see [12]):

(i) The three–dimensional pyramid (which we denote by $C_N : C_2$) is a cone with a square base.

(ii) The ordinary cone (which we denote by $C_N : S_2$) is a cone with a circular base.

(iii) An n–simplex is a cone with an $(n-1)$–simplex as base.
(Our definition of a cone cold be generalized so that the the solid n–sphere is a cone with vertex $(0, \ldots, 0)$ and with its surface as base.)

Let $R_{n-1}(\lambda)$, $\lambda \in [0,1]$, be the intersection of R_n and the plane $x_n = \lambda$. An integral over $R_{n-1}(\lambda)$ can be evaluated by affinely transforming $R_{n-1}(\lambda)$ onto R_{n-1}. Using this fact we have

$$\begin{aligned}
&\int_{R_n} x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n}\, dx_1 \cdots dx_n \\
&= \int_0^1 x_n^{\alpha_n} \left\{ \int_{R_{n-1}} (1-x_n)^{n-1} \prod_{j=1}^{n-1} [x_j(1-x_n)]^{\alpha_j}\, dx_1 \cdots dx_{n-1} \right\} dx_n \\
&= \int_0^1 (1-x_n)^{n-1+\beta} x_n^{\alpha_n}\, dx_n \int_{R_{n-1}} x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}}\, dx_1 \cdots dx_{n-1}, \\
&\qquad \beta = \alpha_1 + \cdots + \alpha_{n-1}.
\end{aligned} \tag{3.2}$$

In the last expression the integral with respect to x_n has the form

$$\int_0^1 (1-x_n)^{n-1} P_\alpha(x_n)\, dx_n,$$

where $P_\alpha(x_n)$ is a polynomial of degree $\alpha = \alpha_1 + \cdots + \alpha_n$.

Hence, if we are given an integration formula of the type (1.3) of degree d for R_{n-1},

$$\int_{R_{n-1}} g(x_1,\dots,x_{n-1})\, dx_1 \dots dx_{n-1} \cong \sum_{j=1}^{N_0} \sum_{k\in K} B_j^{(k)} g^{(k)}(\mu_{j,1},\dots,\mu_{j,n-1}),$$

where $k = (k_1,\dots,k_{n-1})$ is the multiindex with the nonnegative integer components from $\{0,1,\dots,2s\}$, $s \in N \cup \{0\}$, and an integration formula of degree d for $[0,1]$, of the type (1.1),

$$\int_0^1 (1-t)^{n-1} h(t)\, dt \cong \sum_{i=0}^{2s} \sum_{\nu=1}^{m} A_{i,\nu} h^{(i)}(\tau_\nu),$$

where $I = \{0,1,\dots,2s\}$, then the formula

$$\int_{R_n} f(x_1,\dots,x_n)\, dx_1 \dots dx_n$$

$$\cong \sum_{j=1}^{N_0} \sum_{\nu=1}^{m} \sum_{(k,i)=q\in K\times I} A_{i,\nu} B_j^{(k)} f^{(q)}(\mu_{j,1}(1-\tau_\nu),\dots,\mu_{j,n-1}(1-\tau_\nu),\tau_\nu),$$

where

$$f^{(q)}(\mu_{j,1}(1-\tau_\nu),\dots,\mu_{j,n-1}(1-\tau_\nu),\tau_\nu) = \frac{\partial^{|k|+i} g(\mu_{j,1},\dots,\mu_{j,n-1},\tau_\nu)}{\partial x_1^{k_1} \dots \partial x_{n-1}^{k_{n-1}} \partial x_n^{i}}$$

and $g(x_1,\dots,x_{n-1},x_n) = f(x_1(1-x_n),\dots,x_{n-1}(1-x_n),x_n)$, is the "product" cubature of Turán type of degree of exactness d, for R_n.

Using the obtained result, since T_n is a cone with base T_{n-1}, we can construct, by induction, starting with formulas for the interval $T_1 = [0,1]$, the "product" formulas of Turán type for T_n (see [8]). Now, we shall construct this formula directly.

Hence, we construct the "product" cubature formulas (1.3) for T_n, the n–simplex with vertices

$$(0,0,0,\dots,0,0), (1,0,0,\dots,0,0), \dots, (0,0,0,\dots,0,1).$$

T_2 is a triangle and T_3 is a tetrahedron. By an affine transformation we can transform T_n onto any other given n–simplex and therefore integration formulas of the type (1.3) for a given n–simplex can be obtained by an affine transformation of formulas for T_n.

The integral of a monomial over T_n is

$$\int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} \cdots \int_0^{1-x_1-\cdots-x_{n-1}} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\, dx_1 \cdots dx_n. \tag{3.3}$$

Let us transform integral (3.3) using the transformation

$$\begin{aligned} x_1 &= y_1 = y_1, \\ x_2 &= y_2(1-y_1) = y_2(1-x_1), \\ x_3 &= y_3(1-y_2)(1-y_1) = y_3(1-x_1-x_2), \\ &\vdots \\ x_n &= y_n(1-y_{n-1})\ldots(1-y_1) = y_n(1-x_1-\cdots-x_{n-1}). \end{aligned} \tag{3.4}$$

Since the limits of integration for the x_i are $0 \le x_i \le 1 - x_1 - \cdots - x_{i-1}$, $i = 1,\ldots,n$, the limits for the y_i shall be $0 \le y_i \le 1$, $i = 1,\ldots,n$. Since the Jacobian of transformation (3.4) is

$$J = (1-y_1)^{n-1}(1-y_2)^{n-2}\ldots(1-y_{n-1})$$

the monomial integral (3.3) transforms into

$$\begin{aligned} &\int_0^1 \cdots \int_0^1 (1-y_1)^{\beta_1}\ldots(1-y_{n-1})^{\beta_{n-1}} y_1^{\alpha_1}\ldots y_n^{\alpha_n}\, dy_1 \ldots dy_n, \\ &\beta_1 = \alpha_2 + \cdots + \alpha_n + n - 1, \\ &\beta_2 = \alpha_3 + \cdots + \alpha_n + n - 2, \\ &\vdots \\ &\beta_{n-1} = \alpha_n + 1. \end{aligned} \tag{3.5}$$

The integral (3.5) is a product of n single integrals, where the integral with respect to y_k has the form

$$\int_0^1 (1-y_k)^{n-k} P_\alpha(y_k)\, dy_k, \quad k = 1,\ldots n,$$

where $P_\alpha(y_k) = y_k^{\alpha_k}(1-y_k)^{\alpha_{k+1}+\cdots+\alpha_n}$ is a polynomial of degree $\alpha = \alpha_k + \cdots + \alpha_n$ in y_k. Therefore if we have n one–variable formulas, each of degree d, of the type (1.1),

$$\int_0^1 (1-y_k)^{n-k} h(y_k)\, dy_k \cong \sum_{i_k=0}^{2s} \sum_{\nu_k=1}^{m} A_{i_k,\nu_k}^{(k)} h^{(i_k)}(\mu_{k,\nu_k}), \tag{3.6}$$

for $k = 1, \ldots, n$, these can be combined to give a "product" formula of the type (1.3) of degree d for T_n

(3.7)

$$\int_{T_n} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n \cong \sum_{i_1,\ldots,i_n=0}^{2s} \sum_{\nu_1,\ldots,\nu_n=1}^{m} A_{i_1,\nu_1}^{(1)} \cdots A_{i_n,\nu_n}^{(n)} \times$$
$$\times f^{(i_1,\ldots,i_n)}(\mu_{1,\nu_1}, \mu_{2,\nu_2}(1-\mu_{1,\nu_1}), \ldots, \mu_{n,\nu_n}(1-\mu_{n-1,\nu_{n-1}})\cdots(1-\mu_{1,\nu_1})),$$

where

$$f^{(i_1,\ldots,i_n)}(\mu_{1,\nu_1}, \mu_{2,\nu_2}(1-\mu_{1,\nu_1}), \ldots, \mu_{n,\nu_n}(1-\mu_{n-1,\nu_{n-1}})\cdots(1-\mu_{1,\nu_1})) =$$
$$= \frac{\partial^{i_1+\cdots+i_n} g(\mu_{1,\nu_1}, \ldots, \mu_{n,\nu_n})}{\partial y_1^{i_1} \ldots \partial y_n^{i_n}}$$

and $g(y_1, y_2, \ldots, y_n) = f(y_1, y_2(1-y_1), \ldots, y_n(1-y_{n-1})\cdots(1-y_1))$. The formulas (3.6) are Gauss–Jacobi–Turán quadrature formulas.

The above procedure can be generalized to give an integration formulas for T_n with a weight function

$$x_1^{\delta_1} x_2^{\delta_2} \ldots x_n^{\delta_n} (1-x_1)^{\varepsilon_1} \cdots (1-x_1-\cdots-x_n)^{\varepsilon_n}.$$

The "product" formula is exactly analogous to (3.7) except in place of the one–variable formulas (3.6) we must use formulas

$$\int_0^1 (1-y_k)^{\beta_k} y_k^{\gamma_k} h(y_k)\, dy_k \cong \sum_{i_k=0}^{2s} \sum_{\nu_k=1}^{m} A_{i_k,\nu_k}^{(k)} h^{(i_k)}(\mu_{k,\nu_k}), \tag{3.8}$$

for $k = 1, \ldots, n$, where the $\beta_k, \gamma_k, \delta_k, \varepsilon_k$ are related by

$$\begin{aligned}
&-1 < \gamma_k = \delta_k, \quad k = 1, \ldots, n,\\
&-1 < \beta_1 = \delta_2 + \cdots + \delta_n + \varepsilon_1 + \cdots + \varepsilon_n + n - 1,\\
&-1 < \beta_2 = \delta_3 + \cdots + \delta_n + \varepsilon_2 + \cdots + \varepsilon_n + n - 2,\\
&\qquad \vdots\\
&-1 < \beta_{n-1} = \delta_n + \varepsilon_{n-1} + \varepsilon_n + 1,\\
&-1 < \beta_n = \varepsilon_n.
\end{aligned}$$

The formulas (3.8) are Gauss–Jacobi–Turán qudrature formulas.

4. The Torus

Here we discuss "product" formulas of Turán type for regions which we call n–dimensional tori, $n \geq 3$. For $n = 3$ the most familiar such region is the torus defined by

$$(4.1) \qquad 0 \leq (\sqrt{x^2+y^2} - \varrho_1)^2 + z^2 \leq \varrho_2^2, \quad 0 \leq \varrho_2 \leq \varrho_1 < \infty,$$

and denoted by $T_3 : S_2$. We shall say that $T_3 : S_2$ is a 3–torus with cross section S_2, where S_2 is the circle

$$(4.2) \qquad 0 \leq (y - \varrho_1)^2 + z^2 \leq \varrho_2^2.$$

In general, if R_2 is a bounded two–dimensional region in (u, v)–space which lies in the half–space $u \geq 0$, then we show how to obtain an integration formula (1.3) for the 3–torus with cross section R_2 provided we know a formula for R_2. It is easily to carry out the generalisation of that formula for $n > 3$.

Let R_2 denote a bounded two–dimensional region in the (u, v)–plane and assume $(u, v) \in R_2$ implies $u \geq 0$. Let $T_3 : R_2$ denote the set of points (x, y, z) such that

$$x = u \cos\theta, \ y = u \sin\theta, \ z = v,$$

where $(u, v) \in R_2$, $-\pi < \theta \leq \pi$. We have that

$$\int_{T_3:R_2} x^\alpha y^\beta z^\gamma \, dx dy dz = \int_{U_2} x^\alpha y^\beta \, d\sigma \int_{R_2} u^{1+\alpha+\beta} v^\gamma \, du dv,$$

where

$$U_2 = \{(x, y) \in E_2 | \ x^2 + y^2 = 1\}.$$

For the integral over U_2 we construct the corresponding formula from [8], which for $n = 2$ becomes the formula of rectangles

$$\int_{U_2} h(x, y) \, d\sigma \cong \frac{\pi}{(s+1)m} \sum_{k=1}^{2m(s+1)} h(\cos\frac{\pi k}{(s+1)m}, \sin\frac{\pi k}{(s+1)m}),$$

of degree d, and for the integral over R_2 we construct the formula of the type (1.3) (if that the region R_2 allows), for $n = 2$ and $w(x) \equiv 1$, $x = (x_1, \ldots, x_n)$,

$$\int_{R_2} g(u, v) \, du dv \cong \sum_{j \in J} \sum_{i=1}^{N} B_{ij} g^{(j)}(\nu_{i,1}, \nu_{i,2}),$$

of degree at most $d+1$. By combination these formulas we obtain the "product" formula (1.3) for $T_3 : R_2$, of degree d,

$$\int_{T_3:R_2} f(x,y,z)\,dxdydz \cong \frac{\pi}{(s+1)m} \sum_{k=1}^{2m(s+1)} \sum_{j\in J} \sum_{i=1}^{N} B_{ij} \times$$
$$\times f^{(j)}\left(\nu_{i,1}\cos\frac{\pi k}{(s+1)m}, \nu_{i,1}\sin\frac{\pi k}{(s+1)m}, \nu_{i,2}\right),$$

where

$$f^{(j)}\left(\nu_{i,1}\cos\frac{\pi k}{(s+1)m}, \nu_{i,1}\sin\frac{\pi k}{(s+1)m}, \nu_{i,2}\right) = \frac{\partial^{j_1+j_2} F\left(\frac{\pi k}{(s+1)m}, \nu_{i,1}, \nu_{i,2}\right)}{\partial u^{j_1}\partial v^{j_2}}$$

and

$$F(\theta,u,v) = f(u\cos\theta, u\sin\theta, v).$$

5. Numerical results

We shall give numerical results obtained by using the presented methods. Programs are realized in double precision arithmetics in Fortran 77.

Example 1. As example of the computing the integral over the unit of 3–sphere, we consider the integral computed in [12, pp.37] by using the product of Gauss quadratures:

$$J = \int_{S_3} \exp(xyz^2)\,dxdydz \approx 4.190604290.$$

The results of the computing were given in the table 2.4 in [12, pp.37], where m and $N = m^n$ are the number of nodes of the quadrature and the corresponding "product" cubature, respectively.

If we apply product of Turán formulas, i. e., the cubature (2.9), with $s = 1$, then we shall obtain the results in the table 1.

TABLE 1

m	N	J (*approx.*)
2	8	4.191133340
4	64	4.190604280
5	125	4.190604290

Example 2. As example of the computing the integral over 3–simplex, we give the integral computed in [12. pp.31] by using the product of Gauss quadratures:

$$J = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (1+x+y+z)^{-4}\, dzdydx \approx 0.0208333333.$$

The results of the computing were given in the table 2.3 in [10, pp.31].

If we apply the product of Turán formulas, i. e., the cubature (3.7), with $s = 1$, then we shall obtain the results in the table 2.

TABLE 2

m	N	J (*approx.*)
2	8	0.0208325594
4	64	0.0208333333

In this case, the application of the formula (3.7) is not practise, because it is very complicated to find the large number of partial derivatives of integrand.

Example 3. The "product" cubature of the type (1.3) for cylinders were given in [8]. As example of the computing the integral over the cylinder

$$C_1 : S_2 = \{(x, y, z) :\ z \in [-1, 1],\ x^2 + y^2 \leq 1\}, \quad \text{with } w(x, y, z) \equiv 1,$$

we give the integral

$$I = \int_{C_1:S_2} e^{y^2 z}\, dxdydz = \int_{-1}^{1} |r|\, dr \int_{-1}^{1} \frac{dy_1}{\sqrt{1-y_1^2}} \int_{-1}^{1} e^{r^2 y_1 z}\, dz,$$

where we use the formula (2.15) for the circle S_2.

The exact value of the integral, except for rounding errors, is

$$I = 0.641698898791396(+01).$$

The results of the computing the integral, for $s = 0, 1$, and some m, are given in the table 3.

TABLE 3

s	m	I^*	re
0	2	6.348748861273719(+00)	1.1(−02)
	4	6.416818541835027(+00)	2.7(−05)
	6	6.416988784529709(+00)	3.2(−08)
	8	6.416988987765320(+00)	2.3(−11)
	10	6.416988987913896(+00)	1.0(−14)
1	2	6.416554928299226(+00)	6.8(−05)
	4	6.416988987506392(+00)	6.4(−11)
	6	6.416988987913965(+00)	8.0(−15)

Numbers in parentheses denote decimal exponents. With I^* we denoted the value of integral performed with the "product" cubatures of Turán type and re is the relative error.

References

[1] I. P. Mysovskih, *Interpolating cubature formulas*, FM Moskva, 1981. (Russian)

[2] W. Gautschi and G. V. Milovanović, *S-orthogonality and construction of Gauss–Turán type quadrature formulae*, J. Comput. Appl. Math. (to appear).

[3] L. Gori, M. L. Lo Cascio and G. V. Milovanović, *The σ-orthogonal polynomials: a method of construction*, Orthogonal Polynomials and their applications (1991), J.C. Baltzer AG, Scientific Publishing Co ©IMACS, 281–285.

[4] G. V. Milovanović, *Construction of s-orthogonal polynomials and Turán quadrature formulae*, in Numerical Methods and Approximation Theory III, Niš, 1987 (Milovanović, G. V., eds.), Univ. Niš, Niš, 1988., pp. 311–328.

[5] G. V. Milovanović and M. M. Spalević, *A numerical procedure for coefficients in generalized Gauss–Turán quadratures*, Filomat (Niš) **9:1** (1995), 1–8.

[6] G. V. Milovanović and M. M. Spalević, *Construction of Chakalov–Popoviciu's Type Quadrature Formulae*, Rend. Cir. Mat. Palermo (In: Proc. 3rd Internat. Confer. on Func. Analysis and Approx. Theory (Potenza, 1996)) (to appear).

[7] M. M. Spalević, *Computing Turán–Radau and Turán–Lobatto quadratures with multiple nodes*, Zb. Rad. (Kragujevac) **17** (1995), 77–84.

[8] ———, *Product of Turán quadratures for Cube, Simplex, Surface of the Sphere, $\overline{E}_n^r$ and $E_n^{r^2}$* (to appear).

[9] ———, *Product of Turán quadratures for Ellipse and Regions Bounded by Orthogonal Parabolas* (to appear).

[10] D. D. Stancu, *Asupra unor formule generale de integrare numerică*, Acad. R. P. Romîne Stud. Cerc. Mat. **9** (1958), 209–216.

[11] A. H. Stroud and D. D. Stancu, *Qudrature formulas with multiple Gaussian nodes*, J. SIAM Numer. Anal. Ser. B **2** (1965), 129–143.

[12] A. H. Stroud, *Approximate Calculation of Multiple Integrals*, Prentice–Hall, Inc., Englewood Cliffs, N. J., 1971.

[13] G. Vincenti, *On the computation of the coefficients of s–orthogonal polynomials*, SIAM J. Numer. Anal. **23** (1986), 1290–1294.

Faculty of Science, P. O. Box 60, 34000 Kragujevac, Yugoslavia
E-mail: spale@matkg.matf.bg.ac.yu