FILOMAT (Niš) 10 (1996), 187–191

ON REGULARIZATION OF PROBLEM OF MINIMIZATION FOR QUADRATIC FUNCTIONAL ON A HALFSPACE

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Abstract. *In this paper, with the aid of Tihonov's method of regularization we consider the problem of stability and speed of convergence of regulated solution of minimization problem for quadratic functional on a halfspace.*

A very important part of the theory of extremal problems and mathematical physics is the problem of minimization of quadratic functional on the set of solutions of a system of linear differential equations, with some constraints. In these problems very often we have given Hilbert spaces H and F, continuous linear operator $A : H \to F$, closed convex set $U \subseteq H$ and an element $f \in F$, and, we look for the solution of the extremal problem

$$J(u) = \|Au - f\|_F^2 \to \inf, \quad u \in U.$$

Here, we consider the case when the set U is a halfspace. Hence we have the following problem

$$(1) \qquad J(u) = \|Au - f\|_F^2 \to \inf, \quad u \in U = \{u \in H :< c, u >\leq \beta\}$$

where $c \in H, c \neq 0$, is a fixed element and β is a given real number. In real problems, instead of operator A and elements f and c, very often we have only some of their approximations. Here, we assume that $A_\mu \in \mathcal{L}(H, F)$, and $f_\delta \in F$, $c_\delta \in H$ are given and that $\|A - A_\mu\| \leq \mu$, $\|f - f_\delta\| \leq \delta$, $\|c - c_\sigma\| \leq \sigma$ where μ, δ, σ are small positive numbers.

The problem (1) is, in general, incorrect, so, to sol it we'll use Tihonov's method of regularization [1].

The problem with no constraints

$$(2) \qquad J(u) = \|Au - f\|_F^2 \to \inf, \quad u \in H.$$

Received March 11, 1996
1991 *Mathematics Subject Classification:* 49D29.

was considered in [1] and [3], while the extremal problem

$$J(u) = \|Au - f\|_F^2 \to \inf, \quad u \in U = \{u \in H : \| u \| \leq R\}$$

was considered in [2] and [4].

Denote $R(A)$-the range of A, P-the operator of orthogonal projection from H on $\overline{R(A^*)}$, u_* -normal solution of the problem (1) and u_∞-normal solution of the problem (2). Since u_∞ is the solution of the operator-equation $J'(u) = 0$, then

$$A^*Au_\infty - A^*f = 0 \tag{3}$$

while, for u_* there is $\lambda_* \geq 0$ such that

$$A^*Au_* - A^*f + \lambda_* c = 0, \tag{4}$$

$$\lambda_*((c, u_*) - \beta) = 0 \tag{5}$$

The following lemma gives us a coonnection between normal solutions u_∞ and u_*.

Lemma 1. *a) If $c \in R(A^*A)$ then there exist $\lambda_* \geq 0$ and $h \in \overline{R(A^*A)}$ such that $u_* = u_\infty - \lambda_* h$.*

b) If $c \notin R(A^*A)$ then there exists $\gamma_* \geq 0$ such that $u_* = u_\infty - \gamma_*(I - P)c$.

Proof. a) Let $c \in R(A^*A)$ i.e. $c = A^*Ax$ for some $x \in$H. Since $H = \overline{R(A^*A)} \oplus \operatorname{Ker} A$, the element x can be written as $x = p + h$, $h \in \overline{R(A^*A)}$, $p \in \operatorname{Ker} A$. Then $c = A^*Ah$ and, according to (3)-(5) it follows that $A^*A(u_* - u_\infty - \lambda_* h) = 0$. Using the fact that A^*A is injective on the subspace $\overline{R(A^*A)}$ we establish statement a).

b) Since $c \notin R(A^*A)$ it follows from (4) that $\lambda_* = 0$ and therefore the functional J reaches its absolute minimum at the point u_*. Set of absolute minimum points of the functional J is given by $U_\infty = u_\infty + \operatorname{Ker} A$. Hence we conclude that $(I - P)c = 0$ implies $u_* = u_\infty$. Now, assume $(I - P)c \neq 0$. Let $v_* = u_\infty - \gamma_*(I - P)c$, where $\gamma_* = ((u_\infty, c) - \beta)/ \| (I - P)c \|^2$. It is easy to see that v_* is a solution of (1). On the other hand

$$\begin{aligned} \| v_* \|^2 &= \| u_\infty \|^2 + (((u_\infty, c) - \beta)^2 / \| (I - P)c \|^4) \| (I - P)c \|^2 \leq \\ &\leq \| u_\infty \|^2 + \| (I - P)u_* \|^2 = \| u_* \|^2 . \end{aligned}$$

Since u_* is the normal solution it follows that $u_* = u_\infty$, and the lemma is proved.

When we apply Tihonov's method on the problem (1) we take, as a aproximation of the solution of this problem, the unique solution u_α of the extremal problem

$$T_\alpha(u) = \| A_\mu u - f_\delta \|^2 + \alpha \| u \|^2 \rightarrow \inf, \quad u \in U_\sigma = \{u \in H \mid (c_\sigma, u) \leq \beta\}.$$

Since

$$(6) \qquad A_\mu^* A_\mu u_\alpha - A_\mu^* f_\delta + \alpha u_\alpha + p_\alpha c_\sigma = 0, \quad p_\alpha((c_\sigma, u_\alpha) - \beta) = 0$$

it follows that

$$u_\alpha = g_\alpha(A_\mu^* A_\mu)(A_\mu^* f_\delta - p_\alpha c_\sigma) = \bar{u}_\alpha - p_\alpha g_\alpha(A_\mu^* A_\mu) c_\sigma$$

where $g(t) = (\alpha + t)^{-1}$, $\bar{u}_\alpha = g_\alpha(A_\mu^* A_\mu) A_\mu^* f_\delta$ is the solution of the problem $T_\alpha(u) \rightarrow \inf$, $u \in H$, $p_\alpha = 0$ if $(\bar{u}_\alpha, c_\sigma) \leq \beta$ and $p_\alpha = ((c_\sigma, u_\alpha) - \beta)/(g_\alpha(A_\mu^* A_\mu) c_\sigma, c_\sigma)$ for $(\bar{u}_\alpha, c_\sigma) > \beta$.

Theorem 1. *Let $A, A_\mu \in \mathcal{L}(H, F)$, $c, c_\delta \in H$, $f, f_\delta \in F$, $c_\delta \in H$ and $\|A - A_\mu\| \leq \mu$, $\|f - f_\delta\| \leq \delta$, $\|c - c_\sigma\| \leq \sigma$. If the parameter $\alpha = \alpha(\mu, \delta, \sigma)$ is chosen such that*

$$\alpha = \alpha(\mu, \delta, \sigma) \rightarrow 0, \quad (\mu + \delta + \sigma)/\alpha(\mu, \delta, \sigma) \rightarrow 0 \ \text{ as } \ \mu, \delta, \sigma \rightarrow 0\ .$$

then $u_{\alpha = \alpha(\mu, \delta, \sigma)} \rightarrow u$ as $\mu, \delta, \sigma \rightarrow 0$.

Proof: Denote by w_α and v_α respectively, the solutions of the following problems

$$J_\alpha(u) = \| Au - f \|^2 + \alpha \| u \|^2 \rightarrow \inf, \qquad u \in U,$$

$$T_\alpha(u) = \| A_\mu u - f_\delta \|^2 + \alpha \| u \|^2 \rightarrow \inf, \quad u \in U$$

and estimate $\| w_\alpha - v_\alpha \|$. Using the extremal property

$$(A^* A w_\alpha - A^* f + \alpha w_\alpha - A_\mu^* A_\mu v_\alpha - A_\mu^* f_\delta + \alpha v_\alpha, w_\alpha - v_\alpha) \leq 0$$

we get

$$\| A_\mu(w_\alpha - v_\alpha) \|^2 + \| w_\alpha - v_\alpha \|^2 \leq$$

$$\leq ((A^* A - A_\mu^* A_\mu) w_\alpha, w_\alpha - v_\alpha) + ((A^* f - A_\mu^* f, w_\alpha - v_\alpha) + (f - f_\delta, A_\mu(w_\alpha - v_\alpha)).$$

According to [3], since

$$(7) \qquad w_\alpha \rightarrow u_*, \quad (\alpha \rightarrow 0),$$

if follows from (6) that

$$\| w_\alpha - v_\alpha \| \leq c_1(\mu + \delta^2).$$

Similarly, we can prove that

$$\| w_\alpha - u_\alpha \| \leq c_2 \sigma / \alpha.$$

Hence

$$\| u_* - u_\alpha \| \le c(\| u_* - w_\alpha \| + (\mu + \delta^2 + \sigma)/\alpha) \tag{8}$$

According to (7) and the conditions of the theorem it follows that $u_{\alpha=\alpha(\mu,\delta,\sigma)} \to u_*$ as μ, δ, $\sigma \to 0$.

With no additional conditions, the speed of convergence $u_\alpha \to u_*$ can be arbitrary small [3]. Thus, it is very interesting to emphasize some cases where we can guarantee the speed of convergence.

Theorem 2. *If $u_\infty \in R(A^*A)$ and $Pc \in R((A^*A)^2)$ then, with $\alpha = d(\mu + \delta^2 + \sigma)^{1/2}$, $d \equiv const$, the following estimate holds*

$$\| u_* - u_\alpha \| \le c(\mu + \delta^2 + \sigma)^{1/2}$$

Proof: First, let us estimate $\| u_* - w_\alpha \|$. Suppose $c \in R(A^*A)$. According to the lemma 1 $u_* = u_\infty - \lambda_* h$, $h \in \overline{R(A^*A)}$ from $Pc = c \in R((A^*A)^2)$ it follows that $c = (A^*A)^2 h_1$, for some $h_1 \in H$. Since [1]

$$\lim_{\alpha \to 0} (A^*Ag_\alpha(A^*A) - I)h = 0$$

we find that

$$\lim_{\alpha \to 0} (g_\alpha(A^*A)c, c) = \lim_{\alpha \to 0} (A^*Ag_\alpha(A^*A) - I)h, c) + \| Ah \|^2 = \| Ah \|^2 \neq 0.$$

Observe that $w_\alpha = \bar{w}_\alpha - \lambda_\alpha g_\alpha(A^*A)c$, where $\bar{w}_\alpha = g_\alpha(A^*A)A^*f$ is the solution of the problem $J_\alpha(u) \to \inf$, $u \in H$, $\lambda_\alpha = 0$ if $(\bar{w}_\alpha, c) \le \beta$, and $\lambda_\alpha = ((c, \bar{w}_\alpha) - \beta)/(g_\alpha(A^*A)c, c)$ for $(c, \bar{w}_\alpha) > \beta$. It was proved, in [3],that

$$\bar{w}_\alpha \to u_\infty, \quad (\alpha \to 0) \tag{9}$$

while, under conditions of theorem 2, we have

$$\| \bar{w}_\alpha - u_\infty \| \le c\alpha \tag{10}$$

If $(u_*, c) < \beta$ then $u_* = u_\infty$, and, according to (10), $(w_\alpha, c) < \beta$ i.e. $\lambda_\alpha = 0$ for sufficiently small α. Thus, for small enough α, $u_* = u_\infty$, $\bar{w}_\alpha = w_\alpha$ and therefore we have estimation

$$\| w_\alpha - u_\infty \| \le c\alpha \tag{11}$$

Now, suppose that $(u_*, c) = \beta$. Then

$$\lambda_* = ((u_\infty, c) - \beta)/ \| Ah \|^2, \ \ \lambda_\alpha = ((c, \bar{w}_\alpha) - \beta)/(g_\alpha(A^*A)c, c).$$

If is easy to prove the following equality

(12)

$$u_* - w_\alpha = u_\infty - \lambda_\alpha(A^*Ag_\alpha(A^*A) - I)A^*Ah_1 +$$
$$+(\bar{w}_\alpha - u_\infty, c)/(g_\alpha(A^*A)c, c)h +$$
$$+((u_\infty, c) - \beta)/(\| Ah \|^2 (g_\alpha(A^*A)c, c))((A^*Ag_\alpha(A^*A) - I)A^*Ah_1, c)h.$$

Using (11) and the inequality [1]:

$$\| A^*Ag_\alpha(A^*A) - I)A^*Ah_1 \| \leq \gamma\alpha$$

from (13) we obtain the equality (12). Also, inequality (12) can be proved for the other cases described in the lemma 1. Thus, using (8), we find the estimation

$$\| u_* - u_\alpha \| \leq c(\alpha + (\mu + \delta + \sigma)/\alpha)$$

From this inequality, for $\alpha = d(\mu + \delta^2 + \sigma)^{1/2}$ we find estimation (9).

Finally, let us emphasize that, in the lemma 1, we described the normal solution of the problem (1) using the normal solution of the problem with no constraints and the element c. In order to find approximated solution of the problem (1) we suggest Tihonov's method of regulation. In the theorem 1 we proved the convergence of regulated solution with some conditions on α, μ, σ and σ. Finally, in the theorem 2 we gave the estimate for the speed of convergence, with condititionn $u_\infty \in R(A^*A)$ and $Pc \in R((A^*A)^2)$, that is valid if, for example, the range $R(A)$ of the opertator A is closed subspace of the space F. In solving real problems, these conditions are very hard to verify, so, when we don't have any informations about the properties of u_*, it is left to make a choice of parametar α. Also, we can consider application of iterative method of regulation. The problem withaut constraints was considered in [1].

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