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ON COMMON FIXED POINT FOR COMPATIBLE MAPPINGS OF (A) TYPE ON METRIC AND 2-METRIC SPACES

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Abstract. *In this paper, using the concept of compatible mappings of (A) type of G. Jungck [5], we will give a common fixed point theorems for mappings defined on metric and 2-metric spaces.*

1. Introduction

Let us recall the definition of compatible mappings.

Definition 1.1. *Let S and I be mappings from metric space (X,d) into itself. A pair $\{S,I\}$ is said to be compatible on X if whenever $\{x_n\}$ is a sequence in X such that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ix_n = z \in X$$

then

$$d(STx_n, ISx_n) \to 0, \text{ as } n \to \infty.$$

In [5] G. Jungck introduced the concept of compatible mappings of (A) type.

Definition 1.2. *Let S and I be mappings from metric space (X,d) into itself. A pair $\{S,I\}$ is said to be compatible of (A) type on X if whenever $\{x_n\}$ is a sequence in X such that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ix_n = z \in X$$

then

$$d(ISx_n, SSx_n) \to 0 \text{ and } d(SIx_n, IIx_n) \to 0, \text{ as } n \to \infty.$$

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In [5] G. Jungck, P. P. Murthy and Y. J. Cho proved that pairs of compatible mappings and compatible mappings of type (A) on metric space are equivalent each other under some conditions. For instance:

Proposition 1.1. *Let S and I be continuous mappings on X. Then a pair $\{S,T\}$ is compatible on X if and only if it is compatible of type (A) on X.*

The following examples show that this proposition is not true if S and I are not continuous on X.

Example 1. Let $X = R$ with usual metric and define two mapping $S, I : X \to X$ as follows

$$Sx = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \quad \text{and } Ix = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

Both, S and I are discontinuous at $x = 0$. CFor any sequence $\{x_n\} \subset X$ we have

$$d(SIx_n, ISx_n) = 0$$

so the pair $\{S, I\}$ is compatible on X. But now consider a sequence $x_n = n$, $n \in N$. Then $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ix_n = 0 \in X$ but

$$d(SIx_n, IIx_n) = |n^2 - n^4| \to \infty, \text{ as } n \to \infty$$

implies that the pair $\{S, I\}$ is not compatible of type (A) on X.

Example 2. Now, for

$$Ix = \begin{cases} -\frac{1}{x}, & x > 1 \\ 1, & 0 \le x \le 1 \\ 0, & x < 0 \end{cases} \quad \text{and } Sx = \begin{cases} \frac{1}{x}, & x > 1 \\ 1, & 0 \le x \le 1 \\ 0, & x < 0 \end{cases}$$

let us chek that I and S are compatible mappings of (A) type. Since restriction of I and S on $(-\infty, 1]$ are equal it's enough to prove conditions for sequences $\{x_n\} \subset (1, +\infty)$. Then $\{Ix_n\} \subset (-1, 0)$, $\{Sx_n\} \subset (0, 1)$ so for every $n \in N$

$$I(Ix_n) = 0, \quad I(Sx_n) = 1,$$

$$S(Ix_n) = 0, \quad S(Sx_n) = 1,$$

and so

$$S(Ix_n) - I(Ix_n) = 0$$
$$I(Sx_n) - S(Sx_n) = 0, \quad n \in N$$

To prove that I and S are not compatible it's enough to find just one sequence $\{x_n\}$such that $\lim_{n\to\infty} = \lim_{n\to\infty} = z \in X$ and

$$S(Ix_n) - I(Sx_n) \not\to 0, \quad n \to \infty.$$

Let $x_n = n$, $n \in N$. Then $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Sx_n = 0$ but for every $n \in N$, $S(Ix_n) = 0$, $I(Sx_n) = 1$ so

$$S(Ix_n) - I(Sx_n) \not\to 0, \quad n \to \infty.$$

2. Result in a metric space

In [3] next fixed point is proved.

Theorem 2.1. *Let $\{S, I\}$ and $\{T, J\}$ be compatible pairs of mappings of complete metric space (X, d) into itself such that*

a) $T(X) \subset I(X)$, $S(X) \subset J(X)$;

b) *For all x, y in X either*

$$(1)\ d(Sx,Ty) \le \alpha\frac{d(Ix,Sx)d(Ix,Ty) + d(Jy,Ty)d(Jy,Sx)}{d(Ix,Ty) + d(Jy,Sx)} + \beta d(Ix,Jy)$$

if $d(Ix,Ty) + d(Jy,Sx) \ne 0$, where $\alpha, \beta > 0$ $\alpha + \beta < 1$, or

$$(1') \qquad d(Sx,Ty) = 0 \text{ if } d(Ix,Ty) + d(Jy,Sx) = 0$$

If one of S, T, I or J is continuous then S, T, I and J have unique fixed point z. Further, z is the unique common fixed point of S and I and T and J.

Since only one of mappings S, T, I, J have to be continuous it is interesting to see what happens when $\{S, I\}$ and $\{T, J\}$ are compatible mappings of (A) type. With some modifications in proof of Theorem 2.1 we shall show the next theorem.

Theorem 2.2. *Let $\{S, I\}$ and $\{T, J\}$ be compatible of (A)type pairs of mappings of complete metric space (X, d) into itself such that conditions a) and b) of Theorem 2.1 are satisfied.*

If one of S, T, I or J is continuous then S, T, I and J have unique fixed point. Further, z is unique common fixed point of S and I and T and J.

Proof. Let x_0 be an arbitrary point of X. Since $S(X) \subset J(X)$ we can find a point x_1 in X such that $Sx_0 = Jx_1$. Also, since $T(X) \subset I(X)$ we can choose a point x_2 with $Tx_1 = Ix_2$. In general for the point x_{2n} we can pick up a point x_{2n+1} such that $Sx_{2n} = Jx_{2n+1}$ and then a point x_{2n+2} with $Tx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2\dots$

Let us put $u_{2n} = d(Sx_{2n}, Tx_{2n+1})$ and $u_{2n+1} = d(Tx_{2n+1}, Sx_{2n+2})$.

Now, we distinguish two cases:

(i) Suppose $u_{2n} \ne 0$, $u_{2n+1} \ne 0$ for $n = 0, 1, 2...$

Then on using inequality (1), we have

$$(2) \qquad u_{2n+1} \le (\alpha + \beta)u_{2n} \le .... \le (\alpha + \beta)^{2n+1}u_0, \text{ for } n = 0, 1, 2, ...$$

It follows that the sequence

$$\{Sx_0, Tx_1, Sx_2, \dots, Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}, \dots\}$$

is Cauchy sequence in the complete metric space (X, d) and so gets a limit z in X. Hence the sequences $\{Sx_{2n}\} = \{Jx_{2n+1}\}$ and $\{Tx_{2n-1}\} = \{Ix_{2n}\}$ as it's subsequences also converge to the same point z.

Let us now suppose that I is continuous so that the sequences $\{I^2x_{2n}\}$ and $\{ISx_{2n}\}$ converge to the same point Iz.

Since S and I are compatible mappings of (A) type, by continuity of I and because of

$$d(SIx_{2n}, Iz) \leq d(SIx_{2n}, IIx_{2n}) + d(IIx_{2n}, Iz)$$

we have that the sequence $\{SIx_{2n}\}$ also converges to the point Iz.

As in [2] one can see that

$$\begin{aligned} d(SIx_{2n}, Tx_{2n+1}) \leq\ & \alpha(\frac{d(I^2x_{2n}, SIx_{2n}) \cdot d(I^2x_{2n}, Tx_{2n+1})}{d(I^2x_{2n}, Tx_{2n+1}) + d(Jx_{2n+1}, SIx_{2n})} + \\ & + \frac{d(Jx_{2n+1}, Tx_{2n+1}) \cdot d(Jx_{2n+1}, SIx_{2n})}{d(I^2x_{2n}, Tx_{2n+1}) + d(Jx_{2n+1}, SIx_{2n})} + \\ & + \beta d(I^2x_{2n}, Jx_{2n+1}) \end{aligned}$$

which on letting $n \to \infty$ reduced to

$$d(Iz, z) \leq \beta d(Iz, z),$$

giving $Iz = z$.

Now let us prove that $Sz = z$.

$$\begin{aligned} d(Sz, Tx_{2n+1}) \leq\ & \alpha\left(\frac{d(Iz, Sz)d(Iz, Tx_{2n+1})}{d(Iz, Tx_{2n+1}) + d(Jx_{2n+1}, Sz)} + \right. \\ & \left. + \frac{d(Jx_{2n+1}, Tx_{2n+1})d(Jx_{2n+1}, Sz)}{d(Iz, Tx_{2n+1}) + d(Jx_{2n+1}, Sz)}\right) \\ & + \beta d(Iz, Jx_{2n+1}) \end{aligned}$$

When $n \to \infty$ it's reduced to

$$d(Sz, z) \leq \beta d(Iz, z)$$

which meach that $d(Sz, z) = 0$.

Further, since $Sz = z$ and $S(X) \subset J(X)$ there always exists a point z^* such that $Jz^* = z$. Thus

$$d(z,Tz^*) = d(Sz,Tz^*) \leq \alpha \frac{d(Iz,Sz)d(Iz,Tz^*) + d(Jz^*,Tz^*)d(Jz^*,Sz)}{d(Iz,Tz^*) + d(Jz^*,Sz)}$$

$$+\beta d(Iz,Jz^*)$$

giving thereby $Tz^* = z$.

Using compatibility of (A) type, $Tz^* = Jz^* = z$ implies that

$$d(TJz^*,JJz^*) = 0$$

which means tha $Tz = Jz$.

At the end we have that

$$d(z,Tz) = d(Sz,Tz) \leq \alpha \frac{d(Iz,Sz)d(Iz,Tz) + d(Jz,Tz)d(Jz,Sz)}{d(Iz,Tz) + d(Jz,Sz)}$$

$$+\beta d(Iz,Jz)$$

$$= \beta d(z,Tz)$$

which implies that $z = Tz = Jz$.

Thus we have proved that z is a common fixed point of S, T, I and J.

Now suppose that S is continuous, so sequences $\{S^2x_{2n}\}$ and $\{SIx_{2n}\}$ converge to the point Sz. Since S and I are compatible mappings of (A) type one can see that

$$d(ISx_{2n},Sz) \leq d(ISx_{2n},SSx_{2n}) + d(SSx_{2n},Sz)$$

implies that $ISx_{2n} \to Sz$, as $n \to \infty$.

One can prove that $Sz = z$ similarly as in [3].

As $S(X) \subset J(X)$ and $Sz = z$, we can find a point z^* in X such that $Jz^* = z$, and show that $Tz^* = z$. Since T and J are compatible mappings of (A) type it again follows as above that $Tz = Jz$. Further

$$d(Sx_{2n},Tz) \leq \alpha\left(\frac{d(Ix_{2n},Sx_{2n})d(Ix_{2n},Tz)}{d(Ix_{2n},Tz) + d(Jz,Sx_{2n})}\right.$$

$$\left.+\frac{d(Jz,Tz)d(Jz,Sx_{2n})}{d(Ix_{2n},Tz) + d(Jz,Sx_{2n})}\right)$$

$$+\beta d(Ix_{2n},Jz)$$

which on making $n \to \infty$ gives $z = Tz$.

Thus the point z is in the range of T and since $T(X) \subset I(X)$ there always exists a point $\tilde{z}$ in X such that $I\tilde{z} = z$.

Thus, on (1),

$$d(S\tilde{z}, z) = d(S\tilde{z}, Tz) \leq \alpha\Big(\frac{d(I\tilde{z}, S\tilde{z})d(I\tilde{z}, Tz)}{d(I\tilde{z}, Tz) + d(Jz, S\tilde{z})} + \frac{d(Jz, Tz)d(Jz, S\tilde{z})}{d(I\tilde{z}, Tz) + d(Jz, S\tilde{z})}\Big) + \beta d(I\tilde{z}, Jz) = 0$$

so $S\tilde{z} = z$.

Since S and J are compatible mappings of (A) type we have, since $S\tilde{z} = I\tilde{z}$, that

$$d(SI\tilde{z}, II\tilde{z}) = 0,$$

so $Sz = SI\tilde{z} = II\tilde{z} = Iz$.

Thus we have proved again that z is a common fixed point of S, T, I and J.

If the mapping T or J is continuous instead of S or I, then the proof that z is a common fixed point of S, T, I and J is similar.

One can prove that using (1) z is unique common fixed point for S and I and T and J.

(ii) If $u_{2n} = 0$, for some n, then inequality (2) gives $u_{2n+1} = 0$ which implies that

$$Sx_{2n} = Jx_{2n+1} = Tx_{2n+1} = Ix_{2n+2} = Sx_{2n+2} = \cdots = z.$$

As in [3] one can argue that z is a unique fixed point of S, T, I and J.

Remark. By choosing α, β, I, J, S and T suitably, we can derive a multitude of fixed point theorems which generalized well known results for weakly commuting and compatible mappings.

3. Results in 2-metric spaces

At first let us recall the notion of 2-metric space.

Definition 3.3. *A 2-metric space is a nonempty set X with a non negative real value function d on $X \times X \times X$ satisfying the following conditions:*

1) *For two distinct points x, y in X there is a point z in X such that $d(x, y, z) \neq 0$;*
2) *$d(x, y, z) = 0$ if at least two of x, y, z are equal;*
3) *$d(x, y, z) = d(x, z, y) = d(y, x, z)$ for all x, y, z in X;*
4) *$d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all x, y, z, u in X;*

Function d is called a 2-metric on the space X and (X, d) is called 2-metric space.

Remark. It has been shown that although d is a continuous function of any one of its three arguments it need not be continuous in two arguments but if it is continuous in two arguments that it is continuous in all three arguments.

Number of mathematicians have studied the aspects of fixed point theory in the seting of 2-metric spaces. For references see [6]. They have been motivatied by various concepts known for the metric spaces and have thus introduced analogues of various concepts in the frame work of 2-metric spaces.

Let us recall definition of asymptoticaly regular sequence and define the notion of compatible mappings in 2-metric spaces.

Definition 3.4. *Let (X, d) be a 2-metric space, and S and A be mappings from X into itself. Then a sequence $\{x_n\}$ in X is said to be asymptotically $\{S, A\}$-regular if for all a in X*

$$\lim_{n\to\infty} d(Ax_n, Sx_n, a) = 0.$$

In 2-metric space the notion of compatibility has the following form.

Definition 3.5. *Let S and A be mappings from 2-metric space X into itself. Then $\{A, S\}$ are said to be compatible if for every $a \in X$*

$$\lim_{n\to\infty} d(ASx_n, SAx_n, a) = 0$$

whenever $\{x_n\}$ is sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z \in X.$$

Definition 3.6. *Let S and A be mappings from 2-metric space X into itself. Then $\{A, S\}$ are said to be compatible of (A) type if whenever $\{x_n\}$ is sequence in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z \in X$$

then for every $a \in X$

$$\lim_{n\to\infty} d(ASx_n, SSx_n, a) = 0 \text{ and } \lim_{n\to\infty} d(SAx_n, AAx_n, a) = 0.$$

Since $d(x, y, z) = 0$ if at least two of x, y and z are equale it is easy to check that in the case when A and S are continuous the notions of compatible and compatible mappings of (A) type are equale. In oposite case it is not true.

Theorem 3.3. *Let (X, d) be a complete 2-metric space, d continuous and A, S and T mappings from X into itself such that*

1) *S and T are sequentialy continuous;*
2) *$\{A, S\}$ and $\{A, T\}$ are pairs of compatible mappings of (A) type;*
3) *There exists an asymptotically $\{A, S\}$ and $\{A, T\}$-regular sequence;*
4) $$d(Ax, Ay, a) \le a_1 d(Sx, Ax, a) + a_2 d(Tx, Ax, a) + a_3 d(Sy, Ay, a) + a_4 d(Ty, Ay, a) + a_5 d(Sx, Ay, a) + a_6 d(Tx, Ay, a) + a_7 d(Sy, Ax, a) + a_8 d(Ty, Ax, a) + a_9 d(Sx, Ty, a) + a_{10} d(Sy, Tx, a)$$

for all x, y, a in X where a_i, $i = 1, 2, \ldots, 10$ are non-negative real numbers such that

$$\max\{a_5 + a_6 + \cdots + a_{10}, a_2 + a_3 + a_5 + a_8 + a_9 + a_{10}, a_3 + a_4 + a_5 + a_6, a_1 + a_2 + a_7 + a_8\} < 1.$$

Then A, S and T have a unique common fixed point in X.

Proof. Let $\{x_n\}$ be an asymptotically $\{A, S\}$ and $\{A, T\}$-regular sequence. Then by (4)

$$d(Ax_n, Ax_m, a) \le a_1 d(Sx_n, Ax_n, a) + a_2 d(Tx_n, Ax_n, a) + a_3 d(Sx_m, Ax_m, a) + a_4 d(Tx_m, Ax_m, a) + a_5 d(Sx_n, Ax_m, a) + a_6 d(Tx_n, Ax_m, a) + a_7 d(Sx_m, Ax_n, a) + a_8 d(Tx_m, Ax_n, a) + a_9 d(Sx_n, Tx_m, a) + a_{10} d(Sx_m, Tx_n, a)$$

for all a in X and hence, by condition 4) of 2-metric

$$(1 - a_5 - a_6 - a_7 - a_8 - a_9 - a_{10}) d(Ax_n, Ax_m, a) \le (a_1 + a_5 + a_9) d(Sx_n, Ax_n, a) + (a_2 + a_6 + a_{10}) d(Tx_n, Ax_n, a) + (a_3 + a_7 + a_{10}) d(Sx_m, Ax_m, a) + (a_4 + a_8 + a_9) d(Tx_m, Ax_m, a) + (a_8 + a_9) d(Tx_m, Ax_n, Ax_m) + (a_6 + a_{10}) d(Ax_m, Ax_n, Tx_n) + a_5 d(Sx_n, Ax_m, Ax_n) + a_7 d(Sx_m, Ax_n, Ax_m) + a_9 d(Sx_n, Tx_m, Ax_n) + a_{10} d(Sx_m, Tx_n, Ax_m)$$

for all a in X.

Since $\{x_n\}$ is an asymptotically $\{A, S\}$ and $\{A, T\}$-regular sequence, as $m, n \to \infty$, we have

$$(1 - a_5 - a_6 - a_7 - a_8 - a_9 - a_{10}) d(Ax_n, Ax_m, a) \to 0$$

for all a in X.

Therefore, $\{Ax_n\}$ is a Cauchy sequence in X. Since (X, d) is complete 2-metric space, $\{Ax_n\}$ has the limit z in X. It means that $\lim_{n\to\infty} d(Ax_n, z, a) = 0$ for all a in X.

Since

$$d(Sx_n, z, a) \leq \; d(Sx_n, z, Ax_n) + d(Sx_n, Ax_n, a) + \\ + d(Ax_n, z, a) \to 0, \quad n \to \infty,$$

$Sx_n \to z$ as $n \to \infty$. Similarly, we have $Tx_n \to z$, $n \to \infty$. Maps S and T are sequentially continuous, so it follows that

$$SAx_n \to Sz,\; S^2x_n \to Sz,\; STx_n \to Sz,\; TAx_n \to Tz,\; T^2x_n \to Tz,$$

$$TSx_n \to Tz, \;\text{ as } n \to \infty.$$

Since, for any $a \in X$,

$$d(ATx_n, Tz, a) \leq \; d(ATx_n, TTx_n, a) + d(ATx_n, Tz, TTx_n) \\ + d(TTx_n, Tz, a)$$

as $n \to \infty$,

$$d(ATx_n, Tz, a) \to 0$$

we prove that

$$\lim_{n\to\infty} ATx_n = Tz.$$

Similarly, we have also $ASx_n \to Sz$ as $n \to \infty$.

One can prove, just as in [1] or [3], that $Sz = Tz = Az$.

Let us prove that $Az = A^2z$. At first notice that

$$\lim_{n\to\infty} d(A(Sx_n), S(Sx_n), a) = 0 \text{ and}$$

$$\lim_{n\to\infty} d(A(Sx_n), T(Sx_n), a) = 0 \text{ for each } a \in X.$$

so $\{Sx_n\}$ is asymptotically $\{A, S\}$ and $\{A, T\}$-regular sequence. Now, if we repeat the above procedure for sequence $\{Sx_n\}$, using that $\lim_{n\to\infty} A(Sx_n) = Az$, we have that

$$A(Az) = S(Az) = T(Az).$$

Now by 4), for all a in X,

$$\begin{aligned} d(Az, A^2z, a) \leq \; & a_1d(Sz, Az, a) + a_2d(Tz, Az, a) + a_3d(SAz, A^2z, a) \\ & + a_4d(TAz, A^2z, a) + a_5d(Sz, A^2z, a) + \\ & + a_6d(Tz, A^2z, a) + a_7d(SAz, Az, a) + a_8d(TAz, Az, a) + \\ & + a_9d(Sz, TAz, a) + a_{10}d(SAz, Tz, a) \\ = \; & (a_5 + a_6 + a_7 + a_8 + a_9 + a_{10})d(Az, A^2z, a). \end{aligned}$$

Since $a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} < 1$, $d(Az, A^2z, a) = 0$ that is $Az = A^2z$. Putting $p = Az$ we have that

$$p = Ap = Sp = Tp.$$

Thus, p is a common fixed point of A, S and T. For uniqueness see [1].

Remark. Mapping A is sequentially continuous at common fixed point of A, S and T in this case too.

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