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A THEOREM OF IOHVIDOV-FAN'S TYPE FOR MULTIFUNCTIONS

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Abstract. In this note, by O. Hadžić's [4] fixed point theorem for multifunctions in not necessarily locally convex topological vector spaces, we obtain a theorem of Iohvidov-Fan's type for multifunctions and give some its applications. Our results generalizes some results of I. S. Iohvidov [7], Ky Fan [3], Felix Browder [1] and Olga Hadžić [5].

Introduction

Let E be a topological vector space, $\mathcal{U}$ be the fundamental family of open neighbourhoods of zero in E and $K \subseteq E$. We say that the set K is of Zima's type if for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ such that $co(U \bigcap (K - K)) \subseteq V$. An example of such a subset in a non-locally convex space is given in [6].

A multifunction $F : X \to Y$ is a function from a set X into $\mathcal{P}(Y)$, where Y is a nonempty set and $\mathcal{P}(Y)$ is the partitive set of Y. A multifunction $F : X \to X$ has a fixed point if there exists $x_0 \in X$ such that $x_0 \in F(x_0)$. If X and Y are topological spaces, a multifunction $F : X \to Y$ is upper semi-continuous if for each point $x_0 \in X$ and an arbitrary open neighborhood V of $F(x_0)$ there exists an open neighborhood U of x_0 in X, such that $F(x) \subseteq V$ for all $x \in U$. If X and Y are compact spaces then $F : X \to Y$ is an upper semi-continuous multifunction if and only if its graph

$$Gr(F) = \{(x, y) \mid x \in X, y \in F(x)\}$$

is closed in $X \times Y$ (see Browder [1]). If $g : X \to Y$ is a single-valued continuous function then it allso belongs to a class of upper semi-cotinuous multifunction, since we can consider their values as singletone subsets of Y. In 1964. I. S. Iohvidov proved the folowing theorem.

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Theorem A. *(I. S. Iohvidov [7] - Theorem 1) Let X be a Hausdorff locally convex topological vector space and $K \subseteq X$ a nonempty compact and convex subset. Let $g : K \times K \to X$ be a continuous function such that*

$$g(x, \alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 g(x, y_1) + \alpha_2 g(x, y_2)$$

holds for all $x, y_1, y_2 \in K$ and $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 = 1$. If for every $x \in K$ there is a $y \in K$ with $g(x, y) = 0$ then there exists $x_* \in K$ such that $g(x_*, x_*) = 0$.

In a proof of this statement he used one result of Ky Fan [2]. It is easily seen that well known Tychonoff's fixed point theorem follows from Iohvidov's theorem by setting $g(x, y) = f(x) - y$, where $f : K \to K$ is a continuous function.

In 1966. Ky Fan [3] gave an extension of Theorem A. In 1968. Felix Browder [1] gave a new proof's and simple generalizations of Ky Fan's and Iohvidov's result. Futher extension of Browder's result are obtained by Olga Hadžić [6]. In this note we extend Hadžić's result in the case when G is an upper semi-continuous multifunction, and Y is an arbitrary topolological space. Our approach is based on the following fixed point result, which is a generalization of well known Kakutani-Fan's fixed point theorem.

Theorem B. *(O. Hadžić [4]) Let X be a Hausdorff topological vector space, and $K \subseteq X$ a compact and convex nonempty subset of Zima's type. If $T : K \to K$ is an upper semi-continuous multifunction with nonempty closed and convex values. Then T has at least one fixed point.*

Results

In this note the main result is the following theorem.

Theorem 1. *Let K be a nonempty, compact and convex subset of Hausdorff topological vector space X, Y a topological space, $C \subseteq Y$ a nonempty closed subset and $G : K \times K \to Y$ an upper semi-continuous multifunction. Suppose that for each $x \in K$, the set:*

$$\{y \in K | G(x, y) \bigcap C \neq \varnothing\}$$

is nonempty and convex. If K is of Zimma's type, then there exists an element $x_* \in K$ such that $G(x_*, x_*) \bigcap C \neq \varnothing$.

Proof. For each $x \in K$ we define, $T(x) \subseteq K$ in the following way:

$$T(x) = \{y \in K | g(x, y) \bigcap C \neq \varnothing\}.$$

Then T is a multifunction such that, by hypothesis, $T(x)$ is a closed and convex nonempty subset of K, for every $x \in K$. Let

$$Gr(T) = \{(x,y) \mid x \in K, y \in T(x)\},$$

be the graph of T. Then $(x,y) \in Gr(T)$ if and only if $G(x,y) \bigcap C \neq \emptyset$. Suppose that $Gr(T)$ is not a closed set, i.e. that T is not an upper semi-continuous multifunction. Then there exists a convergent generalized sequence (net) $\{(x_i, y_i)\}_{i \in I} \subseteq K \times K$ (where $(I, \geq)$ is a directed partially ordered set) such that $G(x_i, y_i) \bigcap C \neq \emptyset$ for all $i \in I$, $(x_i, y_i) \to (x_*, y_*)$ and $G(x_*, y_*) \bigcap C = \emptyset$. This implies that $G(x_*, y_*) \subseteq Y \backslash C$. Hence there exists $i_0 \in I$ such that $i \geq i_0$ implies $G(x_i, y_i) \subseteq Y \backslash C$ i. e. $G(x_i, y_i) \bigcap C = \emptyset$, because $Y \backslash C$ is the open set and G is the upper semi-continuous multifunction, which is a contradiction. Hence, $Gr(T)$ is a closed set, T is the upper semi-continuous multifunction and by theorem B T has a fixed point. Let x_* be a fixed point of T i.e. $x_* \in T(x_*)$. Then by definition of T $G(x_*, x_*) \bigcap C \neq \emptyset$. The proof is complete.

Remark. *If in the Theorem 1 G is a single-valued continuous function and Y is a Hausdorff topological vector space, then Theorem 1 reduce to the Theorem 6 in [6], which contains as special cases Theorem 17 of [1] (when X is a locally convex space) and Theorem 10 of [3] (when X is a locally convex space and X = Y).*

Next result is a generalization of results of I. S. Iohvidov ([7] - Theorem 1.), F. Browder ([1] - Theorem 18.) and O. Hadžić ([5] - Corollary 5.)

Corollary 1. *Let K be a nonempty compact and convex subset of Hausdorff topological vector space X, Y a topological-vector space and $G : K \times K \to Y$ an upper semi-continuous multifunction such that*

$$\alpha_1 G(x, y_1) + \alpha_2 G(x, y_2) \subseteq G(x, \alpha_1 y_1 + \alpha_2 y_2)$$

holds for all $x, y_1, y_2 \in K$ and $\alpha_1 \geq 0$, $\alpha_2 \geq 0$ with $\alpha_1 + \alpha_2 = 1$. If for each $x \in K$ there is a $y \in K$ with $0 \in G(x,y)$ and K is of Zima's type, then there exists $x_* \in K$ such that $0 \in g(x_*, x_*)$.

Proof. We let $C = \{0\}$ in Theorem, and note that for each $x \in K$, the set $\{y \in K | 0 \in G(x,y)\}$ is nonempty and convex. Hence, by Theorem 1 there exists $x_* \in K$ such that $0 \in G(x_*, x_*)$.

Next result is a generalization of results of Ky Fan ([3] - Corollary.), F. Browder ([1] - Theorem 19.) and O. Hadžić ([5] - Corollary 6.)

Corollary 2. *Let K be a nonempty, compact and convex subset of Zima's type of Hausdorf topological vector space X, C a non-empty closed, convex subset of X. Suppose that $F: K \to X$ is an upper semi-continuous mapping such that $F(x) \bigcap (K + C)$, is convex, nonempty set for each $x \in K$. Then there exists $x_* \in K$ such that $F(x_*) \bigcap (x_* + C) \neq \varnothing$.*

Proof. Let in Theorem 1, $X = Y$ and $G(x,y) = F(x) - y$ for every $x, y \in K$. Then

$$\{y \in K | (F(x) - y) \bigcap C \neq \varnothing\}$$

is a nonempty convex set. Hence, by Theorem 1 there exist $x_* \in K$ such that $C \bigcap G(x_*, x_*) \neq \varnothing$ i.e. $F(x_*) \bigcap (x_* + C) \neq \varnothing$.

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