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THE SEMI-CONTINUITY OF THE BROWDER ESSENTIAL GENERALIZED SPECTRUM

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Abstract. In this paper the points of upper semi-continuity of the Browder essential generalized spectrum and generalized spectrum of bounded operators on Banach space are characterized.

1. Introduction

Let X be an infinite-dimensional Banach space and denote the set of bounded (compact) linear operators on X by $B(X)$ $(K(X))$. For $A \in B(X)$ denote, respectively, by $N(A)$ and $R(A)$ null space and the range space of A. Set $N(A^\infty) = \cup_n N(A^n)$, $R(A^\infty) = \cap_n R(A^n)$, $\alpha(A) = \dim N(A)$, $\beta(A) = \dim X/R(A)$ and $k(A) = \dim N(A)/(N(A) \cap R(A^\infty))$.
Recall that an operator $A \in B(X)$ is semi-Fredholm if $R(A)$ is closed and at least one of $\alpha(A)$ or $\beta(A)$ is finite. For such an operator we define an index $i(A)$ by $i(A) = \alpha(A) - \beta(A)$. Let $\Phi_+(X)$ $(\Phi_-(X))$ denote the set of semi-Fredholm operators with $\alpha(A) < \infty$ $(\beta(A) < \infty)$ and $\sigma_{ek}(A)$ Kato essential spectrum of A, i.e.,

$$\sigma_{ek}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_+(A) \cup \Phi_-(A)\}.$$

Let $\sigma_a(A)$ and $\sigma_d(A)$ denote, respectively, the approximate point spectrum and the approximate defect spectrum of an element $A \in B(X)$, i.e.,

$$\sigma_a(A) = \{\lambda \in \mathbb{C} : \inf_{\|x\|=1} \|(A - \lambda)x\| = 0\},$$
$$\sigma_d(A) = \{\lambda \in \mathbb{C} : A - \lambda \text{ is not onto }\}.$$

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Set

$$V(X) = \{A \in B(X) : R(A) \text{ is closed and } k(A) < \infty\},$$
$$V_n(X) = \{A \in B(X) : R(A) \text{ is closed and } k(A) = n\}, \quad n = 0, 1, 2, \ldots .$$

Recall that $\Phi_+(A) \cup \Phi_-(A) \subset V(X)$ [3], and $k(A) = 0$ if and only if $N(A^\infty) \subset R(A^\infty)$ [4]. Let

$$\sigma_g(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin V_0(X)\}$$
$$\sigma_{gb}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin V(X)\}$$

denote the generalized (regular) spectrum of A and the Browder essential generalized spectrum of A.

If (τ_n) is a sequence of compact subsets of $\mathbb{C}$, then the *limit inferior*, in notation $\liminf \tau_n$, is

$$\liminf \tau_n = \{\lambda \in \mathbb{C} : \text{ there are } \lambda_n \in \tau_n \text{ with } \lambda_n \to \lambda\},$$

and the *limit superior*, in notation $\limsup \tau_n$, is

$$\limsup \tau_n = \{\lambda \in \mathbb{C} : \text{ there are } \lambda_{n_k} \in \tau_{n_k} \text{ with } \lambda_{n_k} \to \lambda\}.$$

If $\liminf \tau_n = \limsup \tau_n$, then $\lim \tau_n$ is said to exist and is equal to this common limit.

Let p be a mapping on $B(X)$, whose values are compact subsets of $\mathbb{C}$. A mapping p is *upper* (*lower*) *semicontinuous* at A, if for every sequence $\{A_n\} \subset B(X)$ such that $A_n \to A$ then $\limsup p(A_n) \subset p(A)$ ($p(A) \subset \liminf p(A_n)$). If p is both upper and lower semicontinuous at A, then p is *contiuous* at A and in this case we write $\lim p(A_n) = p(A)$.

In this note we investigate the sufficient and necessary conditions for upper semi-continuity of the Browder essential generalized spectrum and generalized spectrum of bounded operators on a Banach space which are mentioned in private correspodence between Prof. Laura Burlando and Prof. Vladimir Rakočević (see [6]).

2. Results

Remark 2.1. Reccal that σ_g and σ_{gb} are not upper semi-continuous in the algebras of bounded linear operators on infinite-dimensional Banach space. Example in which σ_g and σ_{gb} are not upper semi-continuous can be constructed using [2, Theorem 1]. By [2, Theorem 1] if $A \in V(X)$, $\alpha(A) = \infty$ and $\beta(A) = \infty$, we can find a sequence $A_n \in B(X)$ with non-closed ranges, such that $A_n \to A$. Thus, we have that $0 \notin \sigma_g(A)(\sigma_{gb}(A))$ and $0 \in \sigma_g(A_n)(\sigma_{gb}(A_n))$ for each n, i.e. $0 \in \limsup \sigma_g(A_n)(\limsup \sigma_{gb}(A_n))$.

Remark 2.2. Let us remark that in the commutative Banach algebra of linear operators $B(X)$ the Browder essential generalized spectrum and generalized spectrum are upper semi-continuous in every $A \in B(X)$ [7, Theorem 4.3].

Theorem 2.3. *Let $A \in B(X)$. Then σ_{gb} is upper semi-continuous at A if and only if $\sigma_{gb}(A) = \sigma_{ek}(A)$.*

Proof. Let $\sigma_{gb}(A) = \sigma_{ek}(A)$ and $A_n \to A$, $A_n \in B(X)$. Since, for every $T \in B(X)$ $\sigma_{gb}(T) \subset \sigma_{ek}(T)$ [7, Coll. 3.4], and σ_{ek} is upper semi-continuous in $B(X)$, we have

$$\limsup \sigma_{gb}(A_n) \subset \limsup \sigma_{ek}(A_n) \subset \sigma_{ek}(A) = \sigma_{gb}(A)\,,$$

i.e. σ_{gb} is upper semi-continuous at A.

Conversely suppose that $\sigma_{gb}(A) \neq \sigma_{ek}(A)$. Then there exists $\lambda \in \sigma_{ek}(A) \setminus \sigma_{gb}(A)$. For that λ we have:

$$\begin{aligned} &1)\quad R(A-\lambda) = \overline{R(A-\lambda)} \\ &2)\quad k(A-\lambda) < \infty \\ &3)\quad \alpha(A-\lambda) = \beta(A-\lambda) = \infty\,. \end{aligned}$$

There exists a sequence $T_n \in B(X)$ with non-closed ranges such that $T_n \to A - \lambda$ [2]. Thus, we have that $0 \in \limsup \sigma_{gb}(T_n)$, i.e. for $A_n = T_n + \lambda$, $\lambda \in \limsup \sigma_{gb}(A_n)$, $A_n \to A$ and $\lambda \notin \sigma_{gb}(A)$. Hence, σ_{gb} is not upper semi-continuous at A. □

Lemma 2.4. *Let $A \in B(X)$. Then $\sigma_a \cap \sigma_d$ is upper semi-continuous at A.*

Proof. Since σ_a and σ_d are upper semi-continuous at A [1], we have that $\sigma_a \cap \sigma_d$ is upper semi-continuous at A. □

Theorem 2.5. *Let $A \in B(X)$. If $\sigma_g(A) = \sigma_a(A) \cap \sigma_d(A)$, then σ_g is upper semi-continuous at A.*

Proof. If $\sigma_g(A) = \sigma_a(A) \cap \sigma_d(A)$, then for every sequence $\{A_n\} \subset B(X)$ such that $A_n \to A$, since for every $T \in B(X)$, $\sigma_g(T) \subset \sigma_a(T) \cap \sigma_d(T)$ [5], we have:

$$\limsup \sigma_g(A_n) \subset \limsup(\sigma_a(A_n) \cap \sigma_d(A_n)) \subset (\sigma_a(A) \cap \sigma_d(A)) = \sigma_g(A)\,,$$

i.e. σ_g is upper semi-continuous at A. □

Theorem 2.6. *Let $A \in B(X)$. If σ_g is upper semi-continuous at A, then $\sigma_{ek}(A) \subset \sigma_g(A)$.*

Proof. Suppose that $\sigma_{ek}(A) \not\subseteq \sigma_g(A)$. Then there exists a $\lambda \in \sigma_{ek}(A) \setminus \sigma_g(A)$. For that λ we have:

$$
\begin{aligned}
&1) \quad R(A-\lambda) = \overline{R(A-\lambda)} \\
&2) \quad k(A-\lambda) = 0 \\
&3) \quad \alpha(A-\lambda) = \beta(A-\lambda) = \infty\,.
\end{aligned}
$$

Then there exists a sequence $T_n \in B(X)$ with non-closed ranges such that $T_n \to A - \lambda$ [2]. Thus, we have that $0 \notin \limsup \sigma_g(T_n)$, i.e. for $A_n = T_n + \lambda$, $\lambda \notin \limsup \sigma_g(A_n)$ and $A_n \to A$. □

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