

A NOTE ON GENERALIZED RESOLVENT

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Abstract. Let T be a Saphar operator on a complex Banach space. In this note we study a generalized resolvent defined on a subset of the Saphar domain of T . Our results are related to the problem posed in [13] [18].

Let X be an infinite-dimensional complex Banach space and denote the set of bounded linear operators on X by $B(X)$. For T in $B(X)$ let $N(T)$, $R(T)$ and $R^\infty(T) = \bigcap_n R(T^n)$ be respectively, the null space, the range space, and the generalized range (hypperrange) of T .

An operator $S \in B(X)$ is a *generalized inverse* (*pseudo inverse*) of T if

$$(0.1) \quad TST = T.$$

We then say that T is *relatively regular*. It is easy to see that if $TST = T$, then the operator $S_1 = STS$ satisfies the equations

$$(0.2) \quad TS_1T = T \quad \text{and} \quad S_1TS_1 = S_1.$$

It is well known that T is relatively regular if and only if $N(T)$ and $R(T)$ are closed, complemented subspaces of X . In this case TS is a projection onto $R(T)$ and $I - ST$ is a projection onto $N(T)$.

$T \in B(X)$ is called an operator of *Saphar type* (*Saphar operator*, or *hyper-regular*, or *regular* see e.g. ([1], [4], [13], [17], [18])) if T is relatively regular and $N(T) \subset R^\infty(T)$. Let us remark that the set of operators $T \in B(X)$ with closed range and $N(T) \subset R^\infty(T)$; ($\dim[N(T)/N(T) \cap R^\infty(T)] < \infty$) is related to the set of Saphar operators, and was studied by many authors; see e.g. ([3], [5], [6], [7], [8], [9], [10], [13], [15], [16]).

Now, set

$$(0.3) \quad \mathcal{S}(X) = \{A \in B(X) : A \text{ is Saphar operator}\},$$

Received June 23, 1996

1991 *Mathematics Subject Classification*: 47A10, 47A53, 47A55.

Key words and phrases: Saphar operator, generalized resolvent

Supported by Grant 04M03 of RFNS through Math. Inst. SANU.

and

$$(0.4) \quad \sigma_s(A) = \{\lambda \in \mathbb{C} : A - \lambda I \notin \mathcal{S}(X)\}.$$

Recall that (see e.g. [13], [18]) $\mathbb{C} \setminus \sigma_s(T)$ is an open set and there exists an analytic function $F : \mathbb{C} \setminus \sigma_s(T) \mapsto B(X)$ such that

$$(T - \lambda)F(\lambda)(T - \lambda) = T - \lambda, \quad \text{and} \quad F(\lambda)(T - \lambda)F(\lambda) = F(\lambda),$$

for all $\lambda \in \mathbb{C} \setminus \sigma_s(T)$.

To the best of my knowledge the next problem is still open

Question 1. ([13, Remark 4.2], [18, Question 3]) Let $T \in B(X)$. Does there exist a holomorphic function $F : \mathbb{C} \setminus \sigma_s(T) \mapsto B(X)$ such that

$$(1.1) \quad (T - \lambda)F(\lambda)(T - \lambda) = T - \lambda, \quad \text{for all } \lambda \in \mathbb{C} \setminus \sigma_s(T),$$

$$(1.2) \quad F(\lambda)(T - \lambda)F(\lambda) = F(\lambda) \quad \text{for all } \lambda \in \mathbb{C} \setminus \sigma_s(T),$$

and

$$(1.3) \quad F(\lambda) - F(\mu) = (\lambda - \mu)F(\lambda)F(\mu), \quad \text{for all } \lambda, \mu \in \mathbb{C} \setminus \sigma_s(T)?$$

Remark 2. Recall that $T \in \mathcal{S}(X)$ if and only if there is a neighbourhood $U \subset \mathbb{C}$ of 0 and a holomorphic function $F : U \mapsto B(X)$ such that

$$(T - \lambda)F(\lambda)(T - \lambda) = T - \lambda, \quad \text{and} \quad F(\lambda)(T - \lambda)F(\lambda) = F(\lambda),$$

for all $\lambda \in U$.

For F it is possible to take

$$(2.1) \quad F(\lambda) = \sum_{i=0}^{\infty} S^{i+1} \lambda^i, \quad \lambda \in U,$$

where $S \in B(X)$ is a generalized inverse of T , i.e., $TST = T$ and $TST = T$, and $U \equiv \{\lambda \in \mathbb{C} : |\lambda| < \|S\|^{-1}\}$. Further

$$(2.2) \quad F(\lambda) - F(\mu) = (\lambda - \mu)F(\lambda)F(\mu), \quad \text{for all } \lambda, \mu \in U,$$

i.e., $F(\lambda)$ satisfies the classic first resolvent equation on U , and $F(\lambda)$ is a *generalized resolvent (pseudoresolvent)* [11].

Let $T \in \mathcal{S}(X)$. Then $T^n \in \mathcal{S}(X)$ for each $n \in \mathbb{N}$, and set ([18])

$$(2.3) \quad \text{dist} \{0, \sigma_s(T)\} = d(T),$$

$$(2.4) \quad \delta_n(T) = \sup\{r(A)^{-1} : A \in B(X), T^n A T^n = T^n\}, \quad \text{for } n \in \mathbb{N},$$

where $r(A)$ is the spectral radius of A , and

$$(2.5) \quad \delta(T) = \sup_{n \geq 1} \delta_n(T)^{1/n}.$$

As an application of the sequence $\delta_n(T)$ C. Schmoegeer [18, Theorem 3] has proved

Theorem (Schmoeger) 3. Suppose that $T \in \mathcal{S}(X)$. Then

$$(3.1) \quad \delta(T) = \lim_{n \rightarrow \infty} \delta_n(T)^{1/n} = d(T).$$

If $F(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n$ is a holomorphic pseudo-inverse of $T - \lambda$ for $|\lambda| < d(T)$, then

$$(3.2) \quad d(T) = \left(\lim_{n \rightarrow \infty} \|A_n\|^{1/n} \right)^{-1} = \lim_{n \rightarrow \infty} (r(A_n)^{1/n})^{-1}.$$

Concerning the question above we can prove the following theorem.

Theorem 4. Suppose that $T \in \mathcal{S}(X)$, $\text{dist}\{0, \sigma_s(T)\} = d(T)$ and $\mathbf{G} = \{\lambda \in \mathbb{C} : |\lambda| < d(T)\}$. Then for any compact subset \mathbb{K} of \mathbf{G} there exists an analytic function $F : U \rightarrow B(X)$, where $U \subset \mathbf{G}$, is a neighbourhood of \mathbb{K} , such that

$$(4.1) \quad (T - \lambda)F(\lambda)(T - \lambda) = T - \lambda, \quad \text{for all } \lambda \in \mathbb{K},$$

$$(4.2) \quad F(\lambda)(T - \lambda)F(\lambda) = F(\lambda), \quad \text{for all } \lambda \in \mathbb{K}$$

and

$$(4.3) \quad F(\lambda) - F(\mu) = (\lambda - \mu)F(\lambda)F(\mu), \quad \text{for all } \lambda, \mu \in \mathbb{K},$$

if and only if

$$(4.4) \quad d(T) = \sup\{r(A)^{-1} : A \in B(X), TAT = T \text{ and } ATA = T\}.$$

Proof. Suppose that (4.4) is true. Let \mathbb{K} be a compact subset of \mathbf{G} . Then there is some $\epsilon > 0$ such that $\mathbb{K} \subset (1 + \epsilon)^{-1} \mathbf{G}$. Since $d(T) > (1 + \epsilon)^{-1} d(T)$, by (4.4) there is some $A_0 \in B(X)$ such that

$$(4.5) \quad \frac{1}{r(A_0)} > \frac{d(T)}{1 + \epsilon}, \quad TA_0T = T \quad \text{and} \quad A_0TA_0 = T.$$

Hence

$$(4.6) \quad F(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_0^{n+1}, \quad \lambda \in (1 + \epsilon)^{-1} \mathbf{G},$$

is a holomorphic function on $(1 + \epsilon)^{-1} \mathbf{G}$. Then for $\lambda \in \mathbb{K}$ we have

$$\begin{aligned}
 (T - \lambda)F(\lambda)(T - \lambda) &= (T - \lambda)\left(\sum_{n=0}^{\infty} \lambda^n A_0^{n+1}\right)(T - \lambda) = \\
 (T - \lambda)\left[I - \sum_{n=0}^{\infty} A_0^n (I - A_0 T) \lambda^n\right] &= \\
 (4.7) \quad (T - \lambda) - \sum_{n=0}^{\infty} (T - \lambda) A_0^n (I - A_0 T) \lambda^n &= \\
 (T - \lambda) - T(I - A_0 T) - \sum_{n=0}^{\infty} (T A_0 - I) A_0^n (I - A_0 T) \lambda^{n+1}.
 \end{aligned}$$

Since $(T A_0 - I) A_0^n (I - A_0 T) = 0, n = 0, 1, \dots$ (see e.g. [13, p.p. 377]), from (4.7) it follows (4.1). Now,

$$\begin{aligned}
 F(\lambda)(T - \lambda)F(\lambda) &= \left[I - \sum_{n=0}^{\infty} A_0^n (I - A_0 T) \lambda^n\right] F(\lambda) = \\
 (4.8) \quad F(\lambda) - \left[\sum_{n=0}^{\infty} \lambda^n A_0^n (I - A_0 T)\right] F(\lambda) &= F(\lambda),
 \end{aligned}$$

and we get (4.2). (4.3) follows by [19, Theorem 2].

Suppose that (4.1), (4.2) and (4.3) hold. By (3.1) we have

$$(4.9) \quad \sup\{r(A)^{-1} : A \in B(X), TAT = T \text{ and } ATA = T\} \leq d(T),$$

and now (4.4) follows by the proof of [19, Theorem 3]. ■

Recall that an operator $T \in B(X)$ is *Fredholm* if $R(T)$ is closed, and both $\dim N(T)$ and $\dim X/R(T)$ are finite. The Fredholm operators $\Phi(X)$ constitute a multiplicative open semigroup in $B(X)$ (see e.g. [2], [4]).

Corollary 5. *Suppose that $T \in \mathcal{S}(X) \cap \Phi(X)$. Then*

$$(5.1) \quad d(T) = \sup\{r(A)^{-1} : A \in B(X), TAT = T \text{ and } ATA = T\}.$$

Proof. By Theorem 4 and [12, Theorem 3.1] ■

Recall that if X is a complex Hilbert space, $T \in B(X)$, then $R(T)$ is closed if and only if there exists a unique operator $T^\dagger \in B(X)$ satisfying the following four Penrose (see e.g. [1], [14]) identities:

$$(5.2) \quad TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad \text{and} \quad (T^\dagger T)^* = T^\dagger T.$$

The operator T^\dagger is called the *Moore-Penrose* inverse of T .

Corollary 6. *Let X be a complex Hilbert space and $T \in \mathcal{S}(X)$. If there is a subsequence $\{n_k\}_{k=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$ such that*

$$(6.1) \quad \|(T^\dagger)^{n_k}\| \leq \|(T^{n_k})^\dagger\|, \quad k = 1, 2, \dots$$

then

$$(6.2) \quad d(T) = \sup\{r(A)^{-1} : A \in B(X), TAT = T \text{ and } ATA = T\}.$$

Proof. By (3.1) we have only to prove " \leq " in (6.2). From (6.1) we get

$$(6.3) \quad \|(T^\dagger)^{n_k}\|^{1/n_k} \leq \|(T^{n_k})^\dagger\|^{1/n_k}, \quad k = 1, 2, \dots$$

and now (see e. g. [18, Corollary 3, Proposition 4] and [14, Theorem 2.3, Corollary 2.4])

$$r(T^\dagger) = \lim_{k \rightarrow \infty} \|(T^\dagger)^{n_k}\|^{1/n_k} \leq \lim_{k \rightarrow \infty} \|(T^{n_k})^\dagger\|^{1/n_k} = \frac{1}{d(T)}. \quad \blacksquare$$

Let us remark that, $T \in \mathcal{S}(X)$ implies $T^n T^\dagger T^n = T^n$ for all $n = 1, 2, \dots$, ([18, Proposition 5(a)]); hence $\|T^n\| \leq \|T^\dagger\|$ for all $n = 1, 2, \dots$, (see e.g. [18, Proposition 4] and [14, Theorem 2.3, Corollary 2.4]) and in fact in (6.1) we get equality.

Acknowledgements.

I am grateful to Professors Vladimir Müller and Christoph Schmoegeer for the helpful conversations and correspondence.

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