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ON A LINEAR POSITIVE OPERATOR AND ITS APPROXIMATION'S PROPERTIES

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Abstract. In this paper we study the approximation's properties of a linear positive operator which was defined using a probabilistic method and is associated with the Pearson's-$\mathcal{X}^2$ distribution.

1. Introduction

Consider the interval $I = (0, \infty)$ and for every $x \in (0, \infty)$ let $(X_{n,x})_{n \in \mathrm{N}^*}$ be a sequence of independet random variables having the same Gaussian distribution with the parameters 0 and x, $N(0, \sqrt{x})$.

Then the sum $\sum_{k=1}^{n} X_{k,x}^2$ has a Pearson-$\mathcal{X}^2$ distribution with n degres of freedom and parameter x, $x > 0$. Its probability density function $\rho_{n,x}$ is given by:

$$(1.1) \qquad \rho_{n,x}(t) = \begin{cases} 0 & , t < 0 \\ \frac{t^{n/2-1} e^{-t/(2x)}}{(2x)^{n/2}\Gamma(n/2)} & , t \geq 0, \ x > 0, \end{cases}$$

where Γ denotes the gamma function. We shall consider the sequence of independent random variables:

$$(12) \qquad Y_{n,x} := \frac{1}{n} \sum_{k=1}^{n} X_{k,x}^2, \ n \in \mathrm{N}^*, x \in D.$$

For every $f \in C_b^*(0, \infty)$ and $x \in (0, \infty)$ we have:

$$(1.3) \qquad (C_n f)(x) = \frac{1}{(2x)^{n/2}\Gamma(n/2)} \int_0^{\infty} t^{n/2-1} e^{-t/(2x)} f(t/n) dt.$$

This operator was defined in our paper [2], using a probabilistic method which was introduced by D.D. Stancu [7] and it was called "the n^{-th} Cismasiu operator" by F. Altomare, M. Campiti [1]. We denote with $C_b^*(0, \infty)$ the

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class of all bounded and uniformly continuous functions on $(0,\infty)$ endowed with norm $\|f\| = \sup\{|f(x)| : x > 0\}$.
In this case, the operator (1.3) is the mean value of the random variables $f(Y_{n,x})$:

$$(C_n f)(x) = E[f(Y_{n,x})] = E\left[f\left(\frac{1}{n}\sum_{k=1}^{n} X_{k,x}^2\right)\right].$$

In Section 2 we shall present the approximation properties of this operator. Next Section 3 give the estimate of the order approximation and then the asymptotic estimate of the remainder.

2. Approximation property

In this section we investigate the approximation properties of the operator (1.3).

Theorem 2.1. *If* $f \in C_b^*(0,\infty)$, *then* $\lim_{n\to\infty} C_n(f) = f$ *uniformly on every compact interval of* $(0,\infty)$.

Proof. In accordance with a result of King [6] the sequence $(C_n f)_{n\in\mathbb{N}^*}$ is uniformly convergent to f iff:

$$\lim_{n\to\infty} E(Y_{n,x}) = x \quad \text{and} \quad \lim_{n\to\infty} D^2(Y_{n,x}) = 0.$$

But

$$E(Y_{n,x}) = E\left(\frac{1}{n}\sum_{k=1}^{n} X_{k,x}^2\right) = \frac{2x}{n\Gamma(n/2)}\Gamma(n/2+1) = x$$

and

$$D^2(Y_{n,x}) = \frac{1}{n^2}E\left(\sum_{k=1}^{n} X_{k,x}^2\right)^2 - x^2 = \frac{4x^2}{n^2\Gamma(n/2)}\Gamma(n/2+2) - x^2 = \frac{2}{n}x^2,$$

where $D^2(Y_{n,x})$ denote the variance of the Pearson's - $\mathcal{X}^2$ distribution with n degrees of freedom and parameter x.

3. Estimate of order of approximation

We shall now proceed to estimate the order of approximation of function f by the operator (1.3). It is convenient to make use of the modulus of continuity, defined as follows:
$\omega(f;\delta) = \sup\{|f(x) - f(y)| : |x-y| < \delta\}$, where $x \in (0,a)$ and $y \in (0,a)$, $a > 0$ and $\delta > 0$.

Theorem 3.1. *If f is a bounded and uniform continuous function on $(0, a)$, $a > 0$, then:*

$$|f(x) - (C_n f)(x)| < (1 + a\sqrt{2})\omega(f; n^{-1/2}) \tag{3.1}$$

and if f has a bounded uniformly continuous derivative on $(0, a)$, $a > 0$, then:

$$|f(x) - (C_n f)(x)| < a\sqrt{2/n}(1 + a\sqrt{2})\omega(f'; n^{-1/2}). \tag{3.2}$$

Proof. Because $\sigma_{n,x}^2 = D^2(Y_{n,x}) = Var(Y_{n,x}) = 2x^2/n$ and $\beta_n = \sup\{\sigma_{n,x} : x \in (0,a), a > 0\} = a\sqrt{2/n}$, using the following result of Stancu [7].

$$|f(x) - (L_n f)(x)| < (1 + \beta_n\sqrt{n})\omega(f; n^{-1/2})$$

respectivly

$$|f(x) - (L_n f)(x)| < \beta_n(1 + \beta_n\sqrt{n})\omega(f'; n^{-1/2})$$

we obtain the inequality (3.1) respectivly (3.2).

For a continuous function f on the real axis, the operator $L_n f$ is

$$(L_n f)(x) = Ef\left[\frac{1}{n}\sum_{k=1}^{n} X_k\right] = \int_{\mathbb{R}} f(t)dF_{n,x}(t) \tag{3.3}$$

where $(X_n)_{n\in\mathbb{N}}$ is a sequence of random variables having distribution function $F_{n,x}(t)$ with mean value $E(X_n) = x$ and variance $\sigma_n^2(x)$. The following well known operators Bernstein, Szasz, Baskakov, Feller, Weienstrass are special case of (3.3), which are obtained when X_n has a binomial, Poisson, geometric, gamma or normal distribution.

4. Asymptotic estimate of the remainder

In this section we shall deal with the problem of asymptotic estimate of the remainder in an approximation formula. Using our results [3],[4] we have:

Theorem 4.1. *If f is a real bounded function on $(0, \infty)$ having the second derivative at a point $x \in (0, \infty)$ then for the sequence (1.3):*

$$\lim_{n\to\infty} n[f(x) - (C_n f)(x)] = -x^2 f''(x). \tag{4.1}$$

Proof. In the case the operator (1.3), $(C_nf)(x) = E\left[f\left(\frac{1}{n}\sum_{k=1}^{n} X_{k,x}^2\right)\right]$, where $(X_k)_{k\in\mathbb{N}^*}$ has the normal distribution $N(0,\sqrt{x})$, $x > 0$, our result of [3] which was gived for the operator (3.3):

$$\lim_{n\to\infty} n\left\{f(x) - E\left[f\left(\frac{1}{n}\sum_{k=1}^{n} X_{k,x}\right)\right]\right\} = -\frac{1}{2}f''(x)Var(X_k)$$

become (4.1), because

$$Var(X_{k,x}^2) = E[X_{k,x}^4] - (E[X_{k,x}^2])^2 = 3x^2 - x^2 = 2x^2.$$

Theorem 4.2. *If f is a real bounded function on $(0,\infty)$ having the derivative $f^{(2k)}$ at a point $x \in (0,\infty)$ then for the sequence (1.3):*

$$\lim_{n\to\infty} n^k\left[(C_nf)(x) - f(x) - \sum_{s=1}^{2k-1}\frac{f^{(s)}(x)}{s!n^s}T_{n,s}(x)\right] = \frac{x^{2k}}{k!}f^{(2k)}(x),$$

where $T_{n,s}(x) = E\left[\left(\sum_{k=1}^{n} X_{k,x}^2 - nx\right)^s\right]$.

Proof. Our result of [4] which was gived for the operator (3.3)

$$\lim_{n\to\infty} n^k\left\{E\left[f\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right) - f(x) - \sum_{s=1}^{2k-1}\frac{f^{(s)}(x)}{s!n^s}T_{n,s}(x)\right]\right\}$$
$$= \frac{1}{2^kk!}Var^k(X_n)f^{(2k)}(x),$$

become (4.2) in the case of the positive linear operator (1.3), because $Var(X_k^2) = 2x^2$.

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