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ON A GENERAL ITERATIVE METHOD FOR SOLVING HEREDITARY DIFFERENTIAL EQUATIONS (I)

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Abstract. In this paper we consider a general iterative method for solving hereditary differential equations. We give sufficient conditions for uniform convergence of a sequence of iterations to the solution of the original equation and we estimate the speed of convergence of these iterations. Also, we construct some concrete iterative methods as special cases of this general iterative procedure.

1. Introduction

In some sense, the idea of the present investigation goes back to the paper of R. Zuber ([10]) treating one general analytic method for solving the ordinary differential equation $y' = f(x,y)$, $y(x_0) = y_0$. The essence is in what follows: suppose that the functions $f(x,y)$ and $F_n(x,y)$, $n = 0,1,\dots$, are defined and continuous on a compact $\Pi = \{(x,y) : |x-x_0| \leq a, |y-y_0| \leq b\}$. Each of these functions satisfies on Π the Lipschitz condition on the second argument with a same constant L. Let $y_{n+1}(x)$ be a solution of the equation $y'_{n+1} = F_n(x, y_{n+1})$, $y_{n+1}(x_0) = y_0$, defined on the segment $[x_0 - a, x_0 + a]$. If $\sum_{n=0}^{\infty} \sup_{|x-x_0|\leq a} |F_n(x, y_n(x)) - f(x, y_n(x))| < \infty$, then there exists a constant h, $o < h \leq a$, such that the sequence of solutions $\{y_n(x), n \in N\}$ converges uniformly in x, $x \in [x_0 - h, x_0 + h]$, to the solution $y(x)$ of the equation $y' = f(x,y)$, $y(x_0) = y_0$. If the choice of the functions $F_n(x,y)$, $n = 0,1,\dots$ is good enough in the sense that the corresponding differential equations can be effectively solved, then we are in a position to find an ϵ-approximation of the solution $y(x)$ of the original equation.

This iterative procedure presents a general algorithm for solving ordinary differential equations, in the paper [10] called the Z-algorythm, because many well-known, historically important iterative methods, are its special cases,

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for example, Picard-Lindelöf method of succesive approximations, Chaplygin methods of secants and tangents, Newton-Kantorovich method and some interpolation methods ([11]). Later, this approach was appropriatelly extended to analyse special classes of stochastic differential and integrodifferential equations of Ito type (for example, see [5], [6], [7]).

The aim of the present paper is to construct an analogous general iterative procedure for solving functional differential equations, called hereditary differential equations. From purely theoretical point of view, and much more from the point of view of various applications, it is very important to make suitable methods for finding at least approximate solutions of these equations and to estimate an error of n-th approximation of the solution of the original equation. Notice that in the paper [10], in general case, such estimation is not given. Also, it is important to make some concrete algorithms as special cases of this general algorithm.

At the beginning we give some known notions from the theory of hereditary differential equations. Let us have in mind that the hereditary phenomena are based on different problems in continuum mechanics of materials with memories, as a version of the theory of "fading memory". Historically, the first fundamental results, such as existence, uniqueness and stability problems of solutions of different classes of these equations, are investigated in the papers [1], [2], [3], [8], [9] and in many others.

Let R^k be the real k-dimensional Euclidian space and L_p^ρ, $1 \leq p \leq \infty$, be the usual space of classes of measurable functions, i.e.,

$$L_p^\rho = \left\{ \varphi \mid \quad \varphi : R^+ \to R^k; \quad \int_0^\infty |\varphi(t)|^p \rho(t)\, dt < \infty \right\},$$

where the function $\rho : R^+ \to R^+$, called an influence function with relaxation properties, is summable on R^+ and for every $\sigma \geq 0$ one has

$$\overline{K}(\sigma) = \operatorname*{ess\,sup}_{s \in R^+} \frac{\rho(s+\sigma)}{\rho(s)} \leq \overline{\overline{K}} < \infty,$$

$$\underline{K}(\sigma) = \operatorname*{ess\,sup}_{s \in R^+} \frac{\rho(s)}{\rho(s+\sigma)} < \infty.$$

Also, ρ is essentially bounded, essentially strictly positive and $s\rho(s) \to 0$ as $s \to 0$ (see [1]).

Let X be *a past-history space*, i.e., a product space $X = R^k \times L_p^\rho$ of elements x, $x = (\varphi(0), \varphi)$, with the norm

$$||x||_X = \left(|\varphi(0)|^p + \int_0^\infty |\varphi(t)|^p \rho(t)\, dt \right)^{1/p} = (|\varphi(0)|^p + ||\varphi||_r^p)^{1/p}.$$

Obviously, X is the Banach space.

An essential property of this space is the existence of the mappings (see [1], [2]): for $\sigma \geq 0$,

$$(T^\sigma \varphi)(s) = \begin{cases} \varphi(0), & s \in [0, \sigma] \\ \varphi(s-\sigma), & s \in [\sigma, \infty), \end{cases}$$

$$(\overline{T}^\sigma \varphi)(s) = \begin{cases} 0, & s \in [0, \sigma] \\ \varphi(s-\sigma), & s \in [\sigma, \infty), \end{cases}$$

for $s \geq 0$, $\varphi \in X$, and $\lim_{\sigma\to\infty} \|T^\sigma\varphi - \varphi(0)^+\|_X = 0$, where $\varphi(0)^+$ is the constant function with value $\varphi(0)$. Because of that, one can formulate the notion of X-admissibility for measurable functions defined on any left semiaxis of R.

The measurable function $x : (-\infty, a] \to R^k$, $a =$const$\in R$, is *X-admissible* if for each $t \in (-\infty, a]$ the function x^t, called *its history up to t* and defined by $x^t(s) = x(t-s)$, $s \in R^+$, is itself an element in X.

Therefore, if x is X-admissible, $x^t = (x(t), x^t_r) \in X$ for each $t \in (-\infty, a]$.

From the definition of the norm on the space X, for each $t \geq t_0$, $t_0 \in (-\infty, a]$, it follows

$$\text{(1)} \qquad \|x^t\|_X \leq \tilde{K}\left[|x(t)| + \overline{\overline{K}}^{\frac{1}{p}}\|x^{t_0}\|_r + \left(\int_{t_0}^t |x(u)|^p \rho(t-u)\,du\right)^{\frac{1}{p}}\right],$$

where $\tilde{K} = 3^{\frac{1}{p}-1} \vee 1$ (see [9]).

In the papers cited earlier, and many others, the functional differential equation, called *the hereditary differential equation*

$$\text{(2)} \qquad \dot{x}(t) = f(t, x^t), \quad x^0 = \varphi, \quad \varphi \in X,$$

is investigated, where $f : R \times X \to R^k$ is the given functional. Its solution consists of a function $x : (-\infty, a] \to R^k$, $a =$const> 0, such that:

a) x is X-admissible on $(-\infty, a]$;

b) x is differentiable for each $t \in (0, a]$;

c) the equation (1) holds for $t \in [0, a]$;

d) $x^0 = \varphi$.

Therefore, in order to determine a solution of the equation (2), we have to find an X-admissible function $x \in C^1((-\infty, a]; R^k)$, such that $x(0) = \varphi(0)$ and for which this equation is valid. Here, $x^t = (x(t), x^t_r)$, where

$$x(t) = \begin{cases} x(t), & 0 \leq t \leq a \\ \varphi(-t), & t \leq 0 \end{cases}$$

$$x_r^t(s) = \begin{cases} x(t-s), & 0 \le s \le t \\ \varphi(s-t), & s > t. \end{cases}$$

The continuity of the function $x(t)$ on $[0,a]$ implies that the function x^t, $t \in [0,a]$, is also continuous with respect to the norm of the space X.

There is a number of papers, for example the papers cited earlier, first of all [2] and [3], in which various sufficient conditions of the existence and uniqueness of a solution of the equation (2) are considered. So, let the functional f be continuous in the pair of arguments and satisfies the local Lipschitz condition on the second argument on some compact $\Omega \subset [0,a] \times X$, i.e., there exists a constant $L > 0$, such that for all $(t,x),(t,y) \in \Omega$,

(3) $$|f(t,x) - f(t,y)| \le L\,||x-y||_t,$$

where $||x||_t = \sup_{s \in [0,t]} ||x^s||_X$. Under these conditions it is proved that there exists a unique solution of the equation (2), defined on an interval $(-\infty, T]$, $0 \le T \le a$.

In what follows, $\tilde{X}$ denotes the Banach space $\tilde{X} = C([0,T];X)$ with the norm $||x|| = ||x^t||_T$.

Let us have in mind that the equation (2) is equivalent to the corresponding hereditary integral equation

$$x(t) = \varphi(0) + \int_0^t f(s,x^s)ds, \quad x^0 = \varphi, \quad t \in [0,T],$$

and that the proof of the existence and uniqueness of its solution is based on Picard-Lindelöf method of succesive approximations

$$x_0 \in \tilde{X}, \quad x_0^0 = \varphi,$$

$$x_n(t) = \varphi(0) + \int_0^t f(s, x_{n-1}^s)ds, \quad x_n^0 = \varphi, \quad t \in [0,T], \quad n \in N.$$

2. Main results

Together with the equation (2) we consider the sequence of hereditary differential equations

(4) $$\dot{x}_{n+1}(t) = F_n(t, x_{n+1}^t), \quad x_{n+1}^0 = \varphi, \quad n \in N,$$

where $F_n : R \times X \to R^k$, $n \in N$, are given functionals. We suppose that the functionals f and F_n, $n \in N$, are continuous in the pair of arguments and satisfy the local Lipschitz condition (3) on a compact Ω. We suppose, also, that solutions of these equations are defined on an interval $(-\infty, T]$, $T \ge 0$.

The main problem is to give some sufficient conditions of closeness of the functionals F_n, $n \in N$, with the functional f, such that the sequence of solutions $\{x_n(t), n \in N\}$ converges to the solution $x(t)$ of the equation (2) as $n \to \infty$.

Theorem. *Let the functionals f and F_n, $n \in N$, be defined as the above and let the condition*

$$\sum_{n=1}^{\infty} \sup_{t\in[0,T]} |F_n(t, x_n^t) - f(t, x_n^t)| < \infty \tag{5}$$

be fulfilled. Then the sequence of solutions $\{x_n(t),\, n \in N\}$ of the equations (4) converges uniformly in t, $t \in [0,T]$, to the solution $x(t)$ of the equation (2) as $n \to \infty$.

Proof. Denote

$$\epsilon_n = \sup_{t\in[0,T]} |F_n(t, x_n^t) - f(t, x_n^t)|, \quad n \in N.$$

So, the condition (5) implies $\sum_{n=1}^{\infty} \epsilon_n < \infty$.

From (2) and (4), using corresponding hereditary integral equations, for $n \in N$ and $0 \le s \le T$ we find

$$x(s) - x_{n+1}(s)$$
$$= \int_0^s [f(u, x^u) - f(u, x_n^u)]\, du + \int_0^s [f(u, x_n^u) - F_n(u, x_n^u)]\, du$$
$$+ \int_0^s [F_n(u, x_n^u) - F_n(u, x^u)]\, du + \int_0^s [F_n(u, x^u) - F_n(u, x_{n+1}^u)]\, du.$$

Applying the Lipschitz condition (3) in Ω, we obtain

$$|x(s) - x_{n+1}(s)|$$
$$\le 2L \int_0^s ||x^u - x_n^u||_X\, du + L \int_0^s ||x^u - x_{n+1}^u||_X\, du + \epsilon_n s.$$

For $0 \le s \le t \le T$ we have

$$\sup_{s\in[0,t]} |x(s) - x_{n+1}(s)|$$
$$\le 2L \int_0^t ||x^u - x_n^u||_X + L \int_0^t ||x^u - x_{n+1}^u||_X\, du + \epsilon_n t.$$

Since

$$||x^0 - x_n^0||_r^p = \int_0^{\infty} |x(-u) - x_n(-u)|^p \rho(u)\, du$$
$$= \int_0^{\infty} |\varphi(u) - \varphi(u)|^p \rho(u)\, du = 0,$$

from the property (1) of the norm we find

$$\sup_{s\in[0,t]} |x(s) - x_{n+1}(s)| \le 2LB \int_0^t \sup_{v\in[0,u]} |x(v) - x_n(v)|\, du$$
$$+ LB \int_0^t \sup_{v\in[0,u]} |x(v) - x_{n+1}(v)|\, du + \epsilon_n t,$$

where $B = \tilde{K}\,(1 + ||\rho||_{L_1}^{\frac{1}{p}})$. Denote $LB = a$ and

$$\sup_{s\in[0,t]} |x(s) - x_n(s)| = \alpha_n(t), \quad t \in [0,T].$$

Then

$$\alpha_{n+1}(t) \le 2a \int_0^t \alpha_n(s)\, ds + a \int_0^t \alpha_{n+1}(s)\, ds + \epsilon_n t, \quad t \in [0,T].$$

If we apply one version of the well-known Gronwall-Bellman lemma, we get the following recurrence formula

$$\alpha_{n+1}(t) \le 2a \int_0^t \alpha_n(s)\, ds + \epsilon_n t \tag{6}$$
$$+a \int_0^t \left[2a \int_0^s \alpha_n(u)\, du + \epsilon_n s\right] e^{a(t-s)}\, ds, \quad t \in [0,T], \quad n \in N.$$

This formula is considered in the paper [6]. The following upper bound for $\alpha_{n+1}(t)$ is obtained by induction, repeating integrations

$$\begin{aligned} \alpha_{n+1}(t) &< \left[2aM\,\frac{(2at)^{n-1}}{(n-1)!} + \sum_{k=1}^{n} \frac{\epsilon_k (2at)^{n-k}}{(n-k)!}\right] \cdot \frac{e^{at}-1}{a} \\ &= P_{n-1}(2at) \cdot \frac{e^{at}-1}{a}, \end{aligned} \tag{7}$$

where $M = \sup_{t\in[0,T]} |\alpha_1(t)|$ and P_{n-1} is a polinomial of degree $n-1$. The proof is based on the fact that $\frac{1}{k!}\int_0^t s^k e^{-s}\, ds = 1 - e^{-t}\left[\frac{t^k}{k!} + \frac{t^{k-1}}{(k-1)!} + \right.$ $\left. + \cdots + t + 1\right]$ and $\frac{t^k}{k!} + \frac{t^{k-1}}{(k-1)!} + \cdots + t + 1 - e^{t} < -\frac{t^{k+1}}{(k+1)!}$, $t > 0$, $k = 0, 1, \ldots,$ what implies

$$2e^{at} \int_0^t \frac{(2as)^{n-k}}{(n-k)!}(1 - e^{-as})\, ds < \frac{(2at)^{n-k+1}}{(n-k+1)!} \cdot \frac{e^{at}-1}{a},$$

$k = 1, 2, \dots n$. Since

$$\sum_{n=1}^{\infty} P_{n-1}(2aT) < \sum_{n=1}^{\infty} \frac{(2aT)^{n-1}}{(n-1)!} \left[2aM + \sum_{n=1}^{\infty} \epsilon_n \right] < \infty,$$

it follows $P_{n-1}(2aT) \to 0$ as $n \to \infty$. Therefore,

$$\alpha_{n+1}(T) = \sup_{t \in [0,T]} |x(t) - x_{n+1}(t)| \to 0 \quad \text{as} \quad n \to \infty, \tag{8}$$

i.e., $x_n(t) \to x(t)$ as $n \to \infty$, uniformly on the inteval $[0, T]$. Because of the inequality (1), it is easy to prove that $x_n^t \to x^t$ as $n \to \infty$, uniformly on $[0, T]$. Let us prove that $\int_0^t F_n(s, x_{n+1}^s)\, ds \to \int_0^t f(s, x^s)\, ds$ as $n \to \infty$, uniformly on $[0, T]$.

Since

$$\left| \int_0^t f(s, x^s)\, ds - \int_0^t F_n(s, x_{n+1}^s)\, ds \right|$$

$$\leq 2a \int_0^t \alpha_n(s)\, ds + a \int_0^t \alpha_{n+1}(s)\, ds + \epsilon_n t, \quad t \in [0, T],$$

from (8) we get

$$\sup_{t \in [0,T]} \left| \int_0^t f(s, x^s)\, ds - \int_0^t F_n(s, x_{n+1}^s)\, ds \right| \to 0 \quad \text{as} \quad n \to \infty.$$

Finally,

$$x_{n+1}(t) - \varphi(0) - \int_0^t F_n(s, x_{n+1}^s)\, ds \to x(t) - \varphi(0) - \int_0^t f(s, x^s)\, ds$$

as $n \to \infty$, uniformly on $[0, T]$. Thus the proof of the Theorem is complete. □

Therefore, the solutions of the equations (4) approximate the solution of the equation (2). From (7) we have obtained an error of n-th approximation

$$\sup_{t \in [0,T]} |x(t) - x_n(t)| < P_{n-2}(2aT) \frac{e^{aT} - 1}{a}, \quad n = 2, 3, \dots . \tag{9}$$

Remark. Notice that the proof of the Theorem is different of the proof of the analogous theorem in the paper [10], in which is shown that the sequence of iterations $\{y_n(x), n \in N\}$, converges uniformly in x, $x \in [0, \min\{T, \frac{1}{3L}\}]$, to the solution $y(x)$ of the original equation. As was stated above, in the general case an error of n-th approximation is not given. So, it will be in the form (9) with suitable constants.

From purely theoretical point of view, the Theorem suggests us an idea how to construct an ϵ-approximation of a solution of the equation (2) by a suitable choice of functionals F_n, $n \in N$. For example, let $x_1^t \in \tilde{X}$, $x_1^0 = \varphi$, be an arbitrary function. We choose a continuous functional $F_1 : R \times X \to R^k$, satisfying the Lipschitz condition (3) on Ω, such that

$$\sup_{t\in[0,T]} |F_1(t, x_1^t) - f(t, x_1^t)| \leq c_1 < \infty.$$

Next we determine a solution $x_2(t)$ of the equation

$$\dot{x}_2(t) = F_1(t, x_2^t), \quad x_2^0 = \varphi, \quad t \in [0, T].$$

Inductively, if we know $x_n(t)$, we choose a continuous functional F_n satisfying the same conditions as F_1 and such that

$$\sup_{t\in[0,T]} |F_n(t, x_n^t) - f(t, x_n^t)| \leq c_n < \infty,$$

where c_n is n-th term of any convergent series $\sum_{n=1}^{\infty} c_n$. So, the function $x_{n+1}(t)$ is defined as a solution of the equation

$$\dot{x}_{n+1}(t) = F_n(t, x_{n+1}^t), \quad x_{n+1}^0 = \varphi, \quad t \in [0, T],$$

etc..Thus, for arbitrary $\epsilon > 0$ there exists $m \in N$ such that

$$\sup_{t\in[0,T]} |x(t) - x_m(t)| < P_{m-2}(2aT)\frac{e^{aT} - 1}{a} < \epsilon.$$

Analogously to the paper [10], we use the notion *Z-algorithm* for the described iterative procedure. Since the functional $F_n(t, x)$ determines $(n+1)$-th approximation, the sequence of the functionals $\{F_n(t, x),\ n \in N\}$ will be called *determined sequence for the Z-algorithm.*

Obviously, the speed of convergence of the Z-algorithm depends on the choice of the determined sequence. The Z-algorithm can be effectively used only if the choice of the determined sequence is good enough, i.e. if the equations (4) can be solved. Certainly, in the case of hereditary differential equations this requirement is too strong and it is difficult to make a such algorithm. Because of that, our study is mostly based on theoretical considerations of some well-known iterative methods for which we prove that they are concrete Z-algorithms. Also, this fact leads to the idea to make simple forms of linearization of the functional $f(t,x)$, as it will be shown in the following examples.

Example 1. Let the functional $f : R \times X \to R^k$ be continuous in the pair of arguments and satisfy the Lipschitz condition (3) on a compact Ω. Also, let $\alpha_n : [0,T] \to R^k$, $n \in N$, be uniformly bounded continuous functions. Then for each $x \in \tilde{X}$ the sequence

$$\{\alpha_n(t)\,||x - x_n^t||_X + f(t, x_n^t), \quad n \in N\}$$

is the determined sequence of the equation (2). Really, the functionals $F_n(t,x) = \alpha_n(t)\,||x - x_n^t||_X + f(t,x_n^t)$, $n \in N$, are continuous and for all $(t,x),(t,y) \in \Omega$ the relation

$$\begin{aligned}|F_n(t,x) - F_n(t,y)| &\le \sup_{t\in[0,T]} |\alpha_n(t)|\,||x-y||_X \\ &\le l\,||x-y||_X \le l\,||x-y||_t, \quad n \in N,\end{aligned}$$

is valid. Therefore, the functionals $f(t,x)$ and $F_n(t,x)$, $n \in N$, satisfy on Ω the Lipschitz condition (3) with the same constant $L_1 = \max\{l, L\}$. The condition (5) is also satisfied because

$$\epsilon_n = \sup_{t\in[0,T]} |F_n(t,x_n^t) - f(t,x_n^t)| = 0, \quad n \in N.$$

So, from the Theorem it follows that the functionals $F_n(t,x)$, $n \in N$, describe the determined sequence of the Z-algorithm with the error of n-th approximation

$$\sup_{t\in[0,T]} |x(t) - x_n(t)| < 2aM \cdot \frac{(2aT)^{n-2}}{(n-2)!} \cdot \frac{e^{aT}-1}{a}, \quad n = 2,3,\ldots,$$

where a and M are suitable constants.

Notice that Picard-Lindelöf method of succesive approximations is a special case of the preceeding algorithm for $\alpha_n(t) \equiv 0$, $n \in N$, with the determined sequence $\{f(t,x_n^t),\ n \in N\}$.

Example 2. Let the functional $f(t,x)$ be defined as in the example 1 and let $\beta_n : [0,T] \to R$, $n \in N$, be continuous uniformly bounded functions. Similarly to the example 1, it is easy to prove that the sequence of functionals

$$\{\beta_n(t)\,[x(t) - x_n(t)] + f(t,x_n^t), \quad n \in N\},$$

is the determined sequence of the Z-algorithm. Therefore, the sequence of solutions of the linearized hereditary differential equations

$$\dot{x}_{n+1}(t) = \beta_n(t)[x_{n+1}(t) - x_n(t)] + f(t, x_n^t),$$
$$x_{n+1}^0 = \varphi, \quad t \in [0,T], \quad n \in N,$$

presents iterations of the solution of the equation (2). In particular, if $\beta_n(t) \equiv 0$, $n \in N$, then this Z-algorithm is reduced to the Picard-Lindelŏf method of iterations.

It could be very interesting to compare speeds of convergences of different Z-algorithms. Moreover, our intention is to form some other determined sequences and to choose the best Z-algorithm. However, it will be a subject of forthcoming papers.

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