FILOMAT (Niš) 10 (1996), 143–147

FUNCTIONAL RELATIONS BEETWEEN ANALYTIC FUNCTIONS IN MULTYPLY CONNECTED REGIONS

Boško Damjanović

Abstract. *In this paper the functions $\Phi(z)$ and $\Phi_j(z)$, $j = 1, 2, ..., m$, being analytic in the multiply connected region S and the simple connected regions D_j respectively are obtained in explicite form if it is presumed that functional relations (1) beetween their limit values on the appropriate contours are given.*

Let S be an n-connected region bounded by n closed disjoint Ljapunov curves $L_1, L_2, ..., L_m$, where the contour L_1 contains in itself all the others and let D_j, $j = 1, 2, ..., n$, be finite regions in the plane of one complex variable z bounded by non intersecting contoures Γj, $j = 1, 2, ..., n$, respectively. Let us suppose that the boundaries of region S and D_j are to be traversed in the positive direction relative to their interiors so that a person traveling in this direction always has the region S and D_j on his left. Denote the simple-connected region, bounded by closed contour L_j by S_j^- where $j = 2, ..., n$, and complement of finite region bounded by L_1 we denote by S_1^-. The complements of regions bounded by contours Γ_j, $j = 1, 2, ..., n$, we denote by D_j^-.

Let $\alpha_j^{-1}(t)$ be the function given on Γ_j that satisfies the following conditions:

a) homemorphically transforms the closed contour Γ_j into some closed contour belonging to the boundary of L of the region S and keeping the direction on movement.

b) $\alpha_j^{-1}(\Gamma_j) \cap \alpha_i^{-1}(\Gamma_i) = 0$, for $j \neq i$, $i = 1, 2, ..., n$.

c) function $\alpha_j^{-1}(t)$ has contionous derivations different from zero at all the points at contour Γ_j.

Let us introduce the function $\omega(t, \Gamma_i) = \delta_{i,j}$, $t \in \Gamma_j$, $i, j = 1, 2, ..., n$, where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$. On the contour L we define the function $\alpha(t)$ by $\alpha(t) = \sum_{j=1}^{n} \omega(t, L_j)\alpha_j(t)$, $t \in L_j$, $j = 1, 2, ..., n$, where $\alpha_j(t)$, $t \in L_j$

Received April 20, 1996
1991 *Mathematics Subject Classification:* 32A90.

are inverse functions of α_j^{-1}. Now, it is obvious that $\alpha(t)$ transforms L on to $\Gamma = \cup_{j=1}^{n}\Gamma_j$.

This paper represents a generalization of work ([2]) in relation to boundary conditions in which coefficient $G(t)$ of problems now is not obligated equal constant. Analogous problem but in the space E considered Ševila ([1]) and Nechaev ([6]). Rasulov ([4]) discusses problem of similar type for polianalytic functions while Čanak ([5]) for metaanalytic functions.

We shall determine the functions $\Phi(z)$ and $\Phi_j(z)$ which are analytic in S and D_j respectively, whose boundary values on the appropriate contours satisfy the following boundary condition:

$$\Phi_j(\alpha(t)) = G(t)\cdot\Phi(t), t\in L \tag{1}$$

where $G(t)$ is a continous function on L in the sense of Holder. First, suppose that $k = [argG(t)]_L/2\pi$ and consider the boundary value problem:

$$\chi_j(\alpha(t)) = \chi(t) + lnG(t), t\in L \tag{2}$$

where $\ln G(t)$ satisfies the Holder's condition on L. It is known that the particular solution of (2) is determined by the formulas:

$$\chi(z) = -\frac{z-a_0}{2\pi i}\int_L \frac{\varphi(t)dt}{(t-a_0)(t-z)}, \quad z\in S$$

$$\chi_j(z) = \frac{z-a_j}{2\pi i}\int_{\Gamma_j} \frac{\varphi(\alpha_j^{-1}(t))dt}{(t-a_j)(t-z)}, \quad z\in D_j, j=1,2,...,n.$$

where a_0 and a_j are arbitrary point in S_1^- and D_j^-, $j = 1,2,...,n$ respectively and $\varphi(t)$ is a solution of the Fredholm integral equation $(F\varphi)(t) = \ln G(t)$ where

$$(F\varphi)(t) = \varphi(t) + \frac{1}{2\pi i}\int_L\left(\frac{k(\sigma,t)\alpha'(\sigma)}{\alpha(\sigma)-\alpha(t)} - \frac{t-a_0}{(\sigma-a_0)(\sigma-t)}\right)\varphi(\sigma)d\sigma$$

$$k(\sigma,t) = \begin{cases}(\alpha(t)-a_j)/(\alpha(\sigma)-a_j), & t\in L_j, \quad j=1,2,...,n.\\ 0, & \sigma\in L_j, t\in L_i, j\neq i, \quad i,j=1,2,...,n.\end{cases}$$

Now, it is easy to cheek that the coeficient $G(t)$ from the problem (2) we can represent in the form: $G(t) = X_{0,j}/(t)X_0(t)$, where $X_0(t) = exp(\chi(z))$, $z\in S$ and $X_{0,j} = exp(\chi_j(z))$, $z\in D_j$. In this way, the boundary condition (1) can be represent in the following way: $\Phi_j(\alpha(t))/X_{0,j}(\alpha(t)) = \Phi(t)/X_0(t)$, $t\in L$. So, the functions $\psi_j(z) = \Phi_j(z)/X_{0,j}(z)$, $z\in D_j$ and $\psi(z) = \Phi(z)/X_0(z)$, $z\in S$, satisfy the following boundary condition:

$$\psi_j(\alpha(t)) = \psi(t), \quad t\in L. \tag{3}$$

The general solution of the problem (3) is given by formulae $\psi(z) = C$ and $\psi_j(z) = C$ where C is an arbitrary complex constant. So, the functions $\Phi_j(z) = C \cdot exp(\chi_j(z))$, $z \in D_j$ and $\Phi(z) = C \cdot exp(\chi(z))$, $z \in S$ are the general solution of the problem (1) in the case $k = 0$.

Let us consider the boundary condition (1) where the index $k = \operatorname{Ind}_L G(t)$ corresponded to the function $G(t)$ is any real number. Assume that the coordinate origin belongs to the region S and define the function $G_0(t)$ in the following way: $G_0(t) = t^{-k} \cdot P(t) \cdot G(t)$, $t \in L$ where $P(t) = (t-b_2)k^2(t-b_3)k^3...(t-b_n)k^n$, b_j arbitrary fixed points in S_j^-, $j = 2, 3, ..., n$, respectively, $k_j = \operatorname{Ind}_{L_j} G(t)$, $j = 1, 2, ..., n$, $k = \operatorname{Ind}_L G(t) = k_1 + (k_2 + k_3 + ... + k_n)$. Now,

$$\frac{[argG_0(t)]_{L_j}}{2\pi} = 0, \quad j = 1, 2, ..., n.$$

Hence, for the homogeneous boundary value problem, with the coefficient $G_0(t)$ there exist the functions $X_0(z)$ and $X_{0,j}(z)$ being analytic in S and D_j, $j = 1, 2, ..., n$ respectively and different from zero successively in $S \cup L$ and $D_j \cup \Gamma_j$, and which on the appropriate contours L and Γ_j have the limits $X_0(t) \in H(L)$ and $X_{0,j}(t) \in H(L)$ satisfying the following boundary value condition: $X_{0,j}(\alpha(t)) = G_0(t) \cdot X_0(t)$, $t \in L$.

Those functions are determined by the formulae:

$$X_0(t) = exp(-\frac{z - a_0}{2\pi i}) \int_L \frac{\varphi(t)dt}{(t - a_0)(t - z)}, z \in S,$$

$$X_{0,j}(z) = exp(-\frac{z - a_j}{2\pi i}) \int_{\Gamma_j} \frac{\varphi(\alpha_j^{-1}(t))dt}{(t - a_j)(t - z)}, z \in D_j$$

where $\varphi(t)$, $t \in L$ is the solution of the equation $(F\varphi)(t) = \ln G_0(t)$. According to all of this it follows that on the contour L, the coefficient $G(t)$ of the problem (1) can be represented in the form:

$$G(t) = \frac{X_{0,j}(\alpha(t))}{t^{-k} \cdot P(t) \cdot X_0(t)}, \quad t \in L \tag{4}$$

From the relations (1) and (4) we get the following boundary conditions:

$$\frac{\Phi_j(\alpha(t))}{X_{0,j}(\alpha(t))} = \frac{\Phi(t)}{t^{-k} \cdot P(t) \cdot X_0(t)}, t \in L \tag{5}$$

Let us denote by $f_j(z)$ the function $\Phi_j(z)/X_{0,j}(z)$, $z \in D_j$, and distinct the cases $k < 0$ and $k \geq 0$.

a) Case $k < 0$.

The function $\Phi(z)/(z^{-k} \cdot P(z) \cdot X_0(z))$, $z \in S$, has the point $z = 0$ as a pole of order $-k$, and it can be represented in the form:

$$\frac{\Phi(z)}{z^{-k}\cdot P(z)\cdot X_0(z)}=\sum_{i=1}^{-k}\frac{c_i}{z^i}+f_0(z), z\in S$$

where $f_0(z)$ is an indefinite analytic function in S and c_i, $i=1,2,...,-k$, are complex constants. If we introduce notation $B_{2j-1}=\operatorname{Re} c_j$, $B_{2j}=\operatorname{Im} c_j$, $g_{2j-1}(t)=1/t^j$, $t\in L$, $g_{2j}(t)=i/t^j$, $t\in L$ then we shall get the solution ofthe problem $f_j(\alpha(t))=\sum_{p=1}^{-2k}B_p\cdot q_p+f_0(t)$ in the form:

$$f_0(z)=B_0-\sum_{p=1}^{-2k}\frac{B_p\cdot(z-a_0)}{2\pi i}\int_L\frac{\varphi_p(dt)}{(t-a_0)(t-z)}, z\in S$$

$$f_j(z)=B_0+\sum_{p=1}^{-2k}\frac{B_p\cdot(z-a_p)}{2\pi i}\int_{\Gamma_j}\frac{\varphi_p(\alpha_j^{-1}(t))dt}{(t-a_p)(t-z)}, z\in D_p$$

where B_0 is an arbitrary complex constant and $\varphi_p(t)$ are the solutions of Fredholm's integral equations: $(F\varphi_p)(t)=g_p(t)$, $p=1,2,...,-2k$. Let us assume that $B_{2k+1}=\operatorname{Re} B_0$, $B_{2k+1}=\operatorname{Im} B_0$ and let us define the functions:

$$U_{2j-1}(z)=\frac{1}{z^j}-\frac{z-a_0}{2\pi i}\int_L\frac{\varphi_{2j-1}(t)dt}{(t-a_0)(t-z)}, z\in S$$

$$U_{2j}(z)=\frac{i}{z^j}-\frac{z-a_0}{2\pi i}\int_L\frac{\varphi_{2j}(t)dt}{(t-a_0)(t-z)},\ z\in S,\quad j=1,2,...,-k.$$

$$U_{-2k+1}(z)=1,\ z\in S,\qquad U_{-2k+1}(z)=i,\ z\in S$$

$$V_j(z)=\frac{z-a_j}{2\pi i}\int_{\Gamma_j}\frac{\varphi_j(\alpha_j^{-1}(t))dt}{(t-a_j)(t-z)},\ z\in D_j,\quad j=1,2,...,-2k,$$

$$V_{-2k+1}(z)=1,\ z\in D_j,\qquad V_{-2k+2}(z)=i,\ z\in D_j$$

Now, we can formulate the general solution of the homogeneous boundary value problem (1) in the case $k<0$ in the following way:

$$\begin{aligned}\Phi(z)&=z^{-k}\cdot P(z)\cdot X_0(z)\sum_{i=1}^{-2k+2}B_i\cdot U_i(z),\ z\in S,\\ \Phi_j(z)&=X_{0,j}(z)\sum_{i=1}^{-2k+2}B_i\cdot V_i(z), z\in D_j.\end{aligned}\tag{6}$$

b) Case $k\geq 0$.

Now, the function $\Phi(z)/(z^{-k} \cdot P(z) \cdot X_0(z))$ is analytic in S and for $k > 0$ is equal to zero at the coordinate origin and the functions $\Phi_j(z)/X_{0,j}(z)$ are analytic in D_j, $j = 1, 2, ..., n$. According to (5) we get that the solution of the problem (1) can be presented in the form: $\Phi(z) = C \cdot z^{-k} \cdot P(z) \cdot X_0(z)$, $z \in S$, $\Phi_j(z) = C \cdot X_{0,j}(z)$, $z \in D_j$, where C is an arbitrary complex constant. For $z = 0$ it is easy to verify that $C = 0$ and consequently $\Phi(z) = 0$, $z \in S$ and $\Phi_j(z) = 0$, $z \in D_j$. If $k = 0$ then $\Phi(z) = C \cdot P(z) \cdot X_0(z)$, $z \in S$, and $\Phi_j(z) = C \cdot X_{0,j}(z)$, $z \in D_j$. This solution can be obtained from the formula (6) assuming that it holds for $k = 0$. Therefore, we have proved the following theorem:

Theorem. *If the index* $k = [argG(t)]_L/2\pi \leq 0$ *then the boundary value problem (1) is solvable and its solution can be represented by the formula (6) containing* $2(-k+1)$ *arbitrary real constants. If, however,* $k > 0$ *then problem (1) has only trivial solution.*

Refrences

[1] Ševila T. A., *Krajevaja zadača Karlemana v prostranstve E*, Vestn. Belorus. Yn-ta. Ser. 1. 1993, N03.

[2] Damjanović B., *Integral equations and boundary value problem for multiply connected regions*, Zbornik radova VII Conference on Applied Mathematics, Osijek, 1990.

[3] Dimitrovski Dragan, Ilievski Borko and Rajović Miloje, *Sup l'equation differentielle areaolaire lineaire aux coefficients " constants". L'equation areolaire d'Euler*, Matematički bilten društva matematičara i informatičara Makedonije, 1992, - 16.

[4] Rasulov K.M., *Ob odnom obščem podhode k rešeniju klasičeskih krajevih zada dlja polianalitičeskih funkcij i ih obobščenij*, Differencijalnije uravnenija, 1993, $N_0 2$.

[5] Čanak M., *Carleman type boundary value problem for n-th order areolar differential equation*, Matematički vesnik, **5(18)(33)**, (1981), 341–347.

[6] Nechaev A. H., *Pro odnu krajevuju zadaču dlja pari funkcij analitičeskih v oblasti*, Donov., AN. URSR, **10** (1979), 891-893.

MILENTIJA POPOVIĆA 19/23, 11 070, NOVI BEOGRAD, YUGOSLAVIA