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A BERGMAN-CARLESON MEASURE CHARACTERIZATION OF $\mathcal{M}$-HARMONIC BLOCH FUNCTIONS

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Abstract. We prove two Bergman-Carleson measure characterizations of the $\mathcal{M}$-harmonic Bloch space

1. Introduction

Let B denote the open unit ball of $\mathbb{C}^n$, and m the $2n$-dimensional Lebesgue measure on B.

As in [5], we say that a $u \in C^2(B)$ is $\mathcal{M}$-harmonic in B, $u \in \mathcal{M}$, if $\widetilde{\Delta}u(z) = 0$ for every $z \in B$. The operator $\widetilde{\Delta}$ is the invariant Laplacian defined by $\widetilde{\Delta}u(z) = \Delta(u \circ \varphi_z)(0)$, $z \in B$, where Δ is the ordinary Laplacian and φ_z the standard automorphism of B taking 0 to z (see [5]).

For $f \in C^1(B)$, $Df = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$ denotes the complex gradient of f, $\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{2n}}\right)$, $z_k = x_{2k-1} + ix_{2k}$, $k = 1, 2, \dots, n$, denotes the real gradient of f.

For $f \in C^1(B)$ let $\widetilde{D}f(z) = D(f \circ \varphi_z)(0)$, $z \in B$, and $\widetilde{\nabla}f(z) = \nabla(f \circ \varphi_z)(0)$, $z \in B$, be the invariant complex gradient of f and the invariant real gradient of f respectively.

If $f \in C^1(B)$ let

$$|\nabla_T f(z)|^2 = 2\left(|Df(z)|^2 - |Rf(z)|^2 + |D\overline{f}(z)|^2 - |R\overline{f}(z)|^2\right)$$

be the tangential gradient of f. As usual, R denotes the radial derivative $R = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$.

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In [1] it is proved that a holomorphic function f on B belongs to the Bloch space $\mathcal{B}$, i.e.

$$\sup_{z\in B}|\widetilde{D}f(z)|<\infty \text{ if and only if } \sup_{a\in B}\int_B|\widetilde{D}f(z)|^2\frac{(1-|a|^2)^{n+1}}{|1-z\overline{a}|^{2n+2}}dm(z)<\infty.$$

In this note we extend this recult to the $\mathcal{M}$-harmonic Bloch space $\mathcal{MB}$, and we also obtain some other characterizations of the space $\mathcal{MB}$. More precisely we prove

Theorem 1. *Let $0<p<\infty$ and let $f\in\mathcal{M}$. Then the following statements are equivalent:*

(*i*) *f is a $\mathcal{M}$-harmonic Bloch function, $f\in\mathcal{MB}$, i.e.* $\sup_{z\in B}|\widetilde{\nabla}f(z)|<\infty$,

(*ii*) $\sup_{a\in B}\int_B(1-|z|^2)^p|\nabla f(z)|^p\frac{(1-|a|^2)^{n+1}}{|1-z\overline{a}|^{2n+2}}dm(z)<\infty$,

(*iii*) $\sup_{a\in B}\int_B|\widetilde{\nabla}f(z)|^p\frac{(1-|a|^2)^{n+1}}{|1-z\overline{a}|^{2n+2}}dm(z)<\infty$,

For $\xi\in S$ and $0<\delta<2$, put $Q_\delta(\xi)=\{z\in B:|1-z\overline{\xi}|<\delta\}$. In what follows, a positive measure μ on B is called a Bergman-Carleson if $\mu(Q_\delta(\xi))=O(\delta^{n+1})$ uniformly in $\xi\in S$ and $\delta>0$.

It is easily seen that a positive measure μ on B is a Bergman-Carleson measure if and only if

$$\sup_{a\in B}\int_B\frac{(1-|a|^2)^{n+1}}{|1-z\overline{a}|^{2n+2}}d\mu(z)<\infty.$$

Thus the following theorem is a corollary of Theorem 1.

Theorem 2. *Let $0<p<\infty$ and let $f\in\mathcal{M}$ then the following statements are equivalent:*

(*i*) $f\in\mathcal{MB}$,

(*ii*) $d\mu(z)=(1-|z|^2)^p|\nabla f(z)|^p dm(z)$ *is a Bergman-Carleson measure.*

(*iii*) $d\nu(z)=|\widetilde{\nabla}f(z)|^p dm(z)$ *is a Bergman-Carleson measure.*

2. Proof of Theorem 1

(*i*) $\Longrightarrow$ (*iii*). By standard estimates $\int_B(1-|a|^2)^{n+1}|1-z\overline{a}|^{-2n-2}dm(z)\le C$, for every $a\in B$ (see [5], p.17). (Here and elsewhere constants a denoted by C which may indicate a different constant from one occurrence to the next.) Thus, if $\sup_{z\in B}|\widetilde{\nabla}f(z)|<\infty$, then (*iii*) holds.

$(iii) \Longrightarrow (ii)$. An application of Cauchy-Schwarz inequality shows that

$$\begin{aligned}|\nabla_T f(z)|^2 &\geq 2(1-|z|^2)(|Df(z)|^2 + |D\overline{f}(z)|^2)\\ &= (1-|z|^2)|\nabla f(z)|^2.\end{aligned}$$

Also,

$$\begin{aligned}|\widetilde{\nabla} f(z)|^2 &= 2(|\widetilde{D} f(z)|^2 + |\widetilde{D}\overline{f}(z)|^2)\\ &= (1-|z|^2)|\nabla_T f(z)|^2 \qquad \text{(see [4])},\end{aligned}$$

and consequently,

$$|\widetilde{\nabla} f(z)| \geq (1-|z|^2)|\nabla f(z)|$$

Thus, $(iii) \Longrightarrow (ii)$.

$(ii) \Longrightarrow (i)$. For $a \in B$ and $0 < r < 1$ let $E_r(a) = \{z \in B : |\varphi_a(z)| < r\}$. By Lemma 3.1 in [3] we have

$$|\nabla f(a)|^p \leq \frac{C}{(1-|a|^2)^{n+1}} \int_{E_r(a)} |\nabla f(z)|^p dm(z), \quad a \in B,$$

(here $r \in (0,1)$ is fixed).
Since $1-|a|^2 \cong 1-|z|^2 \cong |1-z\overline{a}|$, for $z \in E_r(a)$ we have

$$\begin{aligned}(1-|a|^2)^p|\nabla f(a)|^p &\leq C\int_{E_r(a)} (1-|z|^2)^p|\nabla f(z)|^p \frac{(1-|a|^2)^{n+1}}{|1-z\overline{a}|^{2n+2}} dm(z)\\ &\leq C\int_B (1-|z|^2)^p|\nabla f(z)|^p \frac{(1-|a|^2)^{n+1}}{|1-z\overline{a}|^{2n+2}} dm(z).\end{aligned}$$

From this it follows that if (ii) holds then $\sup_{a\in B}(1-|a|^2)|\nabla f(a)| < \infty$. By Theorem 1 ([3]), $f \in \mathcal{MB}$.

Carefully examining the proof of Theorem 1 we conclude that the following is true.

Theorem 3. *Let $f \in \mathcal{M}$ and $p, s > 0$. Then the following statements are equivalent:*

(i) $f \in \mathcal{MB}$,

(ii) $\sup\limits_{a\in B} \int_B (1-|z|^2)^p|\nabla f(z)|^p \frac{(1-|a|^2)^s}{|1-z\overline{a}|^{n+1+s}} dm(z) < \infty$,

(iii) $\sup\limits_{a\in B} \int_B |\widetilde{\nabla} f(z)|^p \frac{(1-|a|^2)^s}{|1-z\overline{a}|^{n+1+s}} dm(z) < \infty$,

3. Remark

The functions annihilated by the operators $\Delta_{\alpha\beta}$, $\alpha, \beta \in C$, defined by

$$\Delta_{\alpha\beta} = (1 - |z|^2)\left\{ \sum_{i,j=1}^{n} (\delta_{ij} - z_i \overline{z}_j) \frac{\partial^2}{\partial z_i \partial \overline{z}_j} + \alpha R + \beta \overline{R} - \alpha\beta \right\}$$

are called (α, β)-harmonic.

Using a similar method of the proof of Theorem 1 we can obtain a Carleson-Bergman characterization of the (α, β)-harmonic Bloch space $\mathcal{B}_{\alpha\beta}$.

Theorem 4. *Let $0 < p < \infty$ and let f be a (α, β)-harmonic function. Then the following statements are equivalent:*

(i) $f \in \mathcal{B}_{\alpha\beta}$, *i.e.* $\sup_{z \in B} |\widetilde{\nabla} f(z)| < \infty$,

(ii) $(1 - |z|^2)^p |\nabla f(z)|^p dm(z)$ *is a Bergman-Carleson measure,*

(iii) $|\widetilde{\nabla} f(z)|^p dm(z)$ *is a Bergman-Carleson measure.*

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