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ON THE CONVOLUTION OF HARMONIC FUNCTIONS AND HARMONIC POLYNOMIALS

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Abstract. We prove an inequality for the integral means of order $p < 1$ of the convolution $u * v$, where v is a harmonic polynomal and u a harmonic function on the unit disc. As an application we obtain some information on Cesáro means of u and of the integral means of v.

1. Introduction and results

Let $h(\Delta)$ denote the class of all complex-valued, harmonic functions in the unit disc Δ of the complex plane. Each $u \in h(\Delta)$ has a unique series expansion $u(z) = \sum_{-\infty}^{\infty} \widehat{u}(k) r^{|k|} e^{ikt}$, $z = re^{it} \in \Delta$. A harmonic polynomial of degree $\leq n$ is a harmonic function u such that $\widehat{u}(k) = 0$ for $|k| > n$. We write, for $p > 0$,

$$M_p(u, \rho) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |u(\rho e^{it})|^p dt \right\}^{1/p}, \quad \rho > 0,$$

whenever the integral exists, and, for $0 < r < 1$,

$$M_p(u, r; \epsilon) = \sup\{ M_p(u, \rho) : |\rho - r| < \epsilon(1 - r) \}. \tag{1}$$

If $u \in h(\Delta)$ is arbitrary, then (1) is defined for $0 < \epsilon < 1$, while if u is a polynomial, then (1) has sense for $\epsilon \geq 1$ as well.

The convolution of $u, v \in h(\Delta)$ is denoted by $u * v$. We have $(u * v)\widehat{\ }(k) = \widehat{u}(k)\widehat{v}(k)$ $(-\infty < k < \infty)$. It was proved in [7] that if $0 < p < 1$ and $\epsilon = 1/2$, then there exists a constant $C < \infty$ such that for all $u, v \in h(\Delta)$ and $r < 1$

$$M_p(u * v, r^2) \leq C(1 - r)^{1-1/p} M_p(u, r; \epsilon) M_p(v, r; \epsilon). \tag{2}$$

In this paper we consider the convolution of a function with a polynomoal. Our main result is

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Theorem 1. *Let $u \in h(\Delta)$ and v be a harmonic polynomial. Let*

$$0 < p < 1,\ \lambda \geq 1,\ 0 < \epsilon < 1,\ n \geq 2,\ r_n = 1 - 1/n. \tag{3}$$

If $\deg(v) \leq n$, *then*

$$M_p(u * v, r_n; \lambda) \leq C n^{1/p-1} M_p(v, 1) M_p(u, r_n; \epsilon), \tag{4}$$

where C is a constant depending only on p, λ and ϵ.

Remark. Throughout the paper the hypotheses (3) are assumed. By C we denote a constant which depends only on p, λ, ϵ and may be different in different contexts.

If $p < 1/2$, then inequality (2), with $r = r_n$ and $\deg(v) \leq n$, is weaker than (4) because then there exists polynomials v_n, $\deg(v_n) \leq n$, such that $M_p(v_n, r_n)/M_p(v_n, 1) \to \infty$ $(n \to \infty)$ (see Section 4). In the case $p > 1/2$ these two inequalities are essentially equivalent, which is a consequence of the following

Corollary 1. *If $u, v \in h(\Delta)$, $\deg(v) \leq n$, then*

$$M_p(v, r_n; \lambda) \leq C M_p(v, 1) \qquad (p > 1/2), \tag{5}$$

$$M_p(v, r_n; \lambda) \leq C (\log n)^2 M_p(v, 1) \qquad (p = 1/2) \tag{6}$$

and

$$M_p(v, r_n; \lambda) \leq C n^{1/p-1} M_p(v, 1) \qquad (p < 1/2). \tag{7}$$

In Section 4 we will show that (6) and (7) are best possible (compare Lemma 1, Section 2).

Inequalities (5),(6) and (7) are obtained from (4) by taking $u = P$, the Poisson kernel, (note that $\widehat{P}(k) \equiv 1$) and using the following estimates (see, e.g. [6] pp.60,61):

$$M_p(P, r) \leq C(1-r)^{1-1/p} \qquad (p > 1/2),$$

$$M_p(P, r) \leq C(1-r)^{1/2} \log^2 \frac{2}{1-r} \qquad (p = 1/2)$$

and

$$M_p(P, r) \leq C(1-r) \qquad (p < 1/2).$$

The proof of Theorem 1 is in Section 3, where we also prove the following result essenntially due to Gwilliam [2].

Theorem 2. *Let $u \in h(\Delta)$, $0 < p < 1$ and $\alpha > 1/p - 1$. Then there is a constant A depending only on p, α, λ and ϵ such that*

$$M_p(\sigma_n^\alpha u, r_n; \lambda) \leq A M_p(u, r_n; \epsilon). \tag{8}$$

Here $\sigma_n^\alpha u = K_n^\alpha * u$*, where* K_n^α *are Fejer's kernels,*

$$K_n^\alpha(re^{it}) = \sum_{-n}^{n} \frac{B(\alpha, n+1)}{B(\alpha, n+1-|k|)} r^{|k|} e^{ikt}, \tag{9}$$

where B *denotes the Euler beta function.*

In the case $p > 1/2$ inequality (8) is a direct consequence of (4) and the inequality

$$M_p(K_n^\alpha, 1) \le A_{p,\alpha} n^{1-1/p} \qquad (\alpha > 1/p - 1)$$

(see [8], p.48). If $p \le 1/2$, then the latter does not hold, which can be seen from the formulas for $K_n(e^{it})$, $1 < \alpha < 2$, given in [8], p.56. Nevertheless (8) holds for $p \le 1/2$ as well.

Note a consequence of Theorem 2.

Corollary 2. *Let* $u \in h(\Delta)$, $0 < p < 1$ *and* $\alpha > 1/p - 1$. *If* $-\infty < \gamma < \infty$ *and*

$$M_p(u, r) = O((1-r)^{-\gamma}), \quad r \to 1^-, \tag{10}$$

then

$$M_p(\sigma_n^\alpha u, r_n; 1) = O(n^\gamma), \quad n \to \infty. \tag{11}$$

Gwilliam [2] proved this for $\gamma = 0$ but his proof could be modified to show that (10) is always equivalent with (11). However it would be more interesting to know when (10) is implied by

$$M_p(\sigma_n^\alpha u, 1) = O(n^\gamma), \quad n \to \infty. \tag{12}$$

(Note that if $p > 1/2$, then (12) is equivalent with (11).)

Finaly it should be mentioned that the above estimates for Poisson's kernel show that condition (10) has sense for certain negative values of γ.

2. Lemmas on polynomials

The following lemma is a consequence of the subharmonicity of $|f|^p$ (f analytic). See, e.g. [5], Lemma 3.1.

Lemma 1. *If* $f(z)$ *is an algebraic polynomial of degree* $\le n$*, then*

$$M_p(f, \rho) \le (\rho/r)^n M_p(f, r) \quad (0 < r < \rho)$$

and consequently

$$M_p(f, r_n; \lambda) \le C M_p(f, r_n).$$

See (3) and Remark following Theorem 1.

Lemma 2. *Let v be the real part of an algebraic polynomial f, $\deg f \le n$. Then*

$$M_p(f',1) \le CnM_p(v,r_n;\epsilon).$$

Proof. Let $\Delta_R(r) = \{w : |w-r| < R\}$, $0 < r < 1$, $R > 0$. From the inequalities

$$\sup_{\Delta_R(r)} |v|^p \le CR^{-2} \int_{\Delta_{2R}(r)} |v|^p dA \quad (A = area)$$

(see [4], Ch.VII, D.2) and

$$|f'(r)| \le CR^{-1} \sup_{\Delta_R(r)} |v|$$

it follows that

$$|f'(r)|^p \le CR^{-p-2} \int_{\Delta_R(r)} |v(w)|^p dA(w).$$

Applying the latter to the functions $z \mapsto f(ze^{i\theta})$ and then integrating the resulting inequality over the interval $0 < \theta < 2\pi$, we obtain

$$M_p^p(f',r) \le CR^{-p-n} \int_{\Delta_{2R}(r)} M_p^p(v,|w|)dA$$
$$\le CR^{-p} \sup\{M_p^p(v,\rho) : |\rho - r| < 2R\}.$$

Now let $r = r_n$ and $2R = \epsilon/n$ to get

$$M_p(f',r_n) \le C(n/\epsilon)M_p(v,r_n;\epsilon)$$

(see (1)). Now Lemma 1 completes the proof. □

Lemma 3. *If v is a harmonic polynomial of degree $\le n$, then we have*

$$M_p(v,r_n;\lambda) \le CM_p(v,r_n;\epsilon).$$

Recall that $\lambda \ge 1$ and $0 < \epsilon < 1$.

Proof. Let $|r - r_n| \le \lambda/n$. Assuming, as we may, that v is the real part of an algebraic polynomial f, we have

$$|v(re^{it})| \le |v(r_ne^{it})| + (\lambda/n)\sup\{|f'(\rho e^{it})| : \rho < r_n + \lambda/n\}.$$

Hence, by the complex maximal theorem,

$$M_p(v,r) \le CM_p(v,r_n) + Cn^{-1}M_p(f',r_n + \lambda/n).$$

Now the result follows from Lemmas 1 and 2. □

As a further application of Lemma 2 we prove the following. Here B denotes the Euler beta function.

Lemma 4. *Let v be a harmonic polynomial,* $\deg(v) \le n$, *and*

$$h(re^{it}) = \sum_{-n}^{n} \frac{B(\alpha, n+1+|k|)}{B(\alpha, n+1)} \widehat{v}(k) r^{|k|} e^{ikt}, \ \alpha > 0.$$

Then

$$M_p(h, r_n; 1) \le C_{p,\alpha} M_p(v, r_n; 1).$$

Proof. Let $v = \operatorname{Re} f$, where f is an algebraic polynomial. Using the integral form of the beta function one easily verifies that

$$h(re^{it}) - v(re^{it}) = \frac{1}{B(\alpha, n+1)} \int_0^1 (1-\rho)^{\alpha-1} \rho^n (v(\rho re^{it}) - v(re^{it}))dt.$$

Hence

$$\begin{aligned} |h(re^{it})| &\le |v(re^{it})| + \frac{1}{B(\alpha, n+1)} \int_0^1 (1-\rho)^\alpha \rho^n \sup_{s<1} |f'(se^{it})| d\rho \\ &\le |v(re^{it})| + \frac{B(\alpha+1, n+1)}{B(\alpha, n+1)} \sup_{s<1} |f'(se^{it})|. \end{aligned}$$

Now the complerx maximal theorem yields

$$M_p(h, r) \le \frac{\alpha}{\alpha+n+1} C M_p(f', 1) + C M_p(v, r).$$

Now the desired result follows from Lemma 2. □

3. Proof of Theorems 1 and 2

For a 2π-periodic function $g(t)$, $-\infty < t < \infty$, let

$$g^+(t, \delta) = \sup\{|g(x)| : |x - t| < 1 - \delta\}, \quad 0 < \delta < 1.$$

If $u \in h(\Delta)$, let $u^+(re^{it}) = u_r^+(t, r)$, where $u_r(t) = u(re^{it})$.

The following lemma is proved in the same way as Lemma 2.2 in [7].

Lemma 5. *If $u \in h(\Delta)$, then $M_p(u^+, r) \le C M_p(u, r; \epsilon)$ for $0 < r < 1$, where C is independent of r.*

We need another "maximal" lemma.

Lemma 6. *If φ is a trigonometric polynomial of degree $\leq n$, then*

$$\int_0^{2\pi} \varphi^+(t,\delta)^p dt \leq C_p \delta^{-2np} \int_0^{2\pi} |\varphi(t)|^p dt$$

for $0 < \delta < 1$.

Proof. Consider the algebraic polynomial

$$f(z) = \sum_{k=0}^{2n} \delta^{-k} c_{n-k} z^k,$$

where c_j, $-n \leq j \leq n$, are coefficients of φ. Since obviously $|\varphi(t)| = |f(\delta e^{it})|$ and $\varphi^+(t,\delta) = |f^+(\delta e^{it})|$, the proof of the lemma reduces to proving that

$$(+) \qquad M_p(f^+,\delta) \leq C\delta^{-2n} M_p(f,\delta).$$

By Lemma 1 we have $M_p(f^+,\delta) \leq CM_p(f,\delta;1) \leq CM_p(f,1)$. (In the last step we have used the "increasing" property of $M_p(f, \ )$.) Now (+) is proved by Lemma 1. □

Proof of Theorem 1. Let $|r - r_n| < \epsilon/3n = (\epsilon/3)(1 - r_n)$, $u \in h(\Delta)$ and $\varphi(t) = v(e^{it})$, where v is a harmonic polynomial of degree $\leq n$. Then

$$(u * v)(re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) \varphi(t - \theta) d\theta.$$

Now we use the inequality

$$\left(\int_0^{2\pi} |g(\theta)| d\theta \right)^p \leq C(1-r)^{p-1} \int_0^{2\pi} g^+(\theta, r)^p d\theta$$

(see [7], Lemma 2.1) to get

$$|(u * v)(re^{it})|^p \leq Cn^{1-p} \int_0^{2\pi} u^+(re^{i\theta})^p \varphi^+(t - \theta, r)^p d\theta.$$

Hence by integration and using Lemmas 5 and 6

$$\begin{aligned} M_p^p(u * v, r) &\leq Cn^{1-p} M_p^p(u, r\epsilon/3) M_p^p(v, 1) \\ &\leq Cn^{1-p} M_p^p(u, r_n; \epsilon) M_p^p(v, 1), \end{aligned}$$

which implies (4) with $\lambda = \epsilon/3$. Now Lemma 2 completes the proof of Theorem 1. □

In order to prove Theorem 2 introduce the polynomials

$$T_n^\alpha(re^{it}) = \sum_{-n}^{n} \frac{B(\alpha, n+1)^2}{B(\alpha, n+1-k)B(\alpha, n+1+k)} r^{|k|} e^{ikt}.$$

It follows from the formula (9) and Lemmas 4 and 3 that inequality (8) is a consequence of

$$M_p(T_n^\alpha * u, r_n; 1) \leq A M_p(u, r_n; \epsilon), \tag{13}$$

where A is a constant. On the other hand, the latter is an immediate consequence of (4) and the estimate

$$M_p(T_n^\alpha, 1) \leq A_{p,\alpha} n^{1-1/p} \qquad (0 < p < 1,\ \alpha > 1/p - 1). \tag{14}$$

To prove this we write $T_n^\alpha(e^{it})$ as

$$T_n^\alpha(e^{it}) = e^{-int} \sum_{k=0}^{2n} \frac{B(\alpha, n+1)^2}{B(\alpha, k+1)B(\alpha, 2n+1-k)} e^{ikt}.$$

For a fixed t consider the analytic function

$$F_t(z) = \alpha^2 (1-z)^{-\alpha-1}(1-e^{it}z)^{-\alpha-1} = \sum_{j=0}^{\infty} a_j(t) z^j.$$

Simple computation shows that

$$T_n^\alpha(e^{it}) = e^{-int} B(\alpha, n+1)^2 a_{2n}(t),$$

whence

$$|T_n^\alpha(e^{it})| \leq C_\alpha n^{-2\alpha} |a_{2n}(t)|.$$

On the other hand, the Hardy-Littlewood theorem on Taylor's coefficients of H^p functions gives

$$|a_{2n}(t)|^p r^{2np} \leq C_p n^{1-p} \int_0^{2\pi} |F_t(re^{i\theta})|^p d\theta$$

(see [1], Theorem 6.4). Hence, by taking $r = r_n$.

$$|T_n^\alpha(e^{it})|^p \le C_{p,\alpha} n^{-2\alpha p+1-p} \int_0^{2\pi} \left(|1 - r_n e^{i(t+\theta)}|\,|1 - r_n e^{i\theta}|\right)^{-\beta} d\theta,$$

where $\beta = (\alpha+1)p > 1$. Now we integrate this from $t = 0$ to $t = 2\pi$ and use the familiar estimate

$$\int_0^{2\pi} |1 - re^{it}|^{-\beta} dt \le C_\beta (1-r)^{1-\beta} \qquad (\beta > 1).$$

It follows that

$$M_p^p(T_n^\alpha, 1) \le C_{p,\alpha} n^{-2\alpha p+1-p-2(1-(\alpha+1)p)} = C_{p,\alpha} n^{p-1},$$

and this proves (14) and completes the proof of Theorem 2. □

4. Remarks

Inequalities (6) and (7) are in a sense best possible. To see this let ω be C^∞-function on $(-\infty, \infty)$ such that $\omega(x) = 1$ for $|x| < 1/2$ and $\omega(x) = 0$ for $|x| > 1$. Let

$$v_n(re^{it}) = \sum_{k=-\infty}^{\infty} \omega(k/n) r^{|k|} e^{ikt}.$$

Observe that v_n is a harmonic polynomial of degree $\le n$. We shall show that if $q \ne 0$, then there is a constant $c > 0$, independent of n $(n > |q|)$, such that

$$(15) \qquad M_p(v_n, 1 - q/n) \ge cn^{-1}(\log n)^2 \qquad (p = 1/2)$$

and

$$(16) \qquad M_p(v_n, 1 - q/n) \ge cn^{-1} \qquad (p < 1/2).$$

On the other hand, it is a familiar fact that

$$(17) \qquad M_p(v_n, 1) \le Cn^{1-1/p} \qquad (0 < p < 1)$$

(see, e.g. [3], p.177).

To verify (15) and (16) consider the trigonometric polynomials

$$\varphi_n(t) = (1 - e^{it})^3 v_n(\rho_n e^{it}), \quad \rho_n = 1 - q/n.$$

The degree of φ_n is less than $n + 3$ and the coefficients $\widehat{\varphi}_n(k)$ satisfy the relation

$$\widehat{\varphi}_n(k) = F_n(k) - 3F_n(k-1) + 3F_n(k-2) - F_n(k-3),$$

where

$$F_n(x) = \omega(x/n)\rho_n^{|x|} \qquad (-\infty < x < \infty,\, n > |q|).$$

After a little work we find that

$$\widehat{\varphi}_n(1)e^{it} + \widehat{\varphi}_n(2)e^{2it} = (1-\rho_n)(\rho_n - 3)e^{it}(1-e^{it})$$

for n large enough. If $k \neq 1, 2$, we can apply Lagrange's theorem for symmetric differences to obtain

$$|\widehat{\varphi}_n(k)| \le \sup_{x\neq 0} |F_n'''(x)| \le Cn^{-3}.$$

Combining these estimates we find that

$$|v_n(\rho_n e^{it})| \ge t^{-3}(at/n - A/n^2) \qquad (0 < t < \pi),$$

where a and A are constants independent of t, n. This implies (15) and (16).

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