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ON CERTAIN UNIVALENCE CONDITIONS IN THE UNIT DISC

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Abstract. We consider the univalence of a function $f(z) = z + a_2z^2 + \cdots$ for which $zf'^{\alpha}(z)$ is starlike function.

1. Introduction and preliminaries

As usually let A denote the class of functions f analytic in the unit disc $U = \{z : |z| < 1\}$ with $f(0) = 0$ and $f'(0) = 1$. We give the following notations and definitions (see [1]):

$$S^* = \left\{f \in A : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\, z \in U\right\},$$

$$C = \left\{f \in A : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\, z \in U\right\},$$

$$K = \left\{f \in A : (\exists g \in C)\,(\exists \beta, -\frac{\pi}{2} < \beta < \frac{\pi}{2})\ \mathrm{Re}\left(\frac{f'(z)}{e^{i\beta}g'(z)}\right) > 0,\, z \in U\right\}.$$

These classes are the classes of starlike, convex and close-to-convex functions, respectively.

It is well-known that

$$C \subset S^* \subset K \subset S,$$

where S, $S \subset A$, denote the class of univalent functions in U.

The object of this paper is to determine complex or real numbers α, $\alpha \neq 0$, such that if $zf'^{\alpha}(z)$ (where we take the principal values) belongs to some of the previously mentioned classes.

For our considerations we need the following lemmas.

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Lemma 1 ([4]). *Let f be analytic in U and*

$$|\{f,z\}| \leq \frac{2}{(1-r^2)^2}$$

for all z, $|z| = r < 1$, where

$$\{f,z\} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is the Schwarzian derivative. Then f is univalent in U.

Lemma 2 ([2]). *Suppose that $f \in A$ and $f'(z) \neq 0$, $z \in U$. A necessary and sufficient condition that $f \in K$ (i.e., f is close-to-convex) is that for every $r \in (0,1)$ and every pair θ_1, θ_2, with $0 \leq \theta_2 - \theta_1 \leq 2\pi$, we have*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right) d\theta > -\pi.$$

Lemma 3 ([5]). *Let $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ be analytic in U with $\mathrm{Re}(f(z)) > 0$. Then we have*

$$|f(z) - 1| \leq \frac{2r}{1-r} \qquad (|z| \leq r < 1).$$

Lemma 4 ([3]). *Suppose that $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ is analytic and satissfies $\mathrm{Re}(f(z)) > 0$ for $z \in U$. Then*

$$\left|\frac{f'(z)}{f(z)}\right| \leq \frac{2}{1-r^2} \qquad (|z| \leq r < 1).$$

2. Results and consequences

The main results are in the following statement:

Theorem 1. *Let $f \in A$ and let $zf'^{\alpha}(z)$ is a starlike function for some complex number α ($\alpha \neq 0$).*

(a) If $\alpha \leq -2$ or $\alpha \geq 2/3$, then f is a close-to-convex function;

(b) If α is complex and

$$|\alpha| \geq \frac{56 + \sqrt{3280}}{9} = 12.5856...$$

then $f \in S$, i.e., f is a univalent function in U.

Proof: (a) Put $g(z) = zf'^{\alpha}(z)$, $\alpha \neq 0$. Then, after the logarithmic differentiation, we have

$$\frac{f''(z)}{f'(z)} = \frac{1}{\alpha z}\left(\frac{zg'(z)}{g(z)} - 1\right) \tag{1}$$

and also

$$1 + \frac{zf''(z)}{f'(z)} = 1 - \frac{1}{\alpha} + \frac{1}{\alpha}\frac{zg'(z)}{g(z)}. \tag{2}$$

From (2) we obtain

$$\begin{aligned}\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right) d\theta = \left(1 - \frac{1}{\alpha}\right)(\theta_2 - \theta_1)\\ + \frac{1}{\alpha}\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(\frac{re^{i\theta}g'(re^{i\theta})}{g(re^{i\theta})}\right) d\theta.\end{aligned} \tag{3}$$

First, if $\theta_2 - \theta_1 = 0$, then the integral of the left hand side of (3) is equal to 0, hence it is greaten this $-\pi$.

If $\alpha \leq -2$, then $-1/2 \leq 1/\alpha < 0$ and $1 < 1 - 1/\alpha \leq 3/2$, and the first term of the right hand side of (3), for such α and $0 < \theta_2 - \theta_1 \leq 2\pi$, is positive. So, from (3) we get

$$\begin{aligned}\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right) d\theta &> \frac{1}{\alpha}\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(\frac{re^{i\theta}g'(re^{i\theta})}{g(re^{i\theta})}\right) d\theta\\ &= \frac{1}{\alpha}\int_{\theta_1}^{\theta_2} \frac{d}{d\theta}(\arg g(re^{i\theta}))\,d\theta\\ &= \frac{1}{\alpha}\left[\arg g(re^{i\theta_2}) - \arg g(re^{i\theta_1})\right]\\ &\geq \left(-\frac{1}{2}\right)2\pi = -\pi,\end{aligned}$$

since g is a starlike function and $-1/2 \leq 1/\alpha < 0$.

If $2/3 \leq \alpha \leq 1$, then $1 \leq 1/\alpha \leq 3/2$ and $-1/2 \leq 1 - 1/\alpha \leq 0$. In this case from (3) we have for $0 < \theta_2 - \theta_1 \leq 2\pi$

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right) d\theta > \left(1 - \frac{1}{\alpha}\right)(\theta_2 - \theta_1) \geq -\pi,$$

since the second term of the right hand side of (3) is positive.

For $\alpha > 1$ and $0 < \theta_2 - \theta_1 \le 2\pi$ the first term of the right hand side is greiten to 0 and the second is positive, hence its summary is positive.

By applying Lemma 2 we conclude that f is close-to-convex.

We observe that for $\alpha \ge 1$, from (2) follows

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) = 1-\frac{1}{\alpha}+\frac{1}{\alpha}\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) > 0, \quad z \in U,$$

i.e., f is a convex function.

(b) From (1) we have

$$\left(\frac{f''(z)}{f'(z)}\right)' = -\frac{1}{\alpha z^2}\left(\frac{zg'(z)}{g(z)}-1\right)+\frac{1}{\alpha z}\left(\frac{zg'(z)}{g(z)}\right)'. \tag{4}$$

Since $\operatorname{Re}\big(zg'(z)/g(z)\big) > 0$, $z \in U$, then

$$\left|\frac{zg'(z)}{g(z)}\right| \le \frac{1+r}{1-r} \quad \text{for } |z| \le r < 1 \tag{5}$$

(see, for example [1], vol I, p.84). Also, by applying Lemmas 3 and 4, we get

$$\left|\frac{zg'(z)}{g(z)}-1\right| \le \frac{2r}{1-r}, \quad |z| \le r < 1 \tag{6}$$

and

$$\left|\left(\frac{zg'(z)}{g(z)}\right)'\right| \le \frac{2}{1-r^2}\left|\frac{zg'(z)}{g(z)}\right| \le \frac{2}{1-r^2}\cdot\frac{1+r}{1-r} = \frac{2}{(1-r)^2}, \; |z| \le r < 1. \tag{7}$$

Now, by (1), (4), (6) and (7) we obtain for $|z| = r$,

$$\begin{aligned}
|\{f,z\}| &\le \left|\left(\frac{f''(z)}{f'(z)}\right)'\right| + \frac{1}{2}\left|\frac{f''(z)}{f'(z)}\right|^2 \\
&\le \frac{1}{|\alpha||z|^2}\left|\frac{zg'(z)}{g(z)}-1\right| + \frac{1}{|\alpha||z|}\left|\left(\frac{zg'(z)}{g(z)}\right)'\right| + \frac{1}{2}\frac{1}{|\alpha|^2|z|^2}\left|\frac{zg'(z)}{g(z)}-1\right|^2 \\
&\le \frac{2}{|\alpha|r(1-r)} + \frac{2}{|\alpha|r(1-r)^2} + \frac{2}{|\alpha|^2(1-r)^2},
\end{aligned}$$

i.e.,

$$|\{f,z\}| \le \frac{2[(1-|\alpha|)r+2|\alpha|]}{|\alpha|^2 r(1-r)^2}, \quad |z| = r < 1. \tag{8}$$

First, if $|z| < 1/4$, then by the maximum modulus principe from (8) we have

$$|\{f,z\}| < \frac{32(7|\alpha|+1)}{9|\alpha|^2} \leq \frac{1}{(1-r^2)^2} \cdot \frac{32(7|\alpha|+1)}{9|\alpha|^2} \leq \frac{2}{(1-r^2)^2}$$

if $16(7|\alpha|+1)/(9|\alpha|^2) \leq 1$, i.e. if

$$|\alpha| \geq \frac{56+\sqrt{3280}}{9} = 12.5856... \tag{9}$$

Further, the relation (8) can be written in the form

$$\begin{aligned} |\{f,z\}| &\leq \frac{2}{(1-r^2)^2}\left[\frac{1}{|\alpha|^2}\left(1+\frac{1}{r}\right)^2((1-|\alpha|)r^2+2|\alpha|r)\right] \\ &= \frac{2}{(1-r^2)^2}P(r), \end{aligned} \tag{10}$$

where

$$P(r) = \frac{1}{|\alpha|^2}\left(1+\frac{1}{r}\right)^2[(1-|\alpha|)r^2+2|\alpha|r], \quad 0 < r < 1. \tag{11}$$

Since

$$P'(r) = \frac{2}{r|\alpha|^2}\left(1+\frac{1}{r}\right)[(1-|\alpha|)r^2+|\alpha|(r-1)] < 0$$

for $|\alpha| > 1$ and $0 < r < 1$, we have that the function $P(r)$ is an decreasing function in the interval $(0,1)$ and for $1/4 \leq r < 1$ takes its maximum for $r = 1/4$. Because $P(1/4) = 25(7|\alpha|+1)/(16|\alpha|^2)$ and $P(1/4) \leq 1$ for $25(7|\alpha|+1)/(16|\alpha|^2) \leq 1$, i.e., for

$$|\alpha| \geq \frac{175+\sqrt{32225}}{32} = 11.0785...,$$

from (10) and α given by (9) we get

$$|\{f,z\}| \leq \frac{2}{(1-r^2)^2}P(r) \leq \frac{2}{(1-r^2)^2}P(1/4) \leq \frac{2}{(1-r^2)^2},$$

where $1/4 \leq |z| = r < 1$.

Finally, by applying Lemma 1 we conclude that for an α given by (9) we have $f \in S$.

We note that a similar method was given earlier in [5].

Theorem 1(a) gives the following

Corollary 1. *Let $f \in A$. Then each of the following conditions*

(i) $\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2}$, $z \in U$;

(ii) $\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{3}{2}$, $z \in U$;

implies that f is a close-to-convex function.

Proof: Since the conditions (i) and (ii) are equivalent to

$$\mathrm{Re}\left(\frac{z(zf'^{2/3}(z))'}{zf'^{2/3}(z)}\right) > 0 \text{ and } \mathrm{Re}\left(\frac{z(zf'^{-2}(z))'}{zf'^{-2}(z)}\right) > 0,$$

respectively, i.e., $zf'^{2/3}(z)$ and $zf'^{-2}(z)$ are starlike which by Theorem 1(a) gives that f is close-to-convex in both cases.

These are the earlier results given by Sakaguchi in [6], using a different method.

Remark. From $g(z) = zf'^{\alpha}(z)$ we have that

$$f(z) = \int_0^z \left(\frac{g(t)}{t}\right)^{1/\alpha} dt$$

for $\alpha \neq 0$, and from Theorem 1(b) that $g \in S^*$ implies $f \in S$ for a complex number α satisfying (9). The similar problem, but for $g \in S$ and real α, was treated earlier by Nunokawa in [5].

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