FILOMAT (Niš) **10** (1996), 117–122

STARLIKENESS OF A CERTAIN CLASS OF UNIVALENT FUNCTIONS

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Abstract. We consider the starlikeness of the class of univalent functions investigated earlier in [2].

1. Introduction and preliminaries

Let A denote the class of functions which are analytic in the unit disk $U = \{z : |z| < 1\}$ and have the form

$$f(z) = z + a_2 z^2 + \dots \tag{1}$$

Further, as usual, let

$$S^*(\alpha) = \left\{ f \in A : Re\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha,\, 0 \leq \alpha < 1,\, z \in U \right\}$$

be the class of *starlike functions of order* α. We put $S^*(0) \equiv S^*$ (the class of *starlike functions* in U). These classes are the subclasses of the class of univalent functions in U (see, for example [1]).

In their paper [3] Ozaki and Nunokawa gave the following

Theorem A. *Let $f \in A$ satisfy the condition*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1, \quad z \in U. \tag{2}$$

Then f is univalent in U.

Later, in the paper [2] the authors considered the class of functions $f \in A$ which satisfy the condition (2). They obtained the representation formula for those functions and other results in connection with starlikeness and close-to-convexity in the unit disk. Among others results we cite the following

Received May 23, 1996
1991 *Mathematics Subject Classification*: 30C45.
* Supported by Grant 04M03 of RFNS through Math. Inst. SANU.

Theorem B. *Let $f \in A$ satisfy* (2). *Then*

(3) $$\left| \frac{z}{f(z)} - 1 \right| \le |z| \left(|a_2| + |z| \right), \quad z \in U.$$

In this paper, by using another method, we will consider the starlikeness in the unit disk in general, not only in the case when $a_2 = 0$ as in [2]. For our results we need the following

Lemma A ([4]). *Let $0 < \lambda_1 < \lambda < 1$ and let Q be analytic in U satisfying*

$$Q(z) \prec 1 + \lambda_1 z, \quad Q(0) = 1.$$

(a) *If p is analytic in U, $p(0) = 1$ and satisfies*

$$Q(z)[\beta + (1-\beta)p(z)] \prec 1 + \lambda z,$$

where

(4) $$\begin{cases} \dfrac{1-\lambda}{1+\lambda_1}, & 0 < \lambda + \lambda_1 \le 1, \\ \dfrac{1-(\lambda^2+{\lambda_1}^2)}{2(1-{\lambda_1}^2}, & \lambda^2 + {\lambda_1}^2 \le 1 \le \lambda + \lambda_1, \end{cases}$$

then $Re\{p(z)\} > 0$, $z \in U$.

(b) *If ω is analytic in U, $\omega(0) = 0$ and*

$$Q(z)[1 + \omega(z)] \prec 1 + \lambda z,$$

then

(5) $$|\omega(z)| < \frac{\lambda + \lambda_1}{1 - \lambda_1} = r \le 1, \quad \lambda + 2\lambda_1 \le 1.$$

The value of β given by (4) *and the bound* (5) *are the best possible.*

2. Results and consequences

Theorem 1. *Let $f(z) = z + a_2 z^2 + \cdots \in A$ satisfy the condition* (2) *with $0 < |a_2| = a \le 2$. Then the following statements are true:*

(a) $$Re\left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad |z| < r_1(a),$$

where $r_1(a) \in (0,1)$ is the positive root of the equation

(6) $$2r^4 + 2ar^3 + a^2 r^2 - 1 = 0.$$

(b) $$\left| \frac{zf'(z)}{f(z)} - \frac{1}{2\beta} \right| < \frac{1}{2\beta}, \quad |z| < r,$$

where β and r are given by

$$(7)\quad \beta = \begin{cases} \dfrac{1-r(a+r)}{1+r^2}, & 0 < r \le \dfrac{2}{\sqrt{a^2+8}+a} = r_2(a), \\ \dfrac{1-\left[(r(a+r))^2+r^4\right]}{2(1-r^4)}, & r_2(a) \le r < r_1(a) \end{cases}$$

($r_1(a)$ *is the same as in* (a)).

(c) $\left|\dfrac{f(z)}{zf'(z)} - 1\right| < \dfrac{r(a+r)+r^2}{1-r^2} \le 1,\ |z| < r,\ 0 < r \le \dfrac{2}{\sqrt{a^2+12}+a}$.

Proof. Since f satisfies the condition (2) then using the estimation (3) and the proof of Theorem 1 in [2], we can write

$$\left|\frac{rz}{f(rz)} - 1\right| \le r(a+r), \quad z \in U,$$

and

$$\left|\frac{(rz)^2 f'(rz)}{f^2(rz)} - 1\right| \le r^2, \quad z \in U,$$

which are the same as the following

$$(8)\qquad \frac{rz}{f(rz)} \prec 1 + r(a+r)z,$$

and

$$(9)\qquad \frac{(rz)^2 f'(rz)}{f^2(rz)} \prec 1 + r^2 z.$$

If we put

$$Q(z) = \frac{(rz)^2 f'(rz)}{f^2(rz)},\ p(z) = \frac{f(rz)}{rzf'(rz)},\ \lambda = r(a+r),\ \lambda_1 = r^2,$$

then we have that $0 < \lambda_1 < \lambda < 1$ and the conditions (9) and (8) are

$$Q(z) \prec 1 + \lambda_1 z,$$
$$Q(z)\left[\frac{p(z)-\beta}{1-\beta}(1-\beta) + \beta\right] \prec 1 + \lambda z,$$

where β is given by (7) and where in the second case, instead of $r_2(a) \le r < r_1(a)$, it should be $r_2(a) \le r \le r_1(a)$. By Lemma A we have that

$Re\{p(z)\} > \beta$, $z \in U$, or equivalently $Re\left\{\dfrac{f(rz)}{rzf'(rz)}\right\} > \beta$, $z \in U$. For $r = r_2(a)$ we get $\beta = 0$ which implies the statements (a) of the theorem. For $r_2(a) \leq r < r_1(a)$ the last result is equivalent to

$$\left|\frac{(rz)f'(rz)}{f(rz)} - \frac{1}{2\beta}\right| < \frac{1}{2\beta}, \quad z \in U,$$

which is the same as the statement (b) of the theorem.

If we write the condition (8) in the following form

$$Q(z) = \left[\left(\frac{f(rz)}{rzf'(rz)} - 1\right) + 1\right] \prec 1 + \lambda z,$$

and if we put $\omega(z) = \dfrac{f(rz)}{rzf'(rz)} - 1$, then, by applying Lemma A (c), we have the statement (c) of the theorem.

We note that the equation (6) we get from from the condition $\lambda^2 + {\lambda_1}^2 = 1$, and since the function $\varphi(r) = 2r^4 + 2ar^3 + a^2r^2 - 1$ has the properties:

$$\varphi(0) = -1, \quad \varphi(1) = (a+1)^2 > 0, \quad \varphi'(r) = 8r^3 + 6ar^2 + 2a^2r > 0$$

for $r \in (0,1)$, it means that the equation has only one root in the interval (0,1). □

If $|a_2| = a \to 0$, then from Theorem 1 we have the following

Corollary 1. *Let $f(z) = z + a_3z^3 + a_4z^4 + \dots$ be analytic in U and satisfy the condition* (2). *Then f is starlike for, at least, $|z| < \dfrac{1}{\sqrt[4]{2}} = 0.84089...$* □

This is the former result given in [2].

For example, for $a = 1$ in Theorem 1, we have

Corollary 2. *Let $f(z) = z + e^{i\theta}z^2 + a_3z^3 + \dots$ satisfy the condition* (2). *Then the following statements are true:*

(a) $\quad Re\left\{\dfrac{zf'(z)}{f(z)}\right\} > 0, \quad |z| < r_1,$

where $r_1 = 0.58975...$ is the positive root of the equation $2r^3 + r - 1 = 0$.

(b) $\quad \left|\dfrac{zf'(z)}{f(z)} - \dfrac{1}{2\beta}\right| < \dfrac{1}{2\beta}, \quad |z| < r,$

where β and r are given by

$$\beta = \begin{cases} \dfrac{1 - r(1+r)}{1+r^2}, & 0 < r \leq \dfrac{1}{2}, \\[2ex] \dfrac{1 - \left[(r(1+r))^2 + r^2\right]}{2(1-r^4)}, & \dfrac{1}{2} \leq r < r_1. \end{cases}$$

(c) $\left|\dfrac{f(z)}{zf'(z)}-1\right|<\dfrac{r+2r^2}{1-r^2},\ |z|<r,\ 0<r\leq\dfrac{\sqrt{13}-1}{6}=0.43425\ldots$ □

We note that the radius of starlikeness for the whole class of univalent functions in U is equal to $\rho=\tanh\dfrac{\pi}{4}=0.65579...$, and in that sense our result in (a) is not so good. But, for a smaller value of a we have a better result. For example, for $0\leq a\leq 0.72$ we get that the radius of starlikeness is greater than ρ.

From Theorem 1 (c) we also get the following

Corollary 3. *Let $f(z)=z+a_2z^2+\cdots\in A$ satisfy the condition* (2) *with* $|a_2|=a$ $(0\leq a\leq 2)$. *Then $f\in S^*\left(\dfrac{1}{2}\right)$ for $|z|<\dfrac{2}{\sqrt{a^2+12}+a}$.* □

Remarks. For $f\in A$ which satisfies (2) and (3) we obtain (where we put $|a_2|=a$):

$$\left|\left(\frac{z}{f(z)}\right)^2-1\right|=\left|\frac{z}{f(z)}-1\right|\left|\frac{z}{f(z)}+1\right|\leq|z|(a+|z|)(|z|(a+|z|)+2),$$

and similarly as in the proof of Theorem 1, we can write that

$$\left(\frac{rz}{f(rz)}\right)^2\prec 1+r(a+r)(r(a+r)+2)z. \tag{10}$$

Now, if we put

$$Q(z)=\frac{(rz)^2f'(rz)}{f^2(rz)},\ p(z)=\frac{1}{f'(rz)},\ \lambda=r(a+r)\,(r(a+r)+2),\ \lambda_1=r^2,$$

and combine the relations (9) and (10), we can have the appropriate results for a function $\dfrac{1}{f'(z)}$ as for $\dfrac{f(z)}{zf'(z)}$ in Theorem 1, but they are more complicated.

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