FILOMAT (Niš) **10** (1996), 1–32

CONVERGENCE WITH RESPECT TO ULTRAFILTERS: A SURVEY

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Abstract. The purpose of this paper is to present some of the most important results concerning some generalizations of the classical notions of sequentiality, Fréchet-Urysohn property, radiality, pseudo-radiality, countable compactness and pseudocompactness.

1. Introduction

All spaces are assumed to be Tychonoff. For $x \in X$, the set of all neighborhoods of x in X will be denoted by $\mathcal{N}(x)$. If X is a continuous function, then $\bar{f} : \beta X \to \beta Y$ will denote the Stone-Čech extension of f. The Greek letter κ will stand for an arbitrary cardinal and the Greek letters α and γ will stand for infinite cardinal numbers. If α is a cardinal, then the Stone-Čech extension $\beta(\alpha)$ of the discrete space α will be identified with the set of all ultrafilters on α and its remainder $\alpha^* = \beta(\alpha) \setminus \alpha$ will be identified with the set of all free ultrafilters on α. If $A \subseteq \alpha$, then $\hat{A} = Cl_{\beta(\alpha)}A = \{p \in \beta(\alpha) : A \in p\}$ and $A^* = \hat{A} \setminus A$. For $p \in \beta(\alpha)$, the norm of p is $\|p\| = \min\{|A| : A \in p\}$. If α is a cardinal and $\gamma \leq \alpha$, then $U_\gamma(\alpha) = \{p \in \beta(\alpha) : \gamma \leq \|p\|\}$. If $\alpha = \gamma$, then we simply write $U(\alpha)$. If α is a cardinal and X a set, then $[X]^\alpha = \{A \subseteq X : |A| = \alpha\}$ and $[X]^{\leq\alpha} = \{A \subseteq X : |A| \leq \alpha\}$.

For a space X, the *tightness* of $x \in X$, denoted by $t(x, X)$, is the smallest cardinal λ with the property that if $x \in Cl_X A$, then there exists $B \in [A]^{\leq\lambda}$ such that $x \in Cl_X B$, and the *tightness* of X is $t(X) = \sup\{t(x, X) : x \in X\}$. For a space X, the *pseudo-character* of $x \in X$ is denoted by $\psi(x, X)$ and the *pseudo-character* of the space X by $\psi(X)$; $d(X)$ will stand for the *density* of X and $c(X)$ for the cellularity of X. For cardinal invariants see [Ho].

Let X be a space, let $(x_n)_{n<\omega}$ be a sequence in X and let $x \in X$. If $\mathcal{F}_r$ is the Fréchet filter on ω, then $x_n \to x$ iff for every neighborhood V of x

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Received October 10, 1996
1991 *Mathematics Subject Classification*: 54A20, 54A25, 54C05, 54D20, 54D55.
* Supported by the Serbian Scientific Foundation, grant N^0 04M01.

we have that $\{n < \omega : x_n \in V\} \in \mathcal{F}_r$. This shows that the usual concept of convergence sequence can be expressed in terms of filters. In 1968, M. Katetov [Ka] introduced the notion of an $\mathcal{F}$-limit point of an α-sequence in a topological space.

Definition 1.1. Let $\mathcal{F}$ be a filter on α and X a space. A point $x \in X$ is said to be an *$\mathcal{F}$-limit point of the α-sequence* $(x_\xi)_{\xi<\alpha}$ in X, written $x = \mathcal{F}-\lim x_\xi$, if for every neighborhood V of x $\{\xi < \alpha : x_\xi \in V\} \in \mathcal{F}$.

A. I. Bernstein [Be] also discovered, in connection with problems in the theory of non-standard analysis, the $\mathcal{F}$-limits for the particular case when $\mathcal{F}$ is an ultrafilter on ω. The concept of $\mathcal{F}$-limit was also used by Z. Frolík in [Fro3] prior to the appearance of Katetov's paper: when Frolík writes $x = \sum_y \{y_n\}$ he means precisely that $x = y - \lim x_n$. We should remark that the $\mathcal{F}$-limit points when they exist they are unique, since our spaces are Hausdorff. To study $\mathcal{F}$-limits for arbitrary filters $\mathcal{F}$ it suffices to consider only ultrafilters as it shown in the following lemma which is taken from [GM; Lemma 1.2].

Lemma 1.2. *Let $\mathcal{F}$ be a filter on α, X a space and $(x_\xi)_{\xi<\alpha}$ an α-sequence in X. Then $x = \mathcal{F} - \lim x_\xi$ iff $x = p - \lim x_\xi$ for every $p \in \beta(\alpha)$ with $\mathcal{F} \subseteq p$.*

In virtue of Lemma 1.2, we will principally be concerned with ultrafilters. The notion of a cluster point of a subset of a topological space may be expressed in terms of ultrafilters as well:

Lemma 1.3. *Let X be a space with $|X| = \alpha$ and $\varnothing \neq A \subseteq X$. Then, $x \in Cl_X A$ if and only if there are an α-sequence $(x_\xi)_{\xi<\alpha}$ in A and $p \in \beta(\alpha)$ such that $x = p - \lim x_\xi$.*

Proof. *Necessity*: Suppose that $x \in Cl_X A$. We may assume that A is infinite. Let $\{x_\xi : \xi < \alpha\}$ be an enumaration of X. Enumerate A as $\{x_{\xi_\nu} : \nu < \alpha\}$, we repeat elements if it is necessary. Then, $(x_{\xi_\nu})_{\nu<\alpha}$ is an α-sequence in A. We have that $\mathcal{B} = \{\{\nu < \alpha : x_{\xi_\nu} \in V\} : V \in \mathcal{N}(x)\}$ is a filter base on α. It then follows that if $p \in \beta(\alpha)$ and $\mathcal{B} \subseteq p$, then $x = p - \lim_{\nu<\alpha} x_{\xi_\nu}$.

Sufficiency: Let $(x_\xi)_{\xi<\alpha}$ be an α-sequence in A and $p \in \beta(\alpha)$ such that $x = p - \lim x_\xi$. If $V \in \mathcal{N}(x)$, then $\varnothing \neq \{\xi < \alpha : x_\xi \in V\}$. This shows that $x \in Cl_X A$. □

V. Saks [Sa2] pointed out that any topological space is characterized by p-limit points. In fact, he proved that for any $A \subseteq X$ one has that

$$Cl_X A = \{x \in X : x \text{ is the } p\text{-limit point of some } \alpha\text{-sequence } (x_\xi)_{\xi<\alpha} \text{ in } X \text{ for some } p \in \alpha^*\};$$

this fact follows from Lemma 1.3.

Some of the classical concepts of general topology have their equivalent forms in tems of p-limit points, for $p \in \omega^*$ as follows:

Theorem 1.4. *Let X be a space.*

(1) *X is sequential iff for every non-closed subset $A \subseteq X$ there are $x \in X \setminus A$ and a sequence $(x_n)_{n<\omega}$ in A such that $x = p-\lim x_n$ for every $p \in \omega^*$.*

(2) *X is Fréchet-Urysohn iff for every $x \in Cl_X A$ there is a sequence $(x_n)_{n<\omega}$ in A such that $x = p - \lim x_n$ for every $p \in \omega^*$.*

(3) *X is countably compact iff for every countable subset $\{x_n : n < \omega\}$ of X there are $p \in \omega^*$ and $x \in X$ such that $x = p - \lim x_n$.*

In 1975, J. Ginsburg and V. Saks [GS] generalized the notion of p-limit point, for ultrafilters on ω by replacing sequences of points by sequences of non-empty subsets. This generalization is included in the next definition.

Definition 1.5. Let p be an ultrafilter on α and X a space. A point $x \in X$ is said to be an *p-limit point of the α-sequence $(S_\xi)_{\xi<\alpha}$ of non-empty subsets of X* if for every neighborhood V of x, $\{\xi < \alpha : S_\xi \cap V \neq \varnothing\} \in p$.

In general, a sequence $(S_n)_{n<\omega}$ of non-empty subsets of a space X could have more than one p-limit point: for instance, if $S_n = \{\frac{1}{n+1}\} \times \mathbb{R}$ for each $n < \omega$, then we have that every point of $\{0\} \times \mathbb{R}$ is a p-limit point of $(S_n)_{n<\omega}$, for every $p \in \omega^*$. Hence, if $(S_\xi)_{\xi<\alpha}$ is an α-sequence of non-empty subsets of a space X and $p \in \alpha^*$, then $L(p, (S_\xi)_{\xi<\alpha})$ will denote the set of all p-limit points of $(S_\xi)_{\xi<\alpha}$ in X.

For pseudocompactness we have the following equivalent statement by using p-limit points.

Theorem 1.6. *A space X is pseudocompact iff for every sequence $(V_n)_{n<\omega}$ of non-empty open subsets of X there is $p \in \omega^*$ such that $L(p, (V_n)_{n<\omega}) \neq \varnothing$.*

By using several ulrafilters at the same time we may define two kinds of convergence.

Definition 1.7. Let $\varnothing \neq M \subseteq \alpha^*$, X a space and $(x_\xi)_{\xi<\alpha}$ an α-sequence. For $x \in X$, we have:

(1) $x = M$-*weak*-$\lim x_\xi$ if there is $p \in M$ such that $x = p - \lim x_\xi$;

(2) $x = M$-*strong*-$\lim x_\xi$ if $x = p - \lim x_\xi$ for all $p \in M$.

In [Koc3] and in [Kom2], the authors consider the following convergence with respect to a set of ultrafilters: For $\varnothing \neq M \subseteq \alpha^*$ and $(x_\xi)_{\xi<\alpha}$ an α-sequence in X, $x = M$-*very-weak*-$\lim x_\xi$ if for every $V \in \mathcal{N}(x)$ there is $p \in M$ such that $\{\xi < \alpha : x_\xi \in V\} \in p$. But this notion coincides with the notion of $Cl_{\beta(\alpha)} M$-weak-limit.

Theorem 1.8. *Let $\varnothing \neq M \subseteq \alpha^*$, X a space and $(x_\xi)_{\xi<\alpha}$ an α-sequence. For $x \in X$, the following are equivalent.*

(1) $x = Cl_{\beta(\alpha)}M$*-weak-*$\lim x_\xi$;

(2) $x = M$*-very-weak-*$\lim x_\xi$.

Proof. **(1) $\Rightarrow$ (2)**: Suppose that $x = Cl_{\beta(\alpha)}M$-weak-$\lim x_\xi$. Then, there is $p \in Cl_{\beta(\alpha)}M$ such that $x = p-\lim x_\xi$. Let $V \in \mathcal{N}(x)$. Then, we obtain that $A = \{\xi < \alpha : x_\xi \in V\} \in p$. Thus, $p \in \hat{A}$ and hence there is $q \in M$ such that $q \in \hat{A}$. It then follows that $\{\xi < \alpha : x_\xi \in V\} \in q$.

(2) $\Rightarrow$ (1): Assume that $x = M$-very-weak-$\lim x_\xi$ and that $x \neq p-\lim x_\xi$ for every $p \in Cl_{\beta(\alpha)}M$. Then, for each $p \in Cl_{\beta(\alpha)}M$ there is $V_p \in \mathcal{N}(x)$ such that $A_p = \{\xi < \alpha : x_\xi \notin V_p\} \in p$. Since $Cl_{\beta(\alpha)}M$ is compact, then there is $\{p_0, \ldots, p_n\} \subseteq Cl_{\beta(\alpha)}M$ such that $Cl_{\beta(\alpha)}M \subseteq \bigcup_{i\leq n} \hat{A}_{p_i}$. Put $V = \bigcap_{i\leq n} V_{p_i}$. By assumption, there is $q \in M$ for which $A = \{\xi < \alpha : x_\xi \in V\} \in q$. Choose $k \leq n$ so that $A_{p_k} \in q$. Then, we have that $A \cap A_{p_k} \neq \varnothing$, but this is a contradiction. $\square$

The Rudin-Keisler (pre)-order on α^* is defined as follows: for $p, q \in \alpha^*$, $p \leq_{RK} q$ if there is $f \in {}^\alpha\alpha$ such that $\bar{f}(q) = p$. If $p \leq_{RK} q$ and $q \leq_{RK} p$, for $p, q \in \alpha^*$, then we say that p and q are *RK-equivalent* and write $p \approx_{RK} q$. It is known that $p \approx_{RK} q$ iff there is a permutation f of α such that $\bar{f}(p) = q$. For $p \in \alpha^*$, the *type* of p is the set $T(p) = \{q \in \alpha^* : p \approx_{RK} q\}$. For $p, q \in \alpha^*$, $p < q$ means that $p \leq_{RK} q$ and p is not Rudin-Keisler equivalent to q. If $p \in \alpha^*$, then $P_{RK}(p) = \{q \in \alpha^* : q \leq_{RK} p\} \cup \alpha$ and if $M \subseteq \alpha^*$, then $P_{RK}(M) = \bigcup_{p\in M} P(p)$. Observe that $|P_{RK}(p)| \leq 2^\alpha$ and $|T_{RK}(p)| \leq 2^\alpha$, for each $p \in \alpha^*$. An other important order on ω^* is the *Rudin-Frolík order* which is defined by $p \leq_{RF} q$ if there is an embedding $e : \omega \to \omega^*$ such that $\bar{e}(p) = q$ for $p, q \in \omega^*$. It is known that $\leq_{RF} \subseteq \leq_{RK}$ and they are completely different each other (see [CN2]).

The relationship between the Rudin-Keisler order and p-limit points is established in the next easy lemma.

Lemma 1.9. *Let $p, q \in \alpha^*$. Then the following are equivalent.*

(1) $p \leq_{RK} q$;

(2) *if $(x_\xi)_\xi$ is an α-sequence in a space X and $x = p-\lim x_\xi$, then there is a function $f \in {}^\alpha\alpha$ such that $x = q-\lim x_{f(\xi)}$.*

H. Gonshor [Go] noticed that the p-limit point of a sequence $(x_n)_{n<\omega}$ exists iff the function $f : \omega \to X$ defined by $f(n) = x_n$, for each $n < \omega$, can be extended to a continuous function $\omega \cup \{p\} \to X$. In this direction, V. Saks [Sa1] also noticed that $x = p-\lim x_n$ iff the function $f : \omega \to X$ defined by $f(n) = x_n$, for each $n < \omega$, satisfies that $\bar{f}(p) = x$. This observation holds for any cardinal α: $x = p - \lim x_\xi$ for $p \in \alpha^*$ iff the function $f : \alpha \to X$ defined by $f(\xi) = x_\xi$, for each $\xi < \alpha$, satisfies that $\bar{f}(p) = x$. This allows to simplify our notation:

• we shall use in some cases "$f \in {}^{\alpha}X$" instead of "$(x_\xi)_{\xi<\alpha}$ is an α-sequence" and "$\bar{f}(p) = x$" instead of "$x = p - \lim x_\xi$". •

Thus, we have that $x = M$-weak-$\lim x_\xi$ iff the function $f \in {}^{\alpha}X$ defined by $f(\xi) = x_\xi$, for each $\xi < \alpha$, satisfies that $\bar{f}(p) = x$ for some $p \in M$, and $x = M$-strong-$\lim x_\xi$ iff the function, $f \in {}^{\alpha}X$ defined by $f(\xi) = x_\xi$, for each $\xi < \alpha$, satisfies that $\bar{f}(p) = x$ for all $p \in M$.

If $p, q \in \beta(\alpha)$, then the *tensor product* of p and q is

$$p \otimes q = \{A \subseteq \alpha \times \alpha : \{\xi < \alpha : \{\zeta < \alpha : (\xi, \zeta) \in A\} \in q\} \in p\}.$$

Then, $p \otimes q$ is an ultrafilter on $\alpha \times \alpha$ which can be considered as an ultrafilter on α via a fixed bijection between α and $\alpha \times \alpha$. It was pointed out by Katetov [Ka] that $p <_{RK} p \otimes q$ and $q <_{RK} p \otimes q$ for every $p, q \in \beta(\alpha)$.

Now, we give the following two concepts of general topology.

Definition 1.10. Let X be a space.

(1) X is *$< \alpha$-bounded* if $Cl_X A$ is compact for every $A \subseteq X$ with $| A |< \alpha$;

(2) X is *initially α-compact* if every open cover $\mathcal{U}$ of X with $| \mathcal{U} |\leq \alpha$ has a finite subcover.

Notice that initial ω-compactness is countable compactness.

2. Generalizations of Frechét-Urysohn and sequential spaces

The following definition is essential in our generalization of Frechét-Urysohn and sequential spaces.

Definition 2.1. Let $\emptyset \neq M \subseteq \alpha^*$ and X a space. For $\emptyset \neq A \subseteq X$, we define:

(1) $A^M_W = \{x \in X : \exists f : \alpha \to A, \exists p \in M(\bar{f}(p) = x)\}$;

(2) $A^M_S = \{x \in X : \exists f : \alpha \to A, \forall p \in M(\bar{f}(p) = x)\}$;

(3) ([Koc3]) $A_W(M, 0) = A$, $A_W(M, \lambda + 1) = (A_W(M, \lambda))^M_W$, for an ordinal λ, and $A_W(M, \lambda) = \bigcup_{\eta<\lambda} A_W(M, \eta)$ if λ is a limit ordinal;

(4) ([Koc3]) $A_S(M, 0) = A$, $A_S(M, \lambda+1) = (A_S(M, \lambda))^M_S$ and $A_S(M, \lambda) = \bigcup_{\eta<\lambda} A_S(M, \eta)$ if λ is a limit ordinal.

We should remark that if $A \subseteq X$, then $x \in A^{\alpha^*}_S$ iff there is an α-sequence $(x_\xi)_{\xi<\alpha}$ in A such that $x_\xi \to x$.

Next, we state the natural generalizations of the concepts of sequential and Frechét-Urysohn spaces.

Definition 2.2. Let $\varnothing \neq M \subseteq \alpha^*$ and X a space. Then:

(1) ([Kom1], [Sav]) X is *weakly M-sequential*, if for every subset A of X, $Cl_X A = \bigcup_{\lambda<\alpha^+} A_W(M,\lambda)$;

(2) ([Kom1], [Sav]) X is *strongly M-sequential*, if for every subset A of X, $Cl_X A = \bigcup_{\lambda<\alpha^+} A_S(M,\lambda)$;

(3) ([Koc3]) X is a *$WFU(M)$-space*, if for every $A \subseteq X$, $Cl_X A = A_W^M = A_W(M,1)$;

(4) ([Koc3]) X is an *$SFU(M)$-space*, if for every $A \subseteq X$, $Cl_X A = A_S^M = A_S(M,1)$.

We remark that a space X is weakly M-sequential (resp., strongly M-sequential) if and only if A is a non-closed subset of X, then $\exists f \in {}^\alpha X$, $\exists x \in (X \setminus A)$, $\exists p \in M$ $(\bar{f}(p) = x \wedge f[\alpha] \subseteq A)$ (resp., $\exists f \in {}^\alpha X$, $\exists x \in (X \setminus A), \forall p \in M$ $(\bar{f}(p) = x \wedge f[\alpha] \subseteq A)$), and X is a $WFU(M)$-space (resp., $SFU(M)$-space) if and only if $A \subseteq X$ and $x \in Cl_X A$, then $\exists f \in {}^\alpha X$, $\exists p \in M$ $(\bar{f}(p) = x \wedge f[\alpha] \subseteq A)$ (resp., $\exists f \in {}^\alpha X$, $\forall p \in M$ $(\bar{f}(p) = x \wedge f[\alpha] \subseteq A)$). If $p \in \alpha^*$, then weakly $\{p\}$-sequential = strongly $\{p\}$-sequential and $WFU(\{p\})$-space = $SFU(\{p\})$-space. In this case, we simply say *p-sequential space* and *$FU(p)$-space*, respectively, and we write A^p for $A_W^{\{p\}} = A_S^{\{p\}}$ and $A(p,\lambda)$ for $A_W(\{p\},\lambda) = A_S(\{p\},\lambda)$, for every ordinal λ.

Let us also remark that in [Koc3] the author considered different kind of *$\mathcal{P}$-pseudo-radial* and *$\mathcal{P}$-radial* spaces, where $\mathcal{P}$ is a class of uniform ultrafilters on various cardinals (see also [N]); but this generalization is obvious and we shall not consider here these notions.

The following lema establishes the connection among Rudin-Keisler order, $FU(p)$-spaces and p-sequential spaces (a proof is available in [G2]). $\xi(p)$ denotes the subspace $\alpha \cup \{p\}$ of $\beta\alpha$.

Lemma 2.3. *For $p, q \in \alpha^*$, the following are equivalent.*

(1) $p \leq_{RK} q$;

(2) *$\xi(p)$ is a $FU(q)$-space;*

(3) *$\xi(p)$ is q-sequential;*

(4) *every p-sequential space is q-sequential;*

(5) *every $FU(p)$-space is a $FU(q)$-space.*

Some of the classical notions of general topology are defined in our context as follows. For a space X, we have that

(1) X is sequential iff X is strongly ω^*-sequential;

(2) $t(X) \leq \alpha$ iff X is weakly α^*-sequential;

(3) X is Frechét-Urysohn iff X is a $SFU(\omega^*)$-space.

(4) We recall that a space X is *pseudoradial* (or *chain-net*) if every non-closed $A \in X$ there is $x \in Cl_X A \setminus A$ and an α-sequence $(x_\xi)_{\xi<\alpha}$ such that $x_\xi \to x$, and X is *radial* (or *Frechét chain-net*) if for every $x \in Cl_X A$ there is an α-sequence $(x_\xi)_{\xi<\alpha}$ in A such that $x_\xi \to x$ (these two classes of spaces were first introduced by Herrlich [He]). We then have that X is pseudo-radial (resp., radial) iff there is a cardinal $\alpha \leq\mid X \mid$ such that X is stronly α^*-sequential (resp., a $SFU(\alpha)$-space).

In [BM], it was proved that there is a filter $\mathcal{F}$ on ω having the property that every sequential space is an $FU(\mathcal{F})$-space, and in [M] Malykhin proved that if the *Novak number* of ω^* (= the smallest cardinality of a cover by nowhere dense sets) is bigger than the continuum, then sequentiality coincides with weakly ω^*-sequentiality (called ultra-sequentiality in [M]).

If $p \in M \subseteq \alpha^*$, then

$$\begin{array}{ccccc} SFU(M)-\text{space} & \Rightarrow & FU(p)-\text{space} & \Rightarrow & WFU(M)-\text{space} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{strong } M-\text{sequentiality} & \Rightarrow & p-\text{sequentiality} & \Rightarrow & \text{weak } M-\text{sequentiality} \end{array}$$

The following examples show that the arrows cannot be reversed.

Example 2.4. **(1)** For a free filter $\mathcal{F}$ on α, $\xi(\mathcal{F}) = \alpha \cup \{\mathcal{F}\}$ will be the space in which α is discrete and the set $F \cup \{\mathcal{F}\}$, for $F \in \mathcal{F}$, is a basic neighborhood of $\mathcal{F}$. If $\mathcal{F}$ is a free filter $\mathcal{F}$ on α, then $M_{\mathcal{F}} = \{p \in \alpha^* : \mathcal{F} \subseteq p\}$ is a closed subset of α^* and $\xi(\mathcal{F})$ is a $SFU(M_{\mathcal{F}})$-space. Indeed, $\xi(\mathcal{F})$ is a $FU(p)$-space for $p \in \alpha^*$ iff $\mathcal{F} \subseteq p$. If $\varnothing \neq M \subseteq \alpha^*$, then $\mathcal{F}_M = \{A \subseteq \alpha : M \subseteq A^*\}$ is a free filter on α. Thus, we have that $M = M_{\mathcal{F}_M}$ and $\mathcal{F} = \mathcal{F}_{M_{\mathcal{F}}}$, for $\varnothing \neq M \subseteq \alpha^*$ and for a free filter $\mathcal{F}$ on α.

(2) For $p \in U(\alpha)$, $\xi(p) = \alpha \cup \{p\}$ is a subspace of $\beta(\alpha)$ and is a $FU(p)$-space. If $p, q \in U(\alpha)$ and $p <_{RK} q$, then $\xi(q)$ is a $FU(q)$-space that is not p-sequential (by Lemma 2.3). Hence, $\xi(p)$ is a $FU(q)$-space that is not sequential and is a $WFU(\{p,q\})$-space that is not a $FU(p)$-space.

(3) Let $\mathcal{M} = (\mathcal{F}_i)_{i \in I}$, we allow repetition, be an I-sequence of free ultrafilters on arbitrary cardinals. Then $\Xi(\mathcal{M}) = \oplus_{i \in I} \xi(\mathcal{F}_i)$ will denote the topological sum of the spaces $\xi(\mathcal{F}_i)'s$. It is not hard to prove that $\Xi(\mathcal{M})$ is weakly $(\bigcup_{i \in I} M_{\mathcal{F}_i})$-sequential.

(4) If $\varnothing \neq M \subseteq \alpha^*$, then the space $\Xi(M) = \oplus_{p \in M} \xi(p)$ is weakly M-sequential and if there are $p, q \in M$ which are RK-incomparable, then $\Xi(M)$ is neither a $FU(p)$-space nor a $FU(q)$-space.

(5) The Arens space $S_2 = \{x\} \cup \{x_n : n < \omega\} \cup \{x_{n,m} : (n,m) \in \omega \times \omega\}$, where $x_n \to x$ and $x_{n,m} \to x_n$ for every $m < \omega$, is a sequential space that is not Frechét-Urysohn.

(6) The sequential spaces S_n, for $n > \omega$ introduced in [AF] have their p-sequential version for $p \in U(\alpha)$. In fact, for $p \in U(\alpha)$ and $1 \leq n < \omega$, we define

$$S_n(p) = \{x\} \bigcup_{1 \leq k \leq n} \{x_{\xi_1,\ldots,\xi_k} : \xi_1, \ldots, \xi_k \in \alpha\}$$

and the topology on $S_n(p)$ satisfies that

(i) $x = p - \lim_{\xi_1 \to \alpha} x_{\xi_1}$;

(ii) $x_{\xi_1,\ldots,\xi_{k-1}} = p - \lim_{\xi_k \to \alpha} x_{\xi_1,\ldots,\xi_{k-1},\xi_k}$ for every $(\xi_1, \ldots, \xi_{k-1}) \in \alpha^{k-1}$ and for every $1 < k < \alpha$;

(iii) The set $\{x_{\xi_1,\ldots,\xi_n} : (\xi_1, \ldots, \xi_n) \in \alpha^n\}$ is discrete.

It is shown in [G1] that $S_n(p)$ can be embedded in α^* and $S_n(p)$ is a p-sequential space that is not a $FU(p)$-space for every $p \in U(\alpha)$ and for every $n < \omega$.

(7) For $p \in \omega^*$ and for each $\nu \leq \omega_1$, we may defined, by transfinite induction, a p-sequential space $S_\nu(p)$ that generalizes the space S_ν defined in [AF] (for the details see [G2]). For $p \in \omega^*$ and for $\nu \leq \omega_1$, it is also possible to embed $S_\nu(p)$ in $\beta(\omega)$. □

We omit the proof of the next theorem.

Theorem 2.5. *Let $\varnothing \neq N \subseteq M \subseteq \alpha^*$. Then:*

(1) *weak N-sequentiality $\Rightarrow$ weak M-sequentiality;*

(2) *strong M-sequentiality $\Rightarrow$ strong M-sequentiality;*

(3) *$WFU(N)$-space $\Rightarrow$ $WFU(M)$-space;*

(4) *$SFU(M)$-space $\Rightarrow$ $SFU(N)$-space;*

(5) *strong M-sequentiality = strong $(Cl_{\beta(\alpha)}(M))$-sequentiality;*

(6) *$SFU(M)$-space = $SFU(Cl_{\beta(\alpha)}M)$-space.*

W. W. Comfort and S. Negrepontis ([CN1], [CN2]) proved that the Rudin-Keisler order is 2^α upward directed. In the next theorem we give two statements which are equivalent to Comfort-Negrepontis' theorem.

Theorem 2.6. *For a cardinal α, the following are equivalent.*

(1) *If $S \subseteq \beta(\alpha)$ with $| S | \leq 2^\alpha$, then there is $q \in U(\alpha)$ such that $p <_{RK} q$ for all $p \in S$;*

(2) *If X is a space with $t(X) \leq \alpha$ and $| X | \leq 2^\alpha$, then there is $p \in U(\alpha)$ such that X is a $FU(p)$-space.*

Proof. (1) $\Rightarrow$ (2): This is Theorem 3.12 of [G2].

(2) $\Rightarrow$ (1): Let $S \subseteq \beta(\alpha)$ with $|S| \leq 2^\alpha$. Then the space $\Xi(S)$ satisfies that $t(\Xi(S)) = \alpha$ and $|\Xi(S)| \leq 2^\alpha$. By hypothesis, there there is $p \in U(\alpha)$ such that $\Xi(S)$ is a $FU(p)$-space. Hence, $\xi(q)$ is a $FU(p)$-space for each $q \in S$ and hence, by Lemma 2.3, $q \leq_{RK} p$ for each $q \in S$. □

3. Generalizations of countable compactness

The characterization of countable compactness given in Theorem 1.4 suggests the study of the following classes of spaces.

Definition 3.1. Let $\varnothing \neq M \subseteq \alpha^*$.

(1) ([Kom2], [Sav]) A space X is *quasi M-compact* if for every $f \in {}^\alpha X$ there is $p \in M$ such that $\bar{f}(p) \in X$;

(2) ([Be], [Sa2], [Wo]) A space X is *M-compact* if for every $f \in {}^\alpha X$, $\bar{f}(p) \in X$ for all $p \in M$;

(3) ([Koc2], [Koc3]) A space X is *strongly M-compact* if there is a point $x \in X$ such that for every $f \in {}^\alpha X$, $\bar{f}(p) = x$ for all $p \in M$.

For $\varnothing \neq M \subseteq \alpha^*$, we have that M-compactness implies quasi M-compactness. We observe that $(\alpha^* \setminus U(\alpha))$-compactness coincides with $< \alpha$-boundedness (see Theorem 1.3 in [G1]) and quasi ω^*-compactness coincides with countable compactness.

We also mention two results from [Koc3].

(1) Every initially α-compact $WFU(M)$-space (resp. $SFU(M)$-space) is quasi M-compact (resp. strongly M-compact);

(2) If X is a strongly M-compact space, then X is weakly M-sequential iff every strongly M-compact subspace A of X is closed in X.

The topological properties which are productive, closed-hereditary and surjective are characterized by using ultrafilters as follows.

Theorem 3.2. ([KS]) *Let $\mathcal{P}$ be a topological property which is productive, closed-hereditary and surjective. A space X of cardinality α has $\mathcal{P}$ if and only* *if X is $\mathcal{P}(\alpha)$-compact, where $\mathcal{P}(\alpha)$ is the maximal (M-compact)-reflexion of the discrete space α.*

As a consequence of Theorem 3.2, we have that M-compactness for $\varnothing \neq M \subseteq \alpha^*$ is productive, closed-hereditary and surjective. But, quasi M-compactness is not productive; for instance, there is a countably compact space X such that $X \times X$ is not pseudocompact (see [GJ; 9.15]).

If $\mathcal{P}$ = M-compactness for $\varnothing \neq M \subseteq \alpha^*$, then the *maximal $\mathcal{P}$-reflexion* of a space X, denoted by $\beta_M(X)$, has the following properties:

(1) $\beta_M(X) = \bigcap\{Y : X \subseteq Y \subseteq \beta(X), Y \text{ is } M - \text{compact }\}$;

(2) $\beta_M(X)$ is M-compact;

(3) X is dense in $\beta_M(X)$;

(4) for every continuous function $f : X \to Z$ such that Z is M-compact, we have that $\bar{f}[\beta_M(X)] \subseteq Z$;

(5) up to a homeomorphism fixing X pointwise, $\beta_M(X)$ is the only space satisfying (2), (3) and (4).

For an arbitrary space X and for $\varnothing \neq M \subseteq \alpha^*$, the M-compact reflexion $\beta_M(X)$ can be constructed step by step:

Let $\Phi_0(M, X) = X$, and $\Phi_{\lambda+1}(M, X) = \{\bar{f}(p) : f : \alpha \to \Phi_\lambda(M, X), p \in M\}$ for an ordinal λ and $\Phi_\lambda(M, X) = \bigcup_{\eta<\lambda} \Phi_\eta(M, X)$ for a limit ordinal λ. Then, we have that $\beta_M(X) = \bigcup_{\lambda<\alpha^+} \Phi_\lambda(M, X)$.

Hence, we have that for every space X and for $\varnothing \neq M \subseteq \alpha^*$, $|\beta_M(X)| \leq |M| \cdot |X|^\alpha$.

The next result is a direct application of Comfort-Negrepontis' theorem (see Theorem 2.6).

Theorem 3.3. ([G1]) *If $\varnothing \neq M \subseteq \alpha^*$ and $|M| \leq 2^\alpha$, then there is $p \in U(\alpha)$ such that p-compacntess implies M-compactness.*

The authors of [GS] proved that X^α is countably compact for all cardinal numbers α iff there is $p \in \omega^*$ such that X is p-compact. Saks [Sa2] extended this result for initial α-compactness and S. Garcia-Ferreira [G1] improved Saks' theorem by using decomposable ultrafilters as it stated in the following theorem. First, we recall that $p \in U(\alpha)$ is *decomposable* if for every $\omega \leq \gamma \leq \alpha$ there is $p_\gamma \in U(\gamma)$ such that $p_\gamma \leq p$.

Theorem 3.4. *Let X be a space. The following are equivalent.*

(1) *X^γ is initially α-compact for all cardinals γ;*

(2) *$X^{2^{2^\alpha}}$ is initially α-compact;*

(3) *$X^{|X|^\alpha}$ is initially α-compact;*

(4) *there is $p \in U(\alpha)$ decomposable such that X is p-compact.*

H. Donder [Do] has shown that in the core model every ultrafilter is decomposable. Thus, in the core model we have that X^γ is initially α-compact for all cardinal γ iff there is $p \in U(\alpha)$ such that X is p-compact. On the other hand, it is shown in [G6] that if $p \in U(\alpha)$, for a strong limit cardinal α, satisfies that $q \leq p$ implies that either $q \in \omega^*$ or $p \in U(\alpha)$ (these ultrafilters are called *indecomposable*), then $\beta_p(\alpha)$ is not initially α-compact. Prikry [P] proved that if $p \in U(\alpha)$ is α-complete, then there is a generic extension in which any ultrafilter extending p is indecomposable and α is a strong limit with $cf(\alpha) = \omega$.

The existence of two initially α-compact spaces whose product is not initially α-compact for a regular cardinal α is still unknown in ZFC: E. K. van Douwen [vD] constructed an example assuming GCH and Nyikos and Vaughan [NV] proved that if $\alpha^{++} \leq 2^\omega$, then there is a family of α^{++} initially α-compact spaces whose product is not countably compact. The following partial answer is taken from [G1].

Theorem 3.5. *If initial α-compactness is productive, then there is $p \in U(\alpha)$ decomposable such that initial α-compactness coincides with p-compactness.*

A characterization of intial α-compacntess is given in the next theorem (a proof is available in [St]; see also [N]).

Theorem 3.6. *For a space X the following are equivalent.*

(1) *X is initially α-compact;*

(2) *for every $\omega \leq \gamma \leq \alpha$ and for every $f : \gamma \to X$ there is $p \in U(\gamma)$ such that $\bar{f}(p) \in X$.*

Theorem 3.6 suggests the next generalization of quasi M-compactess.

Definition 3.7. Let α be a cardinal and let $\mathcal{M} = \{M_i : i \in I\}$ be an arbitrary set of non-empty subsets of α^*. Then, a space X is said to be *quasi $\mathcal{M}$-compact* if X is quasi M_i-compact for every $i \in I$.

If $\emptyset \neq M \subseteq \alpha^*$, then quasi M-compactness agrees with quasi $\{M\}$-compactness, and if $\mathcal{M} = \{p_i : i \in I\} \subseteq \alpha^*$, then X is quasi $\mathcal{M}$-compact iff X is p_i-compact for all $i \in I$. If $p, q \in \omega^*$ satisfy that $r < q$ for all $r \in \beta_p(\omega)$, then $\beta_p(\omega)$ is p-compact, but it is not quasi $\{\{p\}, \{q\}\}$-compact.

For $\omega \leq \gamma \leq \alpha$, $\mathcal{M}(\gamma, \alpha)$ will denote an arbitrary set $\{M_\kappa : \gamma \leq \kappa \leq \alpha\}$ of non-empty subsets of α^* such that $M_\kappa \subseteq U(\kappa)$ for every $\gamma \leq \kappa \leq \alpha$. Using this terminology, we have that a space X is initailly α-compact if and only if there is a set $\mathcal{M}(\omega, \alpha)$ such that X is quasi $\mathcal{M}(\omega, \alpha)$-compact.

For a cardinal α, let $\mathcal{C}_\alpha$ will denote the class of all spaces X with the property that for every initially α-compact space Y, $X \times Y$ is initially α-compact. The class $\mathcal{C}_\omega$ was introduced by Frolík [Fro2] and he characterized the spaces which are in $\mathcal{C}_\omega$. For cardinals higher than ω we have the following theorem.

For $\omega \leq \gamma \leq \alpha$, we set

$$\mathcal{A}(\gamma, \alpha) = \{U(\gamma) \cap K : K \subseteq \alpha^*, \alpha \cup K \text{ is initially } \alpha - \text{compact}\},$$

and $\mathcal{M}_\alpha = \bigcup_{\omega \leq \gamma \leq \alpha} \mathcal{A}(\gamma, \alpha)$.

Theorem 3.8. ([G7]) *For a space X, the following are equivalent:*

(1) *$X \in \mathcal{C}_\alpha$;*

(2) *X is quasi $\mathcal{M}_\alpha$-compact.*

The following theorem is due to Savchenko [Sav] and Kombarov [Kom1] (for $\alpha = \omega$).

Theorem 3.9. *Let $\varnothing \neq M \subseteq \alpha^*$. If X is a paracompact weakly M-sequential space and Y is a collectionwise normal M-compact space, then $X \times Y$ is collectionwise normal.*

The next definition characterizes the spaces X for which X^γ is initially α-compact for some cardinal γ.

Definition 3.10. ([G7]) Let $\varnothing \neq M \subseteq \alpha^*$ and let κ be a cardinal with $1 \leq \kappa$. Then, we say that X is *(κ, M)-compact* if for every κ-sequence $(f_\xi)_{\xi<\kappa}$ of functions in ${}^\alpha X$, there is $p \in M$ such that $\bar{f}_\xi(p) \in X$ for each $\xi < \kappa$.

Theorem 3.11. ([G7]) *Let X be a space and let α and κ be cardinals with $1 \leq \kappa$. The following are equivalent.*

(1) *X^κ is initially α-compact ;*

(2) *for each cardinal γ with $\omega \leq \gamma \leq \alpha$ there is $\varnothing \neq M_\gamma \subseteq U(\gamma)$ such that X is (κ, M_γ)-compact;*

(3) *X is $(\kappa, U(\gamma))$-compact for every cardinal $\omega \leq \gamma \leq \alpha$.*

Now, we give some results about when $\beta_M(\alpha) \in \mathcal{C}_\alpha$, for $\varnothing \neq M \subseteq \alpha^*$.

Theorem 3.12. ([G7]) *If there is $\varnothing \neq M \subseteq \omega^*$ with $\beta_M(\omega) \in \mathcal{C}_\omega$, then $\beta_M(\omega) = \beta(\omega)$.*

It is shown in [G7] that there is $\varnothing \neq M \subseteq \omega^*$ such that $\beta_M(\omega) \neq \beta(\omega)$ and $|\beta_M(\omega)| = 2^{2^\omega}$. Unfortunately, the following question remains open.

Question 3.13. ([G7]) *Let $\alpha > \omega$ be a regular cardinal. Is there $\varnothing \neq M \subseteq \alpha^*$ such that $\beta_M(\alpha) \in C_\alpha$ and $\beta_M(\alpha) \neq \beta(\alpha)$?*

The following (pre)-order on α^* was introduced by W.W. Comfort in [G4] and is a very important tool to study the p-compact like properties.

Definition 3.14. For $p, q \in \alpha^*$, we say that $p \leq_C q$ if every q-compact space is p-compact.

It is evident that $\leq_{RK} \subseteq \leq_C$ (for a proof of the fact that these two orders are different see [G4; Th. 2.8]). If $p \leq_C q$ and $q \leq_C p$, for $p, q \in \alpha^*$, then we write $p \approx_C q$. The *Comfort type* of $p \in U(\alpha)$ is the set $T_C(p) = \{q \in \alpha^* : p \approx_C q\}$. It is proved in [G6; Lemma 3.4] that if $p \in U(\alpha)$, then $T_C(p) \subseteq U(\alpha)$ and $|\{T_{RK}(q) : p \approx_C q\}| \geq \omega$. If $p, q \in \alpha^*$, then $p <_C q$ means that $p \leq_C q$ and p is not Comfort equivalent to q. A useful characterization of the Comfort-order is given in the next theorem.

Theorem 3.15. ([G4], [G6]) *For $p, q \in \alpha^*$, the following are equivalent:*

(1) *$p \leq_C q$;*

(2) *$\beta_p(\alpha) \subseteq \beta_q(\alpha)$;*

(3) *$p \in \beta_q(\alpha)$;*

(4) *$\exists f \in {}^\alpha\beta_q(\alpha)$ $(\bar{f}(q) = p \notin f[\alpha])$;*

(5) *$\beta_q(\alpha)$ is p-compact;*

(6) *$\beta_q(\alpha) \cap \alpha^*$ is p-compact.*

It is a direct consequence of Theorem 3.15 that if $p \in \alpha^*$, then $P_C(p) = \{q \in \alpha^* : q \leq_C p\} \cup \alpha = \beta_p(\alpha)$.

Next, we state some of the properties of $T_C(p)$ for $p \in \alpha^*$ (for definitions see [CN2]).

Theorem 3.16. *For $p \in U(\alpha)$, we have that*

(1) *if $p \in \omega^*$, then $T_C(p)$ contains a subset S such that $(S, \leq_{RF})$ is order isomorphic to the reals;*

(2) ([G4]) *if $p \in \omega^*$ is RK-minimal, then every two points of $T_C(p)$ are RF-comparable;*

(3) ([G4]) *if $p, r, s \in \omega^*$ satisfy that $s \leq_C p$, $r \leq_C p$ and s and r are RF-incomparable, then $(T_C(p), \leq_{RF})$ is not a linearly ordered set;*

(4) ([G4]) *if $p \in \omega^*$, then $T_C(p)$ is countably compact;*

(5) ([G4]) *if $p \in \omega^*$ is a P-point, then $T_C(p)$ is p-compact;*

(6) ([G7]) *if $p, q \in \omega^*$ are RK-minimal and RK-incomparable, then $T_C(p) \times T_C(q)$ is not countably compact;*

(7) ([G8]) *if $p, q, r \in \omega^*$ satisfy that $r \leq_C p$ and $r \leq_C q$ and r is a P-point of ω^*, then $T_C(p) \times T_C(q)$ is r-compact;*

If $c_p = | \{T_{RK}(q) : p \approx_C q\} |$, then

(8) ([G6]) *if $\alpha \leq c_p$, then $\alpha < c_p \leq 2^\alpha$;*

(9) ([G6]) *if p is countably complete, then $2^\omega \leq c_p = c_p^\omega \leq 2^\alpha$;*

(10) ([G6]) *if p is decomposable, then $c_p = 2^\alpha$;*

(11) ([G6]) *if p is regular, then $c_p = 2^\alpha$;*

(12) ([G6]) *if $p \in U(\aleph_n)$, then $c_p = 2^{\aleph_n}$ for every $n < \omega$;*

(13) ([G6]) *if $\omega < \alpha$ and p is RK-minimal, then $c_p = \omega$.*

We do not know the response to the next question in ZFC.

Question 3.17. *If $p \in U(\aleph_\omega)$, must $c_p = 2^{\aleph_\omega}$?*

The property that says "*Rudin-Keisler ordering is downward directed*" is known as the set-theoretic principle *Near Coherence of Filters*, NCF. Shelah [BS] has defined a model of ZFC in which NCF holds and MA implies the negation of NCF. This principle NCF is equivalent to any one of the next assertions.

Theorem 3.18. ([G4]) *The following are equivalent.*

(1) *$\forall p, q \in \omega^* \exists r \in \omega^* (r \leq_{RK} p \wedge r \leq_{RK} q)$;*

(2) *$\forall p, q \in \omega^* \exists r \in \omega^* (r \leq_C p \wedge r \leq_C q)$;*

(3) *if X is p-compact and Y is q-compact, for $p, q \in \omega^*$, then $X \times Y$ is countably compact;*

(4) *$\forall p, q \in \omega^* (\beta_p(\omega) \cap \beta_q(\omega) \neq \omega)$.*

It is well-known [Bl] that if we assume MA, then there are $p, q \in \beta(\omega)$ such that p and q are RK-minimal and RK-incomparable; hence, $\beta_p(\omega) \cap \beta_q(\omega) = \omega$ (see [G4]). Thus, MA implies ths existence of two ultrafilters $p, q \in \omega^*$ for which there are a p-compact space and a q-compact space Y whose product $X \times Y$ is not countably compact. All the statements of Theorem 3.18 hold in a model of NCF.

It is a theorem of W.W. Comfort and Ch. Waiveris ([CW], [Wa]) that if X is either an F-space or realcompact, then there is a set $\{X_\xi : \xi < 2^{2^\omega}\}$ of countably compact (extra countably compact) subspaces of $\beta(X)$ such that $X_\xi \cap X_\zeta = X$ for $\xi < \zeta < 2^{2^\omega}$. Using p-limit points, V. Saks [Sa3] showed that these spaces can be chosen pairwise non-homeomorphic. In this direction, we shall present a similar result for F-spaces that is independent of the axioms of ZFC. We need two lemmas.

Lemma 3.19. *Let X be a space and let $p \in \omega^*$ be RK-minimal. Then*

$$\beta_p(X) = X \cup \{\bar{f}(p) : f \in {}^{\omega}\beta_p(X) \text{ is an embedding}\}.$$

Proof. Put $Y = X \cup \{\bar{f}(p) : f \in {}^{\omega}\beta_p(X)$ is an embedding$\}$. It is evident that $Y \subseteq \beta_p(X)$. We shall verify that Y is p-compact. Indeeed, let $f \in {}^{\omega}Y$ be an arbitrary function. We may assume that $\bar{f}(p) \notin f[\omega]$. By Lemma 2.16 of [G4], there is $A \in p$ such that $f|_A$ is an embedding. Now, we choose $B \subseteq A$ so that $B \in p$ and $|A \setminus B| = |B| = \omega$. Define a bijection $h : \omega \to A$ such that $h|_B$ is the identity and $g = f \circ h$. Then, $g : \omega \to \beta_p(X)$ is an embedding and $\bar{g}(p) = \bar{f}(p)$ and hence $\bar{f}(p) \in Y$. This shows that Y is p-compact and so $\beta_p(X) = Y$. □.

Lemma 3.20. ([FKZ]) *Let X be an F-space and let $f, g : \omega \to X$ be two embeddings. If $\bar{f}(p) = \bar{g}(q)$ for $p, q \in \omega^*$, then p and q are RF-comparable.*

Theorem 3.21. *Assume* MA. *Let X be a compact F-space. Then, there is a set $\{X_\xi : \xi < 2^{2^\omega}\}$ of subspaces of $\beta(X)$ such that*

(i) *X_ξ is p_ξ-compact for some $p_\xi \in \omega^*$ for each $\xi < 2^{2^\omega}$;*

(ii) *$X_\xi \cap X_\zeta = X$ for $\xi < \zeta < 2^{2^\omega}$.*

In addition, if X is not countably compact, then the spaces X_ξ's can be chosen pairwise non-homeomorphic.

Proof. By Theorem 14.25 of [GJ], we have that $\beta(X)$ is an F-space as well. It was shown by A. Blass [Bl] that if we assume MA, then there is a set $\{p_\xi : \xi < 2^{2^\omega}\}$ of RK-minimal points of ω^* which are pairwise RK-incomparable. For $\xi < 2^{2^\omega}$, we define $X_\xi = \beta_{p_\xi}(X)$. We only need to verify clause (ii). Let $\xi < \zeta < 2^{2^\omega}$ and assume that there is $z \in X_\xi \cap X_\zeta \cap (\beta(X) \setminus X)$. By Theorem 3.15, there are two functions $f : \omega \to X_\xi$ and $f : \omega \to X_\zeta$ such that $\bar{f}(p_\xi) = z = \bar{g}(p_\zeta)$. According to Lemma 3.19, we may suppose that f and g are embeddings. So, by Lemma 3.20, p_ξ and p_ζ are RK-comparable, which is a contradiction. Finally, assume that X is not countably compact. Then, $X \neq X_\xi$ for every $\xi < 2^{2^\omega}$. If X_ξ is homeomorphic to X_ζ, for $\xi < \zeta < 2^{2^\omega}$, then X_ξ would be p_ζ-compact, so $X_\zeta \subseteq X_\xi$, a contradiction since $X_\xi \cap X_\zeta = X$. □

We should remark that in a model of NCF, the discrete space ω does not satisfy the conclusion of Theorem 3.21. In fact, assuming NCF, if $p, q \in \omega^*$, then there is $r \in \omega^*$ such that $r \leq_{RK} p$ and $r \leq_{RK} q$ and then $\omega \neq \beta_r(\omega) \subseteq \beta_p(\omega) \cap \beta_q(\omega)$.

For $p \in U(\alpha)$, the spaces which quasi $P_{RK}(p)$-compact are called *almost p-compact* in [G7]. This name is because if X is quasi $P_{RK}(p)$-compact and $f : \alpha \to X$ is a function, then there $\sigma \in {}^{\alpha}\alpha$ such that $\bar{\sigma}(p) \in \alpha^*$ and $\bar{f}(\bar{\sigma}(p)) \in X$.

For every $p \in \omega^*$, it is clear that

p-compactness $\Longrightarrow$ almost p-compactness $\Longrightarrow$ countable compactness.

We give two examples to show that these three concepts are different each other.

Example 3.22. ([G7) Let $p \in \omega^*$.

(1) *An almost p-compact space that is not p-compact*: Our space Γ_p will be constructed by transfinite induction. Put $\Gamma_0 = \omega$ and assume that Γ_μ has been defined for $\mu < \nu < \omega_1$. Then, define

$$\Gamma_\nu = \{\bar{f}(q) : f : \omega \to \bigcap_{\mu<\nu} \Gamma_\mu \text{ is an embedding } \bar{f}(q) \neq p, q \in T_{RK}(p)\}.$$

We set $\Gamma_p = \bigcap_{\nu<\omega_1} \Gamma_\nu$. Since $p \notin \Gamma_p$, then Γ_p cannot be p-compact. It is not hard to see that Γ_p is almost p-compact.

(2) *A countably compact space that is not almost p-compact*: We define $\Delta_p = \omega \cup (\beta(\omega) \setminus P_{RK}(p))$. It is evident that Δ_p cannot be almost p-compact and since $|P_{RK}(p)| \leq 2^\omega$, Δ_p must be countably compact. □

As an other application of Theorem 2.6 is that if a countably compact space has cardinality not bigger than 2^ω, then the space is almost p-compact for some $p \in \omega^*$. A more general statement is the following.

Theorem 3.23. ([G7]) *If X_ξ is initially α-compact and $|X_\xi| \leq 2^\alpha$ for $\xi < 2^\alpha$, then there is $p \in U(\alpha)$ such that X_ξ is almost p-compact for every $\xi < 2^\alpha$.*

The almost p-compactness for a RK-minimal ultrafilter $p \in \omega^*$ has the following property.

Theorem 3.24. ([G7]) *For $p \in \omega^*$, the following are equivalent.*

(1) *p-is RK-minimal;*

(2) *quasi $T_{RK}(p)$-compactness agrees with almost p-compactness.*

By using almost p-compactness, the Rudin-Keisler order has the next equivalent statement:

Theorem 3.25. ([G7]) *For $p, q \in \omega^*$, the following are equivalent.*

(1) $p \leq_{RK} q$;

(2) *every almost p-compact space is almost q-compact.*

It is pointed out in [GS] that the type $T_{RK}(p)$ for $p \in U(\alpha)$ cannot be countably compact, but for the Comfort-types we have:

Theorem 3.26. ([G7]) *For $p \in \omega^*$, we have that $T_C(p)$ is almost p-compact.*

The following questions will provide information about the topological behaviour of the Comfort-types.

Question 3.27. *If $p \in U(\alpha)$ is not RK-minimal, must $T_C(p)$ be countably compact ?*

Question 3.28. *If $p \in \omega^*$ is not RK-minimal, must $T_C(p)$ p-compact ?*

The answer is in the positive fashion for a RK-minimal ultrafilter on ω.

4. Cardinal invariants

Theorem 2.6 leads us to consider the following cardinal invariant.

Definition 4.1. For a space X, we define

$$\tau_{FU}(X) = \min\{\alpha : \exists p \in U(\alpha)\ (X \text{ is a } FU(p)\text{-space})\}.$$

For any space X, we have that $t(X) \leq \tau_{FU}(X)$ and if $\tau_{FU}(X) \leq \alpha$, then there is $q \in U(\alpha)$ such that X is a $FU(q)$-space. According to Theorem 2.6, if $t(X) = \alpha$ and $|X| \leq 2^\alpha$, then $\tau_{FU}(X) \leq \alpha$ and hence $\tau_{FU}(X) = t(X)$. Note that if X is a $FU(p)$-space for $p \in U(\alpha)$, then $\tau_{FU}(X) \leq \|p\|$. The fact that $|X| \leq 2^{2^{d(X)}}$ (see 1.5.3 in [En]) implies that $\tau_{FU}(X) \leq 2^{2^{d(X)}}$. But, if X is a $WFU(M)$-space, $M \subseteq \alpha^*$, then $|X| \leq 2^{d(X)}$ ([Koc3]) so that for such spaces X we have $\tau_{FU}(X) \leq 2^{d(X)}$.

The following example shows that the functions t and τ_{FU} are different.

Example 4.2. Let $X = \Xi(\omega^*)$. Then, $t(X) = \omega$. Suppose that $\tau_{FU}(X) = \omega$. Then, there is $q \in \omega^*$ such that X is a $FU(q)$-space. Hence, $\xi(p)$ is a $FU(q)$-space for every $p \in \omega^*$. In virtue of Lemma 1.9, we have that $p \leq_{RK} q$ for every $p \in \omega^*$; that is, $|P(q)| = 2^{\mathfrak{c}}$, which is a contradiction. □

The proof of the following lemma is a direct application of Theorem 2.6.

Lemma 4.3. *For every $p \in U(\alpha)$ there is a set $\{p_\nu : \nu < \alpha^+\} \subseteq U(\alpha)$ such that*

(1) *$p_0 = p$;*

(2) *$p_{\nu+1} \approx_{RK} p \otimes p_\nu$ for every $\nu < \alpha^+$;*

(3) *$p_\nu < p_\mu$ whenever $\nu < \mu < \alpha^+$.*

Lemma 4.4. ([GMT; Lemma 1.4]) *Let $p, q \in \alpha^*$ and X a space.*

(1) *If $p \leq_{RK} q$, then $A^p \subseteq A^q$ for every $A \subseteq X$.*

(2) *If* $\{p_\nu : \nu < \alpha^+\} \subseteq U(\alpha)$ *satisfies the conclusion of Lemma 4.3 for* p, *then* $A(p,\nu) \subseteq A(p_\nu, 1)$ *for every* $\nu < \alpha^+$ *and for every* $A \subseteq X$.

The next theorem generalizes Theorem 3.5 of [G2] for arbitrarily higher cardinals.

Theorem 4.5. *If* X *is* p*-sequential for* $p \in U(\alpha)$, *then there is* $q \in U(\alpha)$ *such that* X *is a* $FU(p)$*-space.*

Proof. Let $\{p_\nu : \nu < \alpha^+\} \subseteq U(\alpha)$ satisfy the conditions of Lemma 4.3. According to Theorem 2.6, there is $q \in U(\alpha)$ such that $p_\nu < q$ for every $\nu < \alpha^+$. We claim that X is a $FU(q)$-space. In fact, for $A \subseteq X$ we have that $Cl_X A = \bigcup_{\lambda<\alpha^+} A(p,\lambda)$. In virtue of Lemma 4.4, we have that $A(p,\lambda) \subseteq A(p_\lambda, 1)$ for every $\lambda < \alpha^+$. Applying again Lemma 4.4, we have that $A(p_\lambda, 1) \subseteq A(q,1)$. Therefore, $Cl_X A = A(q,1)$. This shows that X is a $FU(q)$-space. □

It then follows from Theorem 4.5 that, for every space X,

$$\tau_{FU}(X) = \min\{\alpha : \exists p \in U(\alpha)\ (X \text{ is } p\text{-sequential})\}.$$

The next result is a corollary of Theorem 2.6.

Corollary 4.6. *For a cardinal* α, *the following are equivalent.*

(1) $\tau_{FU}(U(\alpha)) = \gamma$;

(2) $2^{2^\alpha} \leq 2^\gamma$.

The *degree of sequentiality* of a weakly (strongly) M-sequential space is given in the next definition.

Definition 4.7. Let X be a space. Then:

(1) if X is weakly M-sequential for some $\varnothing \neq M \subseteq \alpha^*$, we define

$$\sigma_W^M(X) = \min\{\lambda \leq \alpha^+ :\ \forall A \subseteq X(Cl_X A = A_W(M,\lambda))\};$$

(2) if X is strongly M-sequential for some $\varnothing \neq M \subseteq \alpha^*$, we define

$$\sigma_S^M(X) = \min\{\lambda \leq \alpha^+ :\ \forall A \subseteq X(Cl_X A = A_S(M,\lambda))\}.$$

Notice that X is weakly (resp., strongly) M-sequential, for some $\varnothing \neq M \subseteq \alpha^*$, if and only if $\sigma_W^M(X)$ (resp., $\sigma_S^M(X)$) exists [Koc3]. For a space X, we have that $\tau_{FU}(X) \leq \alpha$ if and only if $\sigma_S^M(X)$ exists. A space X is $SFU(M)$-space (resp., $WFU(M)$-space) if and only if $\sigma_S^M(X) = 1$ (resp., $\sigma_W^M(X) = 1$). If $M = \{p\}$ for some $p \in U(\alpha)$, then we write $\sigma_p(X) = \sigma_W^M(X) = \sigma_S^M(X)$.

The cardinal invariants stated in the following definition, for (pseudo) radial spaces, were introduced by Lj. Kočinac in [Koc1].

Definition 4.8. Let X be a space and let $\varnothing \neq M \subseteq \alpha^*$.

(1) if $x \in X$, then
$rt_W^M(x,X) = \min\{\lambda : x \in A_W(M,1) \Rightarrow \exists B \in [A]^{\leq\lambda}\ (x \in B_W(M,1))\}$;

(2) if $x \in X$, then
$rt_S^M(x,X) = \min\{\lambda : x \in A_S(M,1) \Rightarrow \exists B \in [A]^{\leq\lambda}\ (x \in B_S(M,1))\}$;

(3) $rt_W^M(X) = \sup\{rt_W^M(x,X) : x \in X\}$;

(4) $rt_S^M(X) = \sup\{rt_S^M(x,X) : x \in X\}$.

If $M = \{p\}$ for some $p \in U(\alpha)$, then we write $rt^p(x,X) = rt_W^M(x,X) = rt_S^M(x,X)$ and $rt^p(X) = rt_W^M(X) = rt_S^M(X)$, for any space X. If X is strongly M-sequential (resp., a $SFU(M)$-space), then $t(X) \leq rt_W^M(X)$ (resp., $t(X) \leq rt_S^M(X)$). For an arbitrary space X, $rt^p(X) \leq \tau_{FU}(X)$, where $p \in U(\alpha)$ is the ultrafilter which witnesses that X is a $FU(p)$-space.

The proof of the next theorem is left to the reader.

Theorem 4.9. *Let X be a space and let $\varnothing \neq M \subseteq \alpha^*$.*

(1) *if X is weakly M-sequential, then $t(x,X) \leq rt_W^M(x,X)$ for every $x \in X$ and*

$$t(X) \leq rt_W^M(X) \leq \min\{\|p\| : p \in M\} \leq \alpha;$$

(2) *if X is strongly M-sequential, then $t(x,X) \leq rt_S^M(X)$ for every $x \in X$ and*

$$t(X) \leq rt_S^M(X) \leq \min\{\|p\| : p \in M\} \leq \alpha;$$

(3) *if X is a $WFU(M)$-space, then $t(X) = rt_W^M(X)$;*

(4) *if X is a $SFU(M)$-space, then $t(X) = rt_S^M(X)$.*

Now, we give an alternative definition of the tightness of a space.

Theorem 4.10. *Let X be a space. If $t(X) = \gamma$, then $t(x,X) = rt_W^{U(\gamma)}(x,X)$ for every $x \in X$ and $t(X) = rt_W^{U(\gamma)}(X)$.*

Proof. First, notice that X is weakly $U(\gamma)$-sequential. Let $x \in X$. If $x \in Cl_X A$, then there is $B \in [A]^{\leq\gamma}$ such that $x \in Cl_X B$. Hence, by Lemmas 1.3 and 1.9, there is $p \in U(\gamma)$ and an γ-sequence $(x_\xi)_{\xi<\gamma}$ in B such that $x = p - \lim x_\xi$. Thus, $x \in B_W(U(\gamma,1)$ and $|B| \leq \gamma$. This, shows that $t(x,X) \geq rt_W^{U(\gamma)}(x,X)$. The equality follows from Theorem 4.9. Therefore, $t(X) = rt_W^{U(\gamma)}(X)$. $\square$

An application of Theorem 4.10 is that if $\Xi(\omega^*)$ is a $FU(p)$-space for $p \in U(\tau_{FU}(\Xi(\omega^*)))$, then $rt^p(\Xi(\omega^*) = \omega < \tau_{FU}(\Xi(\omega^*))$.

Lemma 4.11. *Let X be a space and $(x_\xi)_{\xi<\alpha}$ an α-sequence in X converging to $x \in X$. If $\psi(x, X) \leq \alpha$, then we have that*

(1) *if $\psi(x, X) < cf(\alpha)$, then $(x_\xi)_{\xi<\alpha}$ is eventually constant;*

(2) *if $cf(\alpha) \leq \psi(x, X)$, then there is a sub γ-sequence of $(x_\xi)_{\xi<\alpha}$ that converges to x.*

Proof. Put $\gamma = \psi(x, X)$. Let $\{V_\nu : \nu < \gamma\} \subseteq \mathcal{N}(x)$ be a pseudo-base at x. For each $\nu < \gamma$, we pick $\xi_\nu < \alpha$ such that $\xi_\nu \in U_\nu$ and if $\xi_\nu \leq \zeta < \alpha$, then $x_\zeta \in U_\nu$. Suppose that $(x_\xi)_{\xi<\alpha}$ is not eventually constant. Then, we may assume that $x \neq x_\xi$ for every $\xi < \alpha$. Since $\{x\} = \bigcap_{\nu<\gamma} V_\nu$, we must have that the set $\{\xi_\nu : \nu < \gamma\}$ is cofinal in α and hence $cf(\alpha) \leq \gamma$. Without loss of generality, we suppose that $\xi_\nu < \xi_\mu$ whenever $\nu < \mu < \gamma$. Now, we shall show that $\xi_\nu \to x$. Fix $V \in \mathcal{N}(x)$. Then, there is $\theta < \alpha$ such that if $\theta \leq \zeta < \alpha$, then $x_\zeta \in V$. Choose $\nu < \gamma$ so that $\theta < \xi_\nu$. Hence, if $\nu < \mu < \gamma$, then $x_{\xi_\mu} \in V$. Thus, $x_{\xi_\nu} \to x$ as required. $\square$

Lemma 4.11 implies the next result.

Lemma 4.12. *Let X be a space and $(x_\xi)_{\xi<\alpha}$ an α-sequence in X converging to $x \in X$. If $\psi(x, X) \leq cf(\alpha)$, then $A_S(\alpha^*, \lambda) \subseteq A_S(\gamma^*, \lambda)$ for every $A \subseteq X$ and for every cardinal λ.*

Now, we have two consequences of Lemmas 4.11 and 4.12.

Theorem 4.13. *Let X be a space such that $\gamma = \psi(X) \leq \alpha$. Then, we have that*

(1) *if X is strongly α^*-sequential, then X is strongly γ^*-sequential;*

(2) *if X is a $FU(\alpha^*)$-space, then X is a $SF(\gamma^*)$-space.*

Corollary 4.14. *Let X be a space.*

(1) ([Koc1]) *A pseudo-radial space of countable pseudo-charecter is sequential;*

(2) ([A1]) *A radial space of countable pseudo-charecter is Frechét-Urysohn.*

Definition 4.15. ([Koc3]) Let X be a space and let $\emptyset \neq M \subseteq \alpha^*$. We define

(1) $d_W^M(X) = \min\{|A| : X = A_W(M, 1)\}$;

(2) $d_S^M(X) = \min\{|A| : X = A_S(M, 1)\}$;

In the context of pseudo-radial spaces, $d_S^{\alpha^*}(X)$ was introduced in [Koc1]. For a space X, we have that $d(X) \leq \min\{d_W^M(X), d_S^M(X)\}$.

The density cardinal function can be defined as follows.

Theorem 4.16. *For any space X, we have that*

$$d(X) = d_W^{U(d(X))}(X).$$

Proof. We know that $d(X) \leq d_W^{U(d(X))}(X)$. If D is a dense subset of X with $|D| = d(X)$, then by Lemma 1.3, we obtain that $X = A_W(d(X)^*, 1)$. Hence, $d(X) \geq d_W^{U(d(X))}(X)$. □

Question 4.17. *Is there an example of a space X such that it is $FU(p)$-space for some $p \in U(\tau_{FU}(X))$ and $d(X) < d^p(X)$?*

From Theorem 2.6 it follows that if $|X \setminus D| \leq 2^{d(X)}$ for a dense subset D of X with $|D| = d(X)$, then $d(X) = d^p(X)$ for some $p \in U(d(X))$.

The next theorem is taken from [Koc3] (see also [G2]).

Theorem 4.18. *Let $\varnothing \neq M \subset \alpha^*$ and let X be a weakly M-sequential space. Then:*

(1) *For every $A \subseteq X$, $|A_W^M| \leq 2^{|A|}$ and, in particular, $|X| \leq 2^{d_W^M(X)}$;*

(2) $|X| \leq d_W^M(X)^\alpha$.

If X is a $WFU(M)$-space, we have

(3) $|X| \leq 2^{d(X)}$.

In a similar way one can prove the following result.

Theorem 4.19. *Let $\varnothing \neq M \subseteq \alpha^*$. If X is a strongly M-sequential space, then $|X| \leq 2^{d_S^M(X)}$.*

We end this section by the following result shown independently by Kočinac and Savchenko.

Theorem 4.20. ([Koc2], [Sav]) *If X is a compact strongly M-sequential space, $\varnothing \neq M \subseteq \omega^*$, then $|X| \leq 2^{c(X)}$.*

5. Mappings and sequential properties

Recall that a continuous mapping $f : X \to Y$ is *pseudo-open* if for every $y \in Y$ and for every open subset U of X with $f^{-1}(y) \subseteq U$, $y \in int(f[U])$. Call a mapping $f : X \to Y$ *M-continuous*, $M \subseteq \alpha^*$, if for every α-sequence $(x_\xi : \xi \in \alpha)$ that weakly M-converges to $x \in X$, the α-sequence $(f(x_\xi) : \xi \in \alpha)$ weakly M-converges to $f(x)$. We shall say that a mapping $f : X \to Y$ is *M-sequence covering* if whenever $(y_\xi) : \xi \in \alpha)$ weakly M-converges to a point $y \in Y$, then there are points $x_\xi \in f^{-1}(y_\alpha)$ and $x \in f^{-1}(y)$ such that $(x_\xi) : \xi \in \alpha)$ weakly M-converges to x.

The characterization of sequential and Frechét-Urysohn spaces due to S.P. Franklin [Fr] and F. Siwiec [Si] can be generalized as follows. (Some incorrectnes in papers cited below are corrected here.)

Theorem 5.1. ([Koc2], [Koc3], [Koc4]) *Let $\varnothing \neq M \subseteq \alpha^*$ and let X be a space. Then,*

(1) *X is weakly M-sequential iff there are a set I and an I-sequence $\mathcal{M} = (p_i)_{i\in I}$ of free ultrafilters on α such that $p_i \in M$ for each $i \in I$ and X is a quotient image of the space $\Xi(\mathcal{M})$;*

(1a) *X is weakly M-sequential iff every M-continuous mapping defined on X is continuous;*

(1b) *X is weakly M-sequential iff every M-sequence covering mapping $f : Y \to X$ onto X is quotient;*

(2) *X is strongly M-sequential iff there are a set I and an I-sequence $\mathcal{M} = (\mathcal{F}_i)_{i\in I}$ of free filters on α such that $\mathcal{F}_M \subseteq \mathcal{F}_i$ for each $i \in I$ and X is a quotient image of the space $\Xi(\mathcal{M})$;*

(3) *X is a $WFU(M)$-space iff there are a set I and an I-sequence $\mathcal{M} = (p_i)_{i\in I}$ of free ultrafilters on α such that $p_i \in M$ for each $i \in I$ and X is a pseudo-open image of the space $\Xi(\mathcal{M})$;*

(3a) *X is a $WFU(M)$-space iff every M-sequence covering mapping (from a space Y) onto X is pseudo-open;*

(4) *X a $SFU(M)$-space iff there are a set I and an I-sequence $\mathcal{M} = (\mathcal{F}_i)_{i\in I}$ of free filters on α such that $\mathcal{F}_M \subseteq \mathcal{F}_i$ for each $i \in I$ and X is a pseudo-open image of the space $\Xi(\mathcal{M})$.*

The cardinal function τ_{FU} introduced in Section 4 can also be characterized in terms of mappings.

If $\mathcal{F}$ is a filter on α and λ is a cardinal number, then $\Xi(\mathcal{F}, \lambda)$ will denote the space that is the topological sum of λ-many copies of the space $\xi(\mathcal{F})$. Notice that $|\,\Xi(p, \lambda)\,| = \alpha \cdot \lambda$ and $\tau(\Xi(p, \lambda)) = \alpha$, for every $p \in U(\alpha)$ and for every cardinal λ. The next result is a consequence of Theorems 4.5 and 5.1.

Corollary 5.2. *For a space X the following are equivalent.*

(1) *$\alpha = \tau_{FU}(X)$;*

(2) *there are a cardinal $\lambda \leq |X|^\alpha$ and $p \in U(\alpha)$ such that X is a quotient image of the space $\Xi(p, \lambda)$;*

(3) *there are a cardinal $\lambda \leq |X|^\alpha$ and $p \in U(\alpha)$ such that X is a pseudo-open image of the space $\Xi(p, \lambda)$.*

We should remark that closedness of projections in a topological product may be described by using properties considered in the previous sections.

Theorem 5.3. ([Koc2], [Kom1]) *Let $\varnothing \neq M \subseteq \alpha^*$. If X is a strongly M-sequential space and Y a quasi-$Cl_{\beta\alpha}M$-compact space, then the projection $\pi_X : X \times Y \to X$ is closed.*

We are going now to consider some relations between p-sequential-like properties and cleavability of topological spaces. We shall restrict our atention only to the case $\alpha = \omega$.

Definition 5.4. ([A2], [Koc5]) If $\mathcal{P}$ is a class of topological spaces and $\mathcal{M}$ is a class of continuous mappings, then a space X is said to be *$\mathcal{M}$-cleavable* (resp. *$\mathcal{M}$-pointwise cleavable*) *over $\mathcal{P}$* if for every $A \subset X$ (resp. every $x \in X$) there exist $Y \in \mathcal{P}$ and $f \in \mathcal{M}$, $f : X \to Y$, such that $f(X) = Y$ and $f^{-1}f(A) = A$ (resp. $f^{-1}f(x) = \{x\}$).

We also need the following notion.

Definition 5.5. ([GMT]) Let $p \in \omega^*$. A space X is called *p-closed* if every p-compact subspace of X is closed.

The following simple results is useful in what follows.

Lemma 5.6. *If a space X is cleavable over the class $\mathcal{K}$ of all p-closed spaces, then X is a p-closed space.*

Using the fact that p-compact p-closed spaces are precisely p-sequential spaces [GMT], from Lemma 5.6 we obtain

Theorem 5.7. ([Koc6]) *If a p-compact space X is cleavable over the class of p-closed spaces, then X is p-sequential.*

Every p-sequential space is p-closed. Therefore, we have this corollary.

Corollary 5.7'. ([Koc6]) *If a p-compact space X is cleavable over the class of p-sequential spaces, then X is p-sequential.*

The following theorem is also from [Koc6].

Theorem 5.8. *Let $p \in \omega^*$.*

(1) *If a compact space X is cleavable over the class $\mathcal{K}$ of ccc p-sequential spaces, then X is a $WFU(\omega^*)$-space.*

(2) *If a separable p-compact space X is cleavable over the class $\mathcal{K}$ of p-closed spaces, then X is a $WFU(\omega^*)$-space (and p-sequential).*

(3) *If a space X is closed pointwise cleavable over the class of $FU(p)$-spaces, then X is also a $FU(p)$-space.*

From Theorems 3.4 and 3.5 in [Be], Theorem 5.8 and the fact (2) preceding Theorem 3.2, one derives the following result.

Theorem 5.9. ([Koc6]) *If a countably compact space X is closed pointwise cleavable over the class $\mathcal{C}$ of Fréchet-Urysohn spaces, then X is ω-bounded.*

The following theorem is a special case of Theorem 23 in [A2] (which states that if a countably compact space is cleavable over the class of sequential spaces, then it is also sequential), but under a special assumption the proof is very easy and follows from our considerations.

Theorem 5.10. *Suppose that the Novak number of ω^* exceeds $\mathfrak{c}$. If an ω^*-compact space (in paricular, compact space) X is cleavable over the class $\mathcal{K}$ of sequential spaces, then X is also sequential.*

Every ω-bounded space is p-compact for every $p \in \omega^*$. A.V. Arhangel'skii has remarked that if an ω-bounded space is cleavable over the class of spaces of countable tightness, then it itself has countable tightness [A2]. So, the following question is quite natural.

Question 5.11. *Let a p-compact space X be cleavable over the class of spaces of countable tightness. Is the tightness of X countable?*

At the end of this section we give one result concerning function spaces.

For a space X, let $C_\pi(X)$ denote the space of all continuous real-valued functions on X with the pointwise topology. Sequential-like properties of $C_\pi(X)$ have been studied in [GT3], [GT4] and [T] (see also [GMT]), where some important results of Gerlitz and Nagy [GN] were generalized. Recall that a family $\mathcal{G}$ of subsets of a space X is called an *ω-cover* for X if for every finite subset A of X there is a member $G \in \mathcal{G}$ such that $A \subseteq G$.

Definition 5.12. ([GT3]) Let $p \in \omega^*$. A space X is said to have *property* γ_p if for every open ω-cover $\mathcal{G}$ of X there is a sequence $(G_n : n \in \omega) \subset \mathcal{G}$ such that $X = \bigcup_{A \in p} \bigcap_{n \in A}$.

Theorem 5.13. ([GT3]) *Let $p \in \omega^*$. A space X has γ_p iff the space $C_\pi(X)$ is an $FU(p)$-space.*

The following question remains unsolved.

Question 5.14. ([GMT]) *Is $C_\pi(X)$ an $FU(p)$-space if it is a p-sequential space?*

6. Generalizations of pseudocompactness

The characterization of pseudocompactness given in Theorem 1.6 leads to the study of the following class of spaces.

Definition 6.1. Let $\emptyset \neq M \subseteq \alpha^*$. Then, a space X is called *M-psedocompact* if for every α-sequence $(V_\xi)_{\xi<\alpha}$ of non-empty open subsets of X, there is $p \in M$ such that $L(p,(V_\xi)_{\xi<\alpha}) \neq \emptyset$.

If $p \in U(\alpha)$, we simply say *p-pseudocompact* instead of $\{p\}$-psedocompact. The concept of p-pseudocompactness for $p \in \omega^*$ was introduced by J. Ginsburg and V. Saks in [GS], and for arbitrary cardinals was considered in [G8]. Ginsburg and Saks [GS] showed that if $p \in \omega^*$ is not a P-point, then $T_{RK}(p)$ is a pseudocompact space that is not countably compact (in fact, $T_{RK}(p)$ is never countably compact for every $p \in \omega^*$). This result can be improved as follows.

Theorem 6.2. ([G4]) *If $p \in \omega^*$ is not a P-point, then there is $q \in \omega^*$ such that $T_{RK}(p)$ is q-pseudocompact.*

It is not difficult to see that p-pseudocompactness, for $p \in U(\alpha)$, is productive and preserved under surjections. But, it is not closed-hereditary:

Example 6.3. Let $p \in \omega^*$ be a non-P-point. By Theorem 6.2, we may choose $q \in \omega^*$ for which $T_{RK}(p)$ is q-pseudocompact. Since $T_{RK}(p)$ is not countably compact, there is a discrete closed subset D of $T_{RK}(p)$. Then, we have that $T_{RK}(p)$ is q-pseudocompact and D is a closed subset of $T_{RK}(p)$ that is not r-pseudocompact for any $r \in \omega^*$. □

If $p \in M \subseteq \alpha^*$, then p-compactness implies M-pseudocompactness. But, M-compactness is not in general preserved under arbitrary products (see [GJ; 9.15]). For $p \in U(\alpha)$, we also have that every p-compact space is p-pseudocompact. The next example shows that the converse does not hold.

Example 6.4. ([G6]) Let $p \in U(\alpha)$ and consider $\beta_p(\alpha)$. First, we state some properties of $T_{RK}(p)$ that we shall need:

(i) $|\beta_p(\alpha)| \leq 2^\alpha$;

(ii) ([G6; Th. 2.3], [CN2; 12.21]) $|T_{RK}(p)| = |{}^\alpha\alpha/p| > \alpha$

(iii) (see [GS; Lemma 5.1]) if $D \subseteq T_{RK}(p)$ is strongly discrete, then D does not have any accumulation point in $T_{RK}(p)$;

Now, we put $\Gamma_0 = P_{RK}(p)$. By Lemma 3.5 of [G6], there is $q \in (U(\alpha) \cap \beta_p(\alpha)) - \Gamma_0$. Notice that $T_{RK}(q) \subseteq \beta_p(\alpha) - \Gamma_0$. Assume that Γ_ν has been defined, for $\nu < \theta < \alpha^+$, so that

(1) $\Gamma_\nu \subseteq \beta_p(\alpha) - T_{RK}(q)$ for $\nu < \theta$; and

(2) if $D \in [\Gamma_\nu \cap \alpha^*]^\alpha$ is strongly discrete in $\beta(\alpha)$ and $\nu + 1 < \theta$, then $(Cl_{\beta(\alpha)}D) \cap \Gamma_{\nu+1} \neq \varnothing$ for $\nu < \theta$.

If θ is a limit ordinal, then we set $\Gamma_\theta = \bigcup_{\nu<\theta} \Gamma_\nu$. Suppose that $\theta = \nu + 1$. For each strongly discrete subset $D \in [\Gamma_\nu \cap \alpha^*]^\alpha$ we choose $r_D \in Cl_{\beta(\alpha)}D - T_{RK}(q)$ (this is possible by clause (iii)). Then, we define $\Gamma_\theta = \Gamma_\nu \cup \{r_D : D \in [\Gamma_\nu \cap \alpha^*]^\alpha$ is strongly discrete$\}$. We define $\Gamma(p) = \bigcup_{\theta<\alpha^+} \Gamma_\theta$. Notice that $\alpha \subseteq \Gamma(p) \subseteq \beta_p(\alpha)$. Since $T_{RK}(q) \subseteq \beta_p(\alpha) - \Gamma(p)$, we have that Γ_p cannot be p-compact, but it is countably compact. Let $(V_\xi)_{\xi<\alpha}$ be a sequence of non-empty clopen subsets of Γ_p. By the Disjoint Refinement Lemma (see [CN2; Lemma 7.5]), there is a set $\{A_\xi : \xi < \alpha\}$ of pairwise disjoint infinite subsets of α such that $\varnothing \neq \hat{A}_\xi \cap \Gamma(p) \subseteq V_\xi$. For each $\xi < \alpha$ we pick $f(\xi) \in A_\xi \cap \Gamma(p)$ and consider the function $f \in {}^\alpha\alpha$. Then, we have that $\bar{f}(p) \in \Gamma_0 \subseteq \Gamma_p$ and $\bar{f}(p) \in L(p, (V_\xi)_{\xi<\alpha})$. This shows that Γ_p is p-pseudocompact. □

Question 6.5. *Does there exist a countably compact space X such that X is p-pseudocompact for all $p \in \omega^*$ and X is not p-pseudocompact for any $p \in \omega^*$?*

The proof of the next result resembles the proof of Theorem 1.5 of [G5].

Theorem 6.6. *For $p, q \in \alpha^*$, the following are equivalent.*

(1) $p \leq_{RK} q$;

(2) *every q-pseudocompact space is p-pseudocompact;*

(3) *$P_{RK}(q)$ is p-pseudocompact;*

(4) *there is a partition $\{A_\xi : \xi < \alpha\}$ of α such that $q \in L(p, (\hat{A}_\xi)_{\xi<\alpha})$.*

It is evident that ω^*-pseudocompactness = pseudocompactness. But, if $p \in U(\alpha)$ is ω_1-complete, then p-compactness does not imply pseudocompactness as it is stated in the next theorem; a proof of Theorem 6.7 is available in [G8] and uses Theorem 6.6. Recall that $p \in U(\alpha)$ is *γ-complete* if $\bigcap_{\xi<\kappa} A_\xi \in p$ whenever $A_\xi \in p$ for every $\xi < \kappa$ and for every $\kappa < \gamma$.

Theorem 6.7. ([G8]) *Let $\gamma \leq \alpha$. For $p \in U(\alpha)$, the following are equivalent.*

(1) *every p-compact space is countably compact;*

(2) *every p-compact space is pseudocompact;*

(3) *$\beta_p(X)$ is pseudocompact for every space X;*

(4) *p is not ω_1-complete;*

(5) *every p-pseudocompact space is pseudocompact.*

Thus if $p \in U(\alpha)$ is ω_1-complete, then ω is p-compact and is not pseudocompact.

We saw in Theorem 3.4 that all the powers of a space X are countably compact iff there is $p \in \omega^*$ such that X is p-compact. The following example is due to Ginsburg and Saks gave and it is an example of a space all whose powers are pseudocompact and it is not p-pseudocompact for any $p \in \omega^*$ (an example that is also countably compact can be found in [G6; Ex. 3.6]).

Example 6.8. For each $p \in U(\alpha)$, let $X_p = \beta(\alpha) - \{p\}$. Then, we have that all the powers of $X = \prod_{p\in U(\alpha)} X_p$ are pseudocompact, since X_p is locally compact and pseudocompact for every $p \in U(\alpha)$. But, X_p is not q-pseudocompact for any $q \in \omega^*$. $\square$

We now turn to characterize the spaces in which all powers are pseudocompact.

Definition 6.9. ([G5]) Let $\varnothing \neq M \subseteq \alpha^*$ and let κ be a cardinal with $1 \leq \kappa$. A space X is said to be (κ, M)*-pseudocompact* if for every κ-sequence $\left((V^\xi_\zeta)_{\zeta<\alpha}\right)_{\xi<\kappa}$ of α-sequences of non-empty open subsets of X, there is $p \in M$ such that $L(p, (V^\xi_\zeta)_{\zeta<\alpha}) \neq \varnothing$ for all $\xi < \kappa$.

If $\varnothing \neq M \subseteq \alpha^*$, then $(1, M)$-pseudocompact coincides with M-pseudocompactness. The spaces such that either some finite power of it is pseudocompact or all its powers are pseudocompact are characterized in the next theorem.

Theorem 6.10. ([G5]) *Let $1 \leq \gamma \leq \omega$ and let X be a space. Then, X^γ is pseudocompact if and only if there is $\varnothing \neq M \subseteq \omega^*$ such that (γ, M)-pseudocompact.*

If $p \in \omega^*$, then we the space $\Sigma(p) = \alpha \cup T_{RK}(p)$ satisfies that all its powers are pseudocompact: this fact was shown by W.W. Comfort [C] and Z. Frolík [Fro3]. We should mention that Theorem 2.6 of [G5] which was stated as an improvement of this fact is wrong, but some of its implications are correct:

Lemma 6.11. *Let $\varnothing \neq M \subseteq \omega^*$ and X a space with $\omega \subseteq X \subseteq \beta(\omega)$. Then, X is (ω, M)-pseudocompact if and only if for every sequence $(f_n)_{n<\omega}$ in ${}^\omega\omega$ there is $p \in M$ such that $\bar{f}_n(p) \in X$ for every $n < \omega$.*

Proof. *Necessity*: Let $(f_n)_{n<\omega}$ be a sequence in ${}^\omega\omega$. Then, we have that $((\{f_n(m)\})_{m<\omega})_{n<\omega}$ is a sequence of sequences of non-empty open subsets of X. By assumption, there is $p \in M$ such that for each $n < \omega$ there is $q_n \in L(p, (\{f_n(m)\})_{m<\omega} \cap X$. Hence, we must have that $q_n = p - \lim_{m\to\infty} f_n(m)$ for each $n < \omega$ and then $\bar{f}_n(p) = q_n \in X$ for each $n < \omega$.

Sufficiency: Let $((\{\hat{A}^n_m\})_{m<\omega})_{n<\omega}$ be a sequence of sequences of non-empty basic open subsets of ω^*. For each $n < \omega$, we choose $f_n \in {}^\omega\omega$

so that $f_n(m) \in A^n_m$ for every $m < \omega$. Then, there is $p \in M$ such that $\bar{f}_n(p) \in X$ for every $n < \omega$. If $q_n = \bar{f}_n(p)$ for $n < \omega$, then we have that $q_n \in L(p, (\{\hat{A}^n_m\})_{m<\omega})$. Therefore, X is (ω, M)-pseudocompact. □

Theorem 6.12. *For $\varnothing \neq M \subseteq \omega^*$ and $p \in \omega^*$. If $T_{RK}(p) \cap M$ is dense in ω^*, then $\Sigma(p)$ is (ω, M)-pseudocompact.*

Proof. We apply Lemma 6.11. Let $(f_n)_{n<\omega}$ be a sequence in ${}^\omega\omega$. By induction we may choose $A_n \in [\omega]^\omega$ so that

(1) $A_{n+1} \subseteq A_n$ for every $n < \omega$;

(2) if $n < \omega$, then either $f_n(A_n)$ is singleton or $f_n|_{A_n}$ is one-to-one.

Since ω^* is an almost P-space, the set $\bigcap_{n<\omega} \hat{A}_n$ has non-empty interior. Hence, $\varnothing \neq \bigcap_{n<\omega} \hat{A}_n \cap T_{RK}(p) \cap T_{RK}(M)$. Pick $q \in \bigcap_{n<\omega} \hat{A}_n \cap T_{RK}(p) \cap T_{RK}(M)$ and put $\bar{f}_n(p) = p_n$ for each $n < \omega$. Thus, if $n < \omega$, then we have that either $p_n \in \omega \subseteq \Sigma(p)$ or $p_n \in T_{RK}(p) \subseteq \Sigma(p)$, as required. □

For $\varnothing \neq M \subseteq \omega^*$, we let $T_{RK}(M) = \bigcup_{p\in M} T_{RK}(p)$.

Theorem 6.13. *For $\varnothing \neq M \subseteq \omega^*$ and $p \in \omega^*$. If $\Sigma(p)$ is (ω, M)-pseudocompact, then $T_{RK}(p) \cap T_{RK}(M)$ is dense in ω^*.*

Proof. Suppose that $\Sigma(p)$ is (ω, M)-pseudocompact. Let $A \in [\omega]^\omega$ which is enumerated as $\{a_n : n < \omega\}$. Then there are $q \in M$ and $r \in T_{RK}(p)$ such that $r = q - \lim a_n$. Hence, $r \approx_{RK} q$ and $r \in A^*$. So, $r \in A^* \cap T_{RK}(p) \cap T_{RK}(M)$. □

Corollary 6.14. *Let $\varnothing \neq M \subseteq \omega^*$ be such that $T_{RK}(M) = M$ and $p \in \omega^*$. Then, the following are equivalent.*

(1) *$T_{RK}(p) \cap M$ is dense in ω^*;*

(2) *$\Sigma(p)$ is (ω, M)-pseudocompact.*

It is a consequence of Corollary 6.14 that $\Sigma(p)$ is $T(p)_{RK}$-pseudocompact for every $p \in \omega^*$. The only possibility for $\Sigma(p)$ to be p-pseudocompact is stated in the next theorem. We need a lemma which is a direct application of Lemma 6.11.

Lemma 6.15. ([G5]) *Let $\omega \neq X \subseteq \beta(\omega)$. Then, X is p-pseudocompact if and only if $P_{RK}(p) \subseteq X$.*

Theorem 6.16. ([G5]) *For $p \in \omega^*$, the following are equivalent.*

(1) *$\Sigma(p)$ is p-pseudocompact;*

(2) *$\Sigma(p)$ is q-pseudocompact for some $q \in \omega^*$;*

(3) *p is RK-minimal.*

Thus, if $p \in \omega^*$ is not RK-minimal, then $\Sigma(p)$ is $T_{RK}(p)$-pseudocompact and is not p-pseudocompact. Next, we give an example of a space that is $T_{RK}(p)$-pseudocompact and is not p-pseudocompact without requiring the RK-minimal property.

Theorem 6.17. *For $p \in \omega^*$, the space $\Sigma(p) \setminus \{p\}$ is $T_{RK}(p)$-pseudocompact and it is not p-compact.*

Proof. Put $S_p = \Sigma(p) \setminus \{p\}$. By Lemma 6.15, we have that S_p cannot be p-pseudocompact. Let $(A_n)_{n<\omega}$ be a sequence of non-empty subsets of ω. Let $f \in {}^{\omega}$ such that $f(n) \in A_n$ for every $n < \omega$. If there is $A \in [\omega]^\omega$ such that $f|_A$ is constant and $A \notin p$, then we choose $q \in A^* \cap T_{RK}(p)$ and then we have that $q \in S_p$ and $q \in L(p, (\hat{A}_n \cap S_p)_{n<\omega})$. If this is not the case, then we may find $B \in [\omega]^\omega$ so that $B \notin p$ and $f|_B$ is one-to-one. Hence, if $r \in B^* \cap T_{RK}(p)$, then $r \in S_p$ and $r \in L(p, (\hat{A}_n \cap S_p)_{n<\omega})$. $\square$

We notice that if $p, q \in \omega^*$ are RK-minimal, then $P_{RK}(p) \cap P_{RK}(q) = \omega$ and hence $P_{RK}(p) \times P_{RK}(q)$ is not pseudocompact, but $P_{RK}(p)$ is p-pseudocompact and $P_{RK}(q)$ is q-pseudocompact (by Lemma 6.15). In a model M of NCF, we have that $M \models \omega^*$ *has not* $P-$*points* and so RK-minimal points do not exist in M (see [BS]). Hence, in this model M, if X is p-pseudocompact and Y is q-pseudocompact for $p, q \in \omega^*$, then there is $r \in P_{RK}(p) \cap P_{RK}(q) \cap \omega^*$ such that $X \times Y$ is pseudocompact.

Some generalizations of bisequential, biradial and absolutely countably compact spaces the authors will publish somewhere else.

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