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WEIGHTED WEAK DRAZIN INVERSES

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ABSTRACT: We extend weak Drazin inverses of square matrices to rectangular, and investigate some properties of these inverses. Applications to systems of matrix equations are also given.

1. Introduction and preliminaries

For every square complex matrix B there is a unique solution of the system in X:

$$B^{p+1}X = B^p, \quad XBX = X, \quad BX = XB$$

for $p \geq \operatorname{ind} B$ ($k = \operatorname{ind} B$ is the smallest nonnegative integer such that $\operatorname{rank}(B^k) = \operatorname{rank}(B^{k+1})$). The unique solution X is called the Drazin pseudoinverse (or simply Drazin inverse) of B and is denoted by B^D [1].

The generalization of the Drazin inverse for rectangular matrices has been introduced by R.E. Cline and T.N.E. Greville [4] as the unique solution of the system in X:

$$(BW)^{k+1}XW = (BW)^k, \; XWBWX = X, \; BWX = XWB.$$

This unique solution is called W-weighted Drazin inverse of B.

The concept of a weak Drazin inverse was essentially introduced by S.L.Campbell and C.D.Meyer Jr.[2] (see also [3]) as the solution for $B^{p+1}X = B^p$ and such a solution was called a weak Drazin inverse.

Following the idea of W-weighted Drazin inverse of

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rectangular matrices we define and investigate so called weighted weak Drazin inverses.

DEFINITION 1. Let $B\in C^{m\times n}$, $W\in C^{n\times m}$ and $k=\mathrm{ind}(BW)$.

(i) $X\in C^{m\times n}$ is a weighted right weak Drazin inverse of B if it satisfies

$$(1)\qquad (BW)^{p+1}XW = (BW)^{p}\ ,\ p\geq k\ .$$

(ii) $X\in C^{m\times n}$ is a weighted left weak Drazin inverse of B if it satisfies

$$(2)\qquad XW(BW)^{q+1} = (BW)^{q}\ ,\ q\geq k\ .$$

The set of all weighted right (left) weak Drazin inverses of B is denoted by $(B:W)_{d,r}$ $((B:W)_{d,\ell})$.

Recall that any element of the set of matrices $\{X\,|\,AXA=A\}$ is called a $\{1\}$-inverse of A and is denoted $A^{(1)}$.

The following fact ([6],p.22) we shall use later:

$$(3)\qquad AB(AB)^{(1)}A = A \iff \mathrm{rank}\ (AB) = \mathrm{rank}\ (A).$$

Also, Penrose [5] proved that the equation

$$AXB = C$$

is consistent if and only if

$$(4)\qquad AA^{(1)}CB^{(1)}B = C$$

and the general solution is then given by

$$X = A^{(1)}CB^{(1)} + Y - A^{(1)}AYBB^{(1)}\ ,$$

where Y is arbitrary.

2. Results

THEOREM 1. *Suppose that* $B\in C^{m\times n}$, $W\in C^{n\times m}$ *and* $k=\mathrm{ind}(BW)$. *Then, for positive integers* p *and* q *such that* $p,q\geq k$, *the general solutions of* (1), (2) *and of the sistem* (1)∧(2) *are respectively given by*

$$(5)\qquad X = (BW)^{D}W^{(1)} + Y - (BW)^{D}BWYWW^{(1)}\ ,$$

$$(6)\qquad X = (BW)^{D}W^{(1)} + U - UWBW(BW)^{D}W^{(1)}$$

and

(7) $$X = Z+((I-ZWBW)(BW)^{D}-(BW)^{D}BWZW(I-BW(BW)^{D}))W^{(1)}$$

where Y, U and Z are arbitrary matrices.

PROOF: By (4), the equation (1) is consistent if and only if

$$(BW)^{p+1}((BW)^{p+1})^{(1)}(BW)^{p}W^{(1)}W = (BW)^{p},$$

i.e.

$$(BW)^{p+1}((BW)^{p+1})^{(1)}(BW)^{p} = (BW)^{p}.$$

This condition is satisfied if and only if (see (3)) $\mathrm{rank}((BW)^{p+1}) = \mathrm{rank}((BW)^{p})$, but this holds since $p \geq \mathrm{ind}(BW)$. Now the general solution of (1) is given by

(8) $$X = ((BW)^{p+1})^{(1)}(BW)^{p}W^{(1)}+Y-((BW)^{p+1})^{(1)}(BW)^{p+1}YWW^{(1)},$$

where Y is an arbitrary matrix.

Since $((BW)^{D})^{p+1}$ is an $\{1\}$-inverse of $(BW)^{p+1}$, (8) can be written as

$$X = ((BW)^{D})^{p+1}(BW)^{p}W^{(1)}+Y-((BW)^{D})^{p+1}(BW)^{p+1}YWW^{(1)},$$

and finally, in the form (5).

The general solution of (2) can be obtained in a similar way, and we omit the proof.

In order to solve the system of equations (1) and (2), we substitute the general solution of (1) into (2) and obtain the following equation in Y:

(9) $$(I-(BW)^{D}BW)YW(BW)^{q+1} = 0$$

The general solution of (9) can be written in the form

(10) $$Y = Z-(I-(BW)^{D}BW)ZWBW(BW)^{D}W^{(1)},$$

where Z is an arbitrary matrix.

Substituting (10) into (5) we get (7). It is easily verified that (7) salisfies (1) and (2), which means that (7) is the general solution of the system (1) and (2). This completes the proof.

and

$$(7) \qquad X = Z+((I-ZWBW)(BW)^{D}-(BW)^{D}BWZW(I-BW(BW)^{D}))W^{(1)}$$

where Y, U and Z are arbitrary matrices.

PROOF: By (4), the equation (1) is consistent if and only if

$$(BW)^{p+1}((BW)^{p+1})^{(1)}(BW)^{p}W^{(1)}W = (BW)^{p},$$

i.e.

$$(BW)^{p+1}((BW)^{p+1})^{(1)}(BW)^{p} = (BW)^{p}.$$

This condition is satisfied if and only if (see (3)) $\mathrm{rank}((BW)^{p+1}) = \mathrm{rank}((BW)^{p})$, but this holds since $p \geq \mathrm{ind}(BW)$. Now the general solution of (1) is given by

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and finally, in the form (5).

The general solution of (2) can be obtained in a similar way, and we omit the proof.

In order to solve the system of equations (1) and (2), we substitute the general solution of (1) into (2) and obtain the following equation in Y:

$$(9) \qquad (I-(BW)^{D}BW)YW(BW)^{q+1} = 0$$

The general solution of (9) can be written in the form

$$(10) \qquad Y = Z-(I-(BW)^{D}BW)ZWBW(BW)^{D}W^{(1)},$$

where Z is an arbitrary matrix.

Substituting (10) into (5) we get (7). It is easily verified that (7) salisfies (1) and (2), which means that (7) is the general solution of the system (1) and (2). This completes the proof.

THEOREM 2. *Let* $B\in C^{m\times n}$, $W\in C^{n\times m}$, $k=\text{ind}(BW)$, $X\in(B:W)_{d,r}$, $Y\in(B:W)_{d,\ell}$, $Z\in(B:W)_{d,r}\cap(B:W)_{d,\ell}$ *and* $\ell\in\mathbb{N}$. *Then for each* $m\geq k$ *we have*

a) $$(BW)^{\ell}(XW)^{n}=(YW)^{n}(BW)^{\ell}=(BW)^{\ell}(ZW)^{n}=$$
$$=(ZW)^{n}(BW)^{\ell}=(BW)^{\ell}((BW)^{D})^{n}, \text{ for } n\in\mathbb{N};$$

b) $$(BW)^{\ell}(XW)^{\ell+1}=(YW)^{\ell+1}(BW)^{\ell}=(BW)^{\ell}(ZW)^{\ell+1}=(ZW)^{\ell+1}(BW)^{\ell}=$$
$$=YW(BW)^{\ell}(XW)^{\ell}=(YW)^{\ell}(BW)^{\ell}XW=$$
$$=(YW)^{\ell}(BW)^{2\ell-1}(XW)^{\ell}=(BW)^{D};$$

c) $$YWBW(BW)^{D}=(BW)^{D}BWXW=(BW)^{D}BWZWBW(BW)^{D}=(BW)^{D};$$

d) $$YW((BW)^{D})^{n}=((BW)^{D})^{n}XW=ZW((BW)^{D})^{n}=((BW)^{D})^{n}ZW=$$
$$=((BW)^{D})^{n+1}, \text{ for } n\in\mathbb{N}.$$

PROOF. a) For example, the relation

$$(BW)^{\ell}(XW)^{n} = (BW)^{\ell}((BW)^{D})^{n}$$

can be proved by mathematical induction in n. Note that the relation

$$(BW)^{\ell}XW = (BW)^{\ell}(BW)^{D}$$

follows from (5), i.e. X can be replaced by the general solution (5). The other relations can be proved in a similar way.

Part b) follows directly from a); c) follows from general solutions (5), (6) and (7); finally, d) can be proved by mathematical induction.

THEOREM 3. *Let* $B\in C^{m\times n}$, $W\in C^{n\times m}$ *and* $\text{ind}(BW)=k$. *Then*:

a) *for* $p\geq k$, *the general solution of the system*

(1) $$(BW)^{p+1}XW = (BW)^{p}$$

(11) $$BWXWBW = BW$$

is given by

(12) $$X = (BW)^{D}(I-BWZW)W^{(1)}+Z+(BW)^{(1)}(I-BW(BW)^{D})(I-BWZW)BW(WBW)^{(1)},$$

where Z *is an arbitrary matrix*;

b) *for* $q\geq k$, *the general solution of the system*

(2) $$XW(BW)^{q+1}=(BW)^{q}$$

(11) $$BWXWBW = BW$$

is given by

(13) $$X=(I-UWBW)(BW)^{D}W^{(1)}+U+(BW)^{(1)}(I-BWUW)(I-BW(BW)^{D})BW(WBW)^{(1)},$$

where U *is an arbitrary matrix.*

PROOF. a) By Theorem 1. we have that (5) is the general solution of (1). Substituting (5) into the equation (11), we obtain the following equation in Y:

(14) $$(I-BW(BW)^{D})BWYWBW = (I-BW(BW)^{D})BW.$$

Because of (11) we have that $\operatorname{rank}(WBW)=\operatorname{rank}(BW)$, so that the equation (14) is consistent. The general solution of (14) is

(15) $$Y = Z+(BW)^{(1)}(I-BW(BW)^{D})BW(I-ZWBW)(WBW)^{(1)},$$

where Z is an arbitrary matrix.

Substituting (15) into (5) we get (12). Now (12) satisfies (1) and (11), and it is the general solution of this system.

b) Can be proved in a similar way as a).

This completes the proof of the theorem.

THEOREM 4. *Let* $B\in C^{m\times n}$, $W\in C^{n\times m}$, $\operatorname{ind}(BW)=k$ *and* $p,q\geq k$. *Then the general solution of the system*

(1) $$(BW)^{p+1}XW = (BW)^{p}$$

(2) $$XW(BW)^{q+1} = (BW)^{q}$$

(11) $$BWXWBW = BW$$

is given by

(16) $$X=T+((I-TWBW)(BW)^{D}-(BW)^{D}BWTW(I-BW(BW)^{D}))W^{(1)}+$$
$$+(BW)^{(1)}(I-BW(BW)^{D})(I-BWTW(I-BW(BW)^{D}))BW(WBW)^{(1)},$$

where T *is an arbitrary matrix.*

PROOF. From Theorem 1. it follows that (7) is the general solution of the system (1)$\wedge$(2). The equation in Z

(17) $$(I-BW(BW)^{D})BWZWBW(I-BW(BW)^{D})=(I-BW(BW)^{D})BW$$

is obtained when we substitute (7) into (11). It can be proved that the equation (17) is consistent. Now the general solution of (17) is

$$Z = (BW)^{(1)}(I-BW(BW)^{D})BW(WBW)^{(1)}+T-(BW)^{(1)}\cdot$$
$$\cdot(I-BW(BW)^{D})BWTWBW(I-BW(BW)^{D})(WBW)^{(1)},$$

where T is an arbitrary matrix. We substitute Z into (7) to obtain (16).

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TEŽINSKI SLABI DRAZINOVI INVERZI

Definicija slabog Drazinovog inverza kvadratne matrice proširena je i na pravougaone matrice. Ispitane su neke osobine ovog inverza i data je primena na rešavanje nekih sistema matričnih jednačina.

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