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ON THE COMPATIBILITY OF THE NATURAL PARTIAL ORDER  
ON AN  $r$ -CANCELATIVE  $r$ -SEMIGROUP

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**Abstract.** In paper [9] we define the natural partial order on an  $r$ -cancelative semigroup. In this paper we introduce the notions of locally  $\mathcal{Z}^*$ - ( $\mathcal{R}^*$ -) unipotent semigroup and consider the compatibility of the natural partial order on it. In this way we obtain a generalization of results of Blyth and Gomes,[1].

1. Preliminary results and definitions

A semigroup  $S$  is  $\pi$ -regular if for every  $a \in S$  there exists a positive integer  $m$  such that  $a^m \in a^m Sa^m$ . We shall denote by  $\text{Reg } S$  the set of all regular elements of  $S$ . An element  $a'$  is an inverse for  $a$  if  $a=aa'a$  and  $a'=a'aa'$ . As usually we shall denote by  $V(a)$  the set of all inverses of  $a$ . If  $A$  is a subset of  $S$  then by  $E(A)$  we denote the set of all idempotents of  $A$ . In [8] is defined a mapping  $r:S \rightarrow \text{Reg } S$  with  $r(a)=a^m$  where  $m$  is the smallest positive integer such that  $a^m \in \text{Reg } S$ . Define on a  $\pi$ -regular semigroup  $S$  an equivalence  $\mathcal{Z}^*$ ,  $\mathcal{R}^*$  and  $\mathcal{H}^*$  by

$$a\mathcal{Z}^*b \iff Sr(a)=Sr(b), \quad a\mathcal{R}^*b \iff r(a)S=r(b)S, \quad \mathcal{H}^*=\mathcal{Z}^*\cap\mathcal{R}^*,$$

[4]. Recall that we can define a partial order on the  $\mathcal{Z}^*$ - ( $\mathcal{R}^*$ -) classes by

$$L_a^* \leq L_b^* \iff Sr(a) \subseteq Sr(b) \quad (R_a^* \leq R_b^* \iff r(a)S \subseteq r(b)S).$$

1.1.PROPOSITION [4] Let  $S$  be a  $\mathcal{R}$ -regular semigroup, then

- (i)  $(a,b) \in \mathcal{Z}^* \iff (\exists a' \in V(r(a))) (\exists b' \in V(r(b))) r(a)a' = r(b)b'$ ;
- (ii)  $(a,b) \in \mathcal{Z}^* \iff (\exists a' \in V(r(a))) (\exists b' \in V(r(b))) a'r(a) = b'r(b)$ .

By  $\mathcal{R}$ ,  $\mathcal{L}$  and  $\mathcal{Z}$  we denote the known Greens relations.

A  $\mathcal{R}$ -regular semigroup  $S$  is an  $r$ -semigroup if

$$(\forall a, b \in S) r(ab) = r(a)r(b),$$

[8]. If  $S$  is an  $r$ -semigroup then  $RegS$  is a subsemigroup of  $S$ , [8]. A  $\mathcal{R}$ -regular semigroup  $S$  is an  $r$ -cancelative semigroup if

$$(\forall a, b \in S - RegS) r(a) = r(b) \implies a = b,$$

[9].

1.1.THEOREM [9] Let  $S$  be an  $r$ -cancelative semigroup. For  $a, b \in S$  we define

$$a \leq b \iff R_a \leq R_b \wedge (\forall b' \in V(r(b))) r(a) = r(a)b' \quad r(a) = r(a)b' \quad r(b) = r(b)b'r(a).$$

Then the relation  $\leq$  is a (natural) partial order on  $S$ .

1.1.THEOREM [9] On an  $r$ -cancelative semigroup  $S$  the following statements are equivalent:

- (i)  $a \leq b$  ;
- (ii)  $R_a \leq R_b \wedge (\exists e, f \in E(S)) r(a) = er(b) = r(b)f$ .

If  $S$  is a semigroup and  $E(S) \neq \emptyset$  then relation

$$(\forall e, f \in E(S)) e \leq f \iff e = ef = fe$$

is a natural partial order on  $E(S)$  and denoted by  $\leq_n$ . The natural partial order  $\leq$  on an  $r$ -cancelative semigroup  $S$  is an extension of the natural partial order  $\leq_n$  with  $E(S)$  on  $S$ , [9].

1.2.PROPOSITION Let  $T$  be a  $\mathcal{R}$ -regular subsemigroup of a  $\mathcal{R}$ -regular semigroup  $S$ , then  $\mathcal{Z}_T^* = \mathcal{Z}_S^* \cap T \times T$ .

PROOF. Let  $(a, b) \in \mathcal{Z}_T^*$ , then there exist  $x, y \in T \cap S$  such that  $r(a) = xr(b)$  and  $r(b) = yr(a)$ . Hence,  $(a, b) \in \mathcal{Z}_S^* \cap T \times T$ .

Conversely, let  $(a, b) \in \mathcal{Z}_S^* \cap T \times T$  and  $a \in V(r(a)) \cap T$ ,  $b \in V(r(b)) \cap T$ . Then

$$(a'r(a), a), (a, b), (b, b'r(b)) \in \mathcal{L}_S^*$$

and so  $(a'r(a), b'r(b)) \in \mathcal{L}_S^*$ . Now there exist  $x, y \in S$  such that

$$a'r(a) = xb'r(b) \quad \text{and} \quad b'r(b) = ya'r(a) ,$$

whence it follows that

$$a'r(a) \cdot b'r(b) = a'r(a) \quad \text{and} \quad b'r(b) \cdot a'r(a) = b'r(b) .$$

Since  $a'r(a), b'r(b) \in T$  we have  $(a'r(a), b'r(b)) \in \mathcal{L}_T^*$ . Hence,

$$(a, a'r(a)), (a'r(a), b'r(b)), (b'r(b), b) \in \mathcal{L}_T^*$$

and so  $(a, b) \in \mathcal{L}_T^*$ .

We refer to [3,5] for nondefined notions.

## 2. The compatibility

If  $S$  is a regular semigroup and  $e, f \in E(S)$  then the sandwich set  $S(e, f)$ , introduced by Nambooripad in [7], can variously be defined as follows:

$$(1) \quad S(e, f) = f \cdot V(ef) \cdot e = E(S) \cap V(ef) \cap fSe \\ = \{g \in E(S) : ge = g = fg, egf = ef\}.$$

Note that  $S(e, f) \neq \emptyset$  since it contains  $fxe$  for every  $x \in V(ef)$ .

If  $S$  is a  $\mathbb{K}$ -regular semigroup and  $e, f \in E(S)$ , then in general case  $ef \notin \text{Reg } S$  so  $V(ef) \neq \emptyset$  and  $S(ef)$  is empty.

2.1.LEMMA Let  $S$  be a  $\mathbb{K}$ -regular semigroup,  $e, f \in E(S)$  and  $\text{Reg } S$  is a subsemigroup of  $S$ . Then

- (i)  $S(e, f) \neq \emptyset$ .
- (ii)  $S(e, f)$  is a rectangular band.
- (iii)  $(\forall a \in V(r(a))) (\forall b \in V(r(b))) (\forall g \in S(a'r(a), r(b)b')) b'ga \in V(r(a)r(b))$ .
- (iv)  $\begin{cases} e \mathcal{R}^* f \implies (\forall g \in E(S)) S(e, g) = S(f, g) \\ e \mathcal{L}^* f \implies (\forall g \in E(S)) S(g, e) = S(g, f) \end{cases}$

PROOF. As in paper [1].

2.2.LEMMA Let  $S$  be an  $r$ -cancelative semigroup,  $a, b \in S$ ,  $a \leq b$  and  $b \in V(r(b))$ . Then there exist  $a', a'' \in V(r(a))$  such that

$$(2) \quad r(a)b' = r(a)a' \leq_n r(b)b' \quad \text{and} \quad r(a) = r(a)a'r(b),$$

$$(3) \quad b'r(a) = a''r(a) \leq_n b'r(b) \quad \text{and} \quad r(a) = r(b)a''r(a).$$

PROOF. Let  $a \leq b$ , then for each  $b' \in V(r(b))$

$$r(a) = r(a)b'r(a) = r(a)b'r(b) = r(b)b'r(a).$$

Then  $R_a^* = R_{r(a)}^* = R_{r(a)b'}^* r(a) \leq R_{r(a)}^* b' \leq R_{r(a)}^*$  implies  $R_{r(a)b'}^* = R_{r(a)}^*$  and so  $r(a)b' \in R_{r(a)}^*$ . Clearly  $r(a)b' \in E(S) \cap V(r(a)b')$ . From  $r(a)b' \not\leq r(a)$ , by 1.1. Proposition, there exists  $a' \in V(r(a))$  such that  $r(a)b' = r(a)a'$ . Now

$$r(a)b'r(b)b' = r(a)b', \quad r(b)b'r(a)b' = r(a)b'$$

and so  $r(a)b' = r(a)a' \leq_n r(b)b'$ . Also,  $r(a)a'r(b) = r(a)b'r(b) = r(a)$ . In the similar way we prove (3).

2.3. LEMMA Let  $S$  be a  $\mathcal{K}$ -regular semigroup, then  $eSe$  is a  $\mathcal{K}$ -regular submonoid of  $S$  for all  $e \in E(S)$ .

PROOF. Clearly  $eSe$  is a submonoid of  $S$ . Let  $a \in eSe$ , then  $a = exe$  for some  $x \in S$  and there exist a positive integer  $m$  and  $t \in S$  such that  $a^m = ata^m t$ . Now

$$a^m = a^m t a^m = a^{m-1} a t a^{m-1} = a^{m-1} e x e \cdot e t e \cdot e x e a^{m-1} = a^m \cdot e z e \cdot a^m.$$

Since  $eze \in eSe$  it follows that  $eSe$  is a  $\mathcal{K}$ -regular submonoid of  $S$ .

We shall say that  $eSe$  is a local submonoid of  $S$  for every  $e \in E(S)$ .

2.1. DEFINITION A  $\mathcal{K}$ -regular semigroup  $S$  is said to be  $\mathcal{Z}^*$ -unipotent ( $\mathcal{R}^*$ -unipotent) if every  $\mathcal{Z}^*$ -class ( $\mathcal{R}^*$ -class) of  $S$  contains a unique idempotent.

2.2. DEFINITION A  $\mathcal{K}$ -regular semigroup  $S$  is said to be locally  $\mathcal{Z}^*$ -unipotent ( $\mathcal{R}^*$ -unipotent) if every local submonoid of  $S$  is  $\mathcal{Z}^*$ -unipotent ( $\mathcal{R}^*$ -unipotent).

2.1. EXAMPLE A semigroup  $S = \{1, 2, 3, 4, 5\}$  given by the table

	1	2	3	4	5
1	2	3	1	1	2
2	3	1	2	2	3
3	1	2	3	3	1
4	1	2	3	4	1
5	2	3	1	1	2

is a local  $\mathcal{R}$ -unipotent semigroup. This semigroup satisfies the condition

$$(4) \quad (\forall a, b \in S) \quad R_a \leq R_b \iff R_{ac} \leq R_{bc}.$$

Also,  $S$  is an  $r$ -cancelative  $r$ -semigroup.

2.1.THEOREM Let  $S$  be an  $r$ -cancelative  $r$ -semigroup. Then the following statements are equivalent:

(i)  $S$  is locally  $\mathcal{R}$ -unipotent and satisfies the condition (4) ;

(ii) The natural partial order  $\leq$  is compatible on the right with multiplication ;

(iii) For every  $e, f \in E(S)$  the set  $S(e, f)$  is right-zero band and  $S$  satisfies the condition (4).

PROOF. (i)  $\implies$  (ii) Let  $a, b, c \in S$  and  $a \leq b$ . Then by (2) there exists  $a' \in V(r(a))$  such that

$$(5) \quad r(a) = r(a)a'r(b).$$

Also,  $a'r(a) \cdot r(c)c' \in \text{Reg } S$  since  $S$  is an  $r$ -semigroup and so  $S(a'r(a), r(c)c') \neq \emptyset$ . If  $c' \in V(r(c))$  and  $g \in S(a'r(a), r(c)c')$  then we have

$$\begin{aligned} r(ac) \cdot c'ga' \cdot r(bc) &= r(a)r(c)c'g \cdot a' \cdot r(b)r(c) = r(a)ga'r(b)r(c) \\ &= r(a)ga'r(a)a'r(b)r(c) = r(a)ga'r(a)r(c) = r(a)gr(c) \\ &= r(a)a'r(a)gr(c)c'r(c) = r(a)r(c) = r(ac) \end{aligned}$$

and so

$$(6) \quad r(ac) = er(bc),$$

where  $e = r(ac)c'ga' \in E(S)$  since  $c'ga' \in V(r(ac))$  by 2.1.Lemma (iv). Now let  $b' \in V(r(b))$  and choose  $a'' \in V(r(a))$  such that, by (3),

$$(7) \quad a''r(a) = a''r(a)b'r(b) = b'r(b)a''r(a), \quad r(a) = r(b)a''r(a).$$

Choose  $c'' \in V(r(c))$  and  $h \in S(a''r(a), r(c)c'')$ . Then we note

first that  $a''r(a)h, b'r(b)h \in E(S)$ . Since

$$\begin{aligned} a''r(a)h &= b'r(b)a''r(a)h = b'r(b)a''r(a)ha''r(a) \\ &= b'r(b)a''r(a)ha''r(a)b'r(b) \end{aligned}$$

and

$$b'r(b)h = b'r(b)ha''r(a) = b'r(b)ha''r(a)b'r(b),$$

we have that  $a''r(a)h$  and  $b'r(b)h$  are idempotents in  $b'r(b)Sb'r(b)$ . Since

$$a''r(a)h \cdot b'r(b)h = a''r(a)ha''r(a)b'r(b)h = a''r(a)ha''r(a)h = a''r(a)h$$

and

$$b'r(b)ha''r(a)h = b'r(b)h \cdot h = b'r(b)h,$$

we have that  $a''r(a)h$  and  $b'r(b)h$  are  $\alpha^*$ -related idempotents in the local submonoid  $b'r(b)Sb'r(b)$ . By 2.2.Definition we have  $a''r(a)h = b'r(b)h$ . Consequently

$$(8) \quad r(bc)c''ha''r(ac) = r(b)r(c)c''ha''r(a)r(c) = r(b)hr(c)$$

$$= r(b)b'r(b)hr(c) = r(b)a''r(a)hr(c) = r(a)hr(c) = r(a)r(c) = r(ac)$$

so that  $r(ac) = r(bc)f$  where  $f = c''ha''r(ac) \in E(S)$  since  $c''ha'' \in V(r(ac))$ . Since  $a \leq b$  implies  $R_a \leq R_b$  and  $S$  satisfies the statement (4), we have  $R_{ac} \leq R_{bc}$ . By (6),(8) and 1.2.Theorem (iv) we have  $ac \leq bc$  and  $\leq$  is right compatibility.

(ii)  $\Rightarrow$  (iii) Since  $S$  is  $r$ -semigroup then  $RegS$  is a subsemigroup of  $S$ . Hence, if  $e, f \in E(S)$  then by 2.1.Lemma  $S(e, f) \neq \emptyset$ . Let  $g, h \in S(e, f)$ , then  $gf \cdot gf = ggf = gf$  (by (1)) and so  $gf \in E(S)$ . Now  $gf = gf \cdot f = f \cdot gf$  and so  $gf \leq_n f$ . Because  $\leq|_{E(S)} = \leq_n$  and since  $\leq$  is right compatibility, we have  $gh = gfh \leq_n fh = h$ . By 2.1.Lemma (ii)  $S(e, f)$  is a rectangular band and so  $gh = hgh = h$ . Thus  $S(e, f)$  is a right zero band.

Condition (4) holds since  $\leq$  is right compatible.

(iii)  $\Rightarrow$  (i) If  $e \in E(S)$ ,  $a \in eSe$ ,  $f, g \in E(S) \cap eSe$  and  $f, g \in L_a^*$  in  $eSe$ , then by 1.1.Proposition (ii) there exist  $a', a'' \in V(r(a)) \cap eSe$  such that  $f = a'r(a)$ ,  $g = a''r(a)$ . By 1.2.Proposition it follows that  $e \alpha^* f$  in  $S$  and by 2.1.Lemma (iv) we have  $S(f, e) = S(g, e)$ . Also,  $f, g \in S(f, e) = S(g, e)$  and  $f = fg = g$  since  $S(f, e)$  is a right zero band. Hence,  $S$  is a locally  $\alpha^*$ -unipotent semigroup.

2.1.COROLLARY Let  $S$  be an  $r$ -cancelative  $r$ -semigroup. Then the following statements are equivalent:

- (i)  $S$  is locally  $\mathcal{R}^*$ -unipotent ;
- (ii) The natural partial order  $\leq$  is compatible on the left with multiplication ;
- (iii) For every  $e, f \in E(S)$  the set  $S(e, f)$  is left-zero band.

PROOF. On arbitrary semigroup holds

$$(\forall a, b \in S) R_a \leq R_b \implies R_{ca} \leq R_{cb} .$$

2.3.DEFINITION A  $\mathcal{K}$ -regular semigroup  $S$  is said to be locally  $\mathcal{K}$ -inverse if every local submonoid is  $\mathcal{K}$ -inverse semigroup.

2.3.LEMMA A  $\mathcal{K}$ -regular semigroup  $S$  is a locally  $\mathcal{K}$ -inverse if and only if for each  $e \in E(S)$  submonoid  $eSe$  is  $\mathcal{K}^*$ -unipotent and  $\mathcal{R}^*$ -unipotent.

PROOF. By Theorem 4.6.[4] and Theorem 1.[2].

2.2.THEOREM Let  $S$  be an  $r$ -cancelative  $r$ -semigroup. Then the following statements are equivalent:

- (i)  $S$  is locally  $\mathcal{K}$ -inverse and satisfies the condition (4) ;
- (ii) The natural partial order  $\leq$  is compatible with multiplication ;
- (iii) For each  $e, f \in E(S)$  the set  $S(e, f)$  is a singleton and  $S$  satisfies the condition (4).

PROOF. By 2.1.Theorem, 2.1.Corollary and 2.3.Lemma.

The idea of the next theorem which closely determines the structure of an  $r$ -cancelative  $r$ -semigroup was given by D. Blagojević.

2.3.THEOREM Let  $S$  be an  $r$ -cancelative  $r$ -semigroup and  $a \in S - \text{Reg } S$ , then  $|\langle a \rangle| = 2$  or  $|\langle a \rangle| = 4$ .

PROOF. Let  $a \in S\text{-Reg}S$ , then  $r(a)=a^m$  for some positive integer  $m$  and  $m \geq 2$ . Then  $a, a^2, \dots, a^{m-1} \in S\text{-Reg}S$ . Let  $a^i=a^j$  for some  $1 \leq i < j \leq m-1$ , then  $j=i+k$  and  $1 \leq k \leq m-i-1$ . Now  $a^i=a^j=a^{i+k}=a^{i+2k}=\dots=a^{i+ik}=a^{i(l+k)}$ . If  $k=1$ , then  $a^i=a^{2i}$  and  $a^i \in E(S)$ . If  $k>1$  then  $k=p+1$  and

$$a^i=a^{i(l+p+1)}=a^i a^p a^i$$

and so  $a^i \in \text{Reg}S$ . In both cases  $a^i \in \text{Reg}S$  what is a contradiction. Hence

$$(9) \quad a^i \neq a^j \quad (1 \leq i < j \leq m-1).$$

Now we prove that  $r(a^i)=a^{m+i-1}$  ( $i=2, 3, \dots, m-1$ ). Clearly, since  $S$  is an  $r$ -semigroup we have  $r(a^i)=(r(a))^i=a^{mi}$  and  $r(a^i)=r(a \cdot a^{i-1})=r(a)r(a^{i-1})=r(r(a)r(a^{i-1}))=r(a^{m+i-1})=r(a^{m+i-1})$ . If  $a^{m+i-1} \in S\text{-Reg}S$  then since  $S$  is  $r$ -cancelative semigroup we have  $a^i=a^{m+i-1}$ . Since  $a^m \in a^m Sa^m$  then  $a^i=a^{m+i-1}=a^m a^{i-1} \in a^m Sa^{m+i-1}=a^i a^{m-i} Sa^{m-1} a^i \in a^i Sa^i$  and so  $a^i \in \text{Reg}S$  what is a contradiction. Hence,  $a^{m+i-1} \notin \text{Reg}S$  and

$$(10) \quad r(a^i)=a^{m+i-1} \quad (i=2, 3, \dots, m-1).$$

Since  $S$  is  $r$ -cancelative and (8) holds, we have  $a^p \neq a^q$  for  $m \leq p < q \leq 2m-2$ . We have  $a^{2m}=r(a^2)=a^{m+1}$  by (10) for  $i=2$  and it follows that

$$\begin{aligned} a^m &= r(a^m) = r(a^{m-1} \cdot a) = r(a^{m-1})r(a) = a^{2m-2} a^m = a^{2m} a^{m-a} \\ &= a^{m+1} a^{m-2} = a^{2m-1}. \end{aligned}$$

Hence,  $|ka| = 2m-2$ . If  $m=2$ , then  $|ka|=2$ . If  $m>2$ , then by (10)  $r(a^{m-2})=a^{2m-3}$ . Also,  $r(a^{m-2})=a^{(m-2)p}$  where  $p$  is the smallest positive integer such that  $a^{(m-2)p} \in \text{Reg}S$ . Now  $a^{2m-3}=a^{(m-2)p}$  and  $2m-3=mp-2p$ , i.e.  $m=2 + \frac{1}{p-2}$  what implies  $p=m=3$ . Hence,  $|ka|=4$ .

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STABILNOST PRIRODNOG PARCIJALNOG UREĐENJA NA  
 $r$ -KANCELATIVNOJ  $r$ -POLUGRUPI

U radu [9] uvodimo pojam  $r$ -kancelativne polugrupe i dokazujemo da na njoj relacija  $\leq$  definisana sa

$a \leq b \iff R_a \leq R_b \wedge (\forall b' \in V(r(b)))(r(a) = r(a)b'r(a) = r(a)b'r(b) = r(b)b'r(a))$   
 jeste (prirodno) parcijalno uređenje. U ovom radu uvodimo pojam lokalno  $\mathcal{L}^*$ - (  $\mathcal{K}^*$ - ) unipotentne polugrupe i na njoj ispitujemo desnu (levu) stabilnost relacije  $\leq$  (2.1.Teorema i 2.1.Posledica).

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