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SYMMETRY PROPERTIES OF CURVATURE TENSORS OF THE SPACE WITH NON-SYMMETRIC AFFINE CONNEXION AND GENERALIZED RIEMANNIAN SPACE

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Abstract. In some previous works we have obtained several curvature tensors in the space L_M with non-symmetric affine connexion and generalized Riemannian space as a particular case of the space L_M.

In the present work we study symmetry properties of these tensors (the anti-symmetry with respect to two indices, the cyclic symmetry, the symmetry with respect to two pairs of indices).

0. Introduction

In the space L_M with non-symmetric affine connexion L^i_{jk} one can define four kinds of covariant derivative for a tensor. For example, for a tensor a^i_j we have

$$
\begin{array}{lll}
a^i_{j\underset{1}{|}m} = a^i_{j,m} + L^i_{pm} a^p_j - L^p_{jm} a^i_p , & & \\
\phantom{a^i_{j}}\ 2 & mp & mj \\
\phantom{a^i_{j}}\ 3 & pm & mj \\
\phantom{a^i_{j}}\ 4 & mp & jm
\end{array}
\tag{0.1}
$$

where for particular cases we have inscribed only kinds of derivatives on the left side and corresponding indices on the right side. So, for example, $a^i_{j\underset{1}{|}m}$ signifies covariant derivative of the first kind with respect to x^m, and comma (,) on the right side in (0.1) denotes partial derivative on x^m. In this manner, by 1st and 2nd kind of derivative we can obtain in all 10 kinds of Ricci type identities for alternated covariant derivative of the second degree, and also 10 kinds of such identities by 3rd and 4th kind of covariant derivative (see [1], [2]).

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For example, for a tensor $a^{r_1\dots r_u}_{t_1\dots t_v}$ are possible next cases of forming corresponding difference by the 1st and the 2nd kind of covariant derivative

$$
(0.2)\qquad
\begin{array}{ccc}
a^{r_1\dots r_u}_{t_1\dots t_v|m|n} & - & a^{r_1\dots r_u}_{t_1\dots t_v|n|m} \quad , \\
1\ 1 & & 1\ 1 \\
2\ 2 & & 2\ 2 \\
1\ 2 & & 1\ 2 \\
2\ 1 & & 2\ 1 \\
1\ 1 & & 2\ 2 \\
1\ 1 & & 1\ 2 \\
1\ 1 & & 2\ 1 \\
2\ 2 & & 1\ 2 \\
2\ 2 & & 2\ 1 \\
1\ 2 & & 2\ 1
\end{array}
$$

where, for example, $a^{\cdots}_{\cdots\underset{1}{|}m\underset{2}{|}n}$ denotes $(a^{\cdots}_{\cdots\underset{1}{|}m})_{\underset{2}{|}n}$. One can express the differences (0.2) by general formulas from [1] in such a manner that in the right side appear three tensors, which we have called curvature tensors respectively of the 1st, 2nd and 3rd kind, and also 15 magnitudes, which have the role and the form of the curvature tensors, but they are not tensors and we have called them "curvature pseudotensors" respectively 1st to 15th kind. The curvature tensors $\underset{1}{R}{}^i{}_{jmn}, \underset{2}{R}{}^i{}_{jmn}, \underset{3}{R}{}^i{}_{jmn}$ appear in the 1st, 2nd and last case in (0.2), while in the rest of cases appear the curvature pseudotensors $\underset{1}{A}{}^i{}_{jmn}, \dots, \underset{15}{A}{}^i{}_{jmn}$. So, in the first two cases (for $\theta=1,2$) we have

$$
(0.3)\qquad a^{r_1\dots r_u}_{t_1\dots t_v\underset{\theta}{|}m\underset{\theta}{|}n} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{\theta}{|}n\underset{\theta}{|}m} = \sum_{\ell=1}^{u} \underset{\theta}{R}{}^{r_\ell}{}_{pmn}\binom{p}{r_\ell}a^{\cdots}_{\cdots} - \sum_{f=1}^{v}\underset{\theta}{R}{}^{p}{}_{t_f mn}\binom{t_f}{p}a^{\cdots}_{\cdots} + (-1)^{\theta}\,2L^{p}_{\underset{\vee}{mn}}\,a^{r_1\dots r_u}_{t_1\dots t_v\underset{\theta}{|}p} ,
$$

where $\binom{p}{r_\ell}a^{\cdots}_{\cdots}$ denotes that in $a^{r_1\dots r_u}_{t_1\dots t_v}$ the index r_ℓ is supstituted by p, and $\binom{t_f}{p}a^{\cdots}_{\cdots}$ has an analogical meaning, $\underset{\vee}{mn}$ denotes antisymmetrisation over the indices m,n, while

$$
(0.4)\qquad \underset{1}{R}{}^i{}_{jmn} = L^i_{jm,n} - L^i_{jn,m} + L^p_{jm}L^i_{pn} - L^p_{jn}L^i_{pm} ,
$$

$$
(0.5)\qquad \underset{2}{R}{}^i{}_{jmn} = L^i_{mj,n} - L^i_{nj,m} + L^p_{mj}L^i_{np} - L^p_{nj}L^i_{mp} .
$$

In the last case in (0.2) we have

$$
(0.6)\qquad a^{r_1\dots r_u}_{t_1\dots t_v\underset{1}{|}m\underset{2}{|}n} - a^{r_1\dots r_u}_{t_1\dots t_v\underset{2}{|}n\underset{1}{|}m} = \sum_{\ell=1}^{u}\underset{3}{R}{}^{r_\ell}{}_{pmn}\binom{p}{r_\ell}a^{\cdots}_{\cdots} - \sum_{f=1}^{v}\underset{3}{R}{}^{p}{}_{t_f mn}\binom{t_f}{p}a^{\cdots}_{\cdots} ,
$$

where

$$
(0.7)\qquad \underset{3}{R}{}^i{}_{jmn} = L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{nm}(L^i_{pj} - L^i_{jp})
$$

is curvature tensor of the 3rd kind of the space L_M.
If in (0.2) we change the first kind of covariant derivative

with the third kind, and the second with the fourth kind, in the Ricci type identities, which we obtain in this way, the same curvature tensors and pseudotensors appear as by the first and second kind, but in another arrangement (see [2]). Beside this, a new curvature tensor $\underset{4}{R}{}^{i}{}_{jmn}$ appears, and this tensor we have called curvature tensor of the fourth kind:

$$(0.8)\quad \underset{4}{R}{}^{i}{}_{jmn} = L^{i}_{jm,n} - L^{i}_{nj,m} + L^{p}_{jm}L^{i}_{np} - L^{p}_{nj}L^{i}_{pm} + L^{p}_{mn}(L^{i}_{pj} - L^{i}_{jp}).$$

In the work [3] (the equations (47),(50),(51),(62),(79),(80),(91),(92)) by curvature pseudotensors we have obtained new curvature tensors, which we have called "derived curvature tensors" $\underset{1}{\widetilde{R}}{}^{i}{}_{jmn},\dots,\underset{8}{\widetilde{R}}{}^{i}{}_{jmn}$ of the space L_M.

Let us remark that in the case of symmetric connexion all of 20 Ricci type identities reduce to well known Ricci identity and all curvature tensors and curvature pseudotensors reduce to the curvature tensor of this space.

As we have shown in [4], from all tensors $\underset{1}{R},\dots,\underset{4}{R},\underset{1}{\widetilde{R}},\dots,\underset{8}{\widetilde{R}}$ only five of them are independent, and the others can be expressed as linear combinations by five of them and by tensor $R^{i}{}_{jmn}$ obtained by symmetric connexion $L^{i}_{\underline{jk}}$ (symmetric part of L^{i}_{jk}). By virtue of (2.25a-g) from [4], we have (leaving out the indices):

$$(0.9a,b)\quad \underset{1}{\widetilde{R}} = 2R - \tfrac{1}{2}(\underset{1}{R}+\underset{2}{R}),\quad \underset{3}{\widetilde{R}} = 2R - \underset{2}{\widetilde{R}},$$

$$(0.9c)\quad \underset{4}{\widetilde{R}} = \tfrac{1}{6}(8R - 2\underset{1}{R} + \underset{3}{R} - \underset{4}{R}),$$

$$(0.9d,e)\quad \underset{5}{\widetilde{R}} = 4R - \underset{1}{R} - 2\underset{2}{\widetilde{R}},\quad \underset{6}{\widetilde{R}} = -\underset{1}{R} + 2\underset{2}{\widetilde{R}},$$

$$(0.9f,g)\quad \underset{7}{\widetilde{R}} = -\underset{2}{R} + 2\underset{2}{\widetilde{R}},\quad \underset{8}{\widetilde{R}} = 4R - \underset{2}{R} - 2\underset{2}{\widetilde{R}}.$$

Accordingly, by $\underset{1}{R},\underset{2}{R},\underset{3}{R},\underset{4}{R},\underset{2}{\widetilde{R}},R$ one can express all remaining curvature tensors. Because by virtue of (0.9a,c) the tensors $\underset{1}{\widetilde{R}},\underset{4}{\widetilde{R}}$ are independent of $\underset{2}{\widetilde{R}}$, one cannot use $\underset{1}{\widetilde{R}}$, or $\underset{4}{\widetilde{R}}$, instead of $\underset{2}{\widetilde{R}}$, to express the remaining curvature tensors $\underset{2}{\widetilde{R}}$, but one can use any of the tensors $\underset{3}{\widetilde{R}},\underset{5}{\widetilde{R}},\underset{6}{\widetilde{R}},\underset{7}{\widetilde{R}},\underset{8}{\widetilde{R}}$.

1. The mixed curvature tensors of the space

As it is known, the curvature tensor of the space of symmetric affine connexion possesses the next symmetry properties (see, for example, [5], §61, the eq.(3),(4)):

$$(1.1,2)\quad R^{i}{}_{jmn} = -R^{i}{}_{jnm},\quad R^{i}{}_{jmn} + R^{i}{}_{mnj} + R^{i}{}_{njm} = 0.$$

Introducing the denotation

$$(1.3)\quad \underset{jmn}{\mathrm{Cycl}}\, R^{i}{}_{jmn} = R^{i}{}_{jmn} + R^{i}{}_{mnj} + R^{i}{}_{njm},$$

and similarly in other cases, the equation (1.2) one can write in the form

(1.2′) $\quad \underset{jmn}{\mathrm{Cycl}}\, R^{i}{}_{jmn} = 0 .$

As the tensors $\underset{1}{R}, \dots, \underset{4}{R}, \underset{1}{\widetilde{R}}, \dots, \underset{8}{\widetilde{R}}$ are generalizations of the tensor R and reduce to this tensor in the case of symmetric connexion, we have to investigate the properties (1.1,2) for these tensors.

From (0.4,5,7,8) one gets

(1.4a,b) $$\underset{1}{R}{}^{i}{}_{jmn} = -\underset{1}{R}{}^{i}{}_{jnm}\, , \quad \underset{2}{R}{}^{i}{}_{jmn} = -\underset{2}{R}{}^{i}{}_{jnm}\, ,$$

while

(1.5a,b) $$\underset{3}{R}{}^{i}{}_{jmn} \neq \underset{3}{R}{}^{i}{}_{jnm}\, , \quad \underset{4}{R}{}^{i}{}_{jmn} \neq \underset{4}{R}{}^{i}{}_{jnm}$$

and further

(1.6) $$\underset{jmn}{\mathrm{Cycl}}\, \underset{1}{R}{}^{i}{}_{jmn} = 2 \underset{jmn}{\mathrm{Cycl}} \left(L^{i}_{\underset{\vee}{jm},n} + L^{p}_{\underset{\vee}{jm}} L^{i}_{pn} \right),$$

(1.7) $$\underset{jmn}{\mathrm{Cycl}}\, \underset{2}{R}{}^{i}{}_{jmn} = 2 \underset{jmn}{\mathrm{Cycl}} \left(L^{i}_{\underset{\vee}{mj},n} + L^{p}_{\underset{\vee}{mj}} L^{i}_{np} \right),$$

(1.8) $$\underset{jmn}{\mathrm{Cycl}}\, \underset{3}{R}{}^{i}{}_{jmn} = 4 \underset{jmn}{\mathrm{Cycl}}\, L^{p}_{\underset{\vee}{jm}} L^{i}_{\underset{\vee}{np}}\, ,$$

(1.9) $$\underset{jmn}{\mathrm{Cycl}}\, \underset{4}{R}{}^{i}{}_{jmn} = 0 .$$

We see that only $\underset{4}{R}{}^{i}{}_{jmn}$ possesses the cyclic symmetry of the form (1.2′).

Crossing to the derived curvature tensors, we remark that one can use the relations (0.9a-g) too. Firstly, by virtue of (0.9a),(1.1,2),(1.4a,b),(1.6,7) we have

(1.10,11) $$\underset{1}{\widetilde{R}}{}^{i}{}_{jmn} = -\underset{1}{\widetilde{R}}{}^{i}{}_{jnm}\, , \quad \underset{jmn}{\mathrm{Cycl}}\, \underset{1}{\widetilde{R}}{}^{i}{}_{jmn} = 2 \underset{jmn}{\mathrm{Cycl}}\, L^{p}_{\underset{\vee}{mj}} L^{i}_{\underset{\vee}{pn}} .$$

From (2.12,5b,6b) in [4], we get

(1.12) $$\underset{2}{\widetilde{R}}{}^{i}{}_{jmn} = R^{i}{}_{jmn} + L^{p}_{\underset{\vee}{jm}} L^{i}_{\underset{\vee}{pn}} + L^{p}_{\underset{\vee}{jn}} L^{i}_{\underset{\vee}{pm}}\, ,$$

from where is

(1.13,14) $$\underset{2}{\widetilde{R}}{}^{i}{}_{jmn} \neq -\underset{2}{\widetilde{R}}{}^{i}{}_{jnm}\, , \quad \underset{jmn}{\mathrm{Cycl}}\, \underset{2}{\widetilde{R}}{}^{i}{}_{jmn} = 0 .$$

Using the previous symmetry properties and (0.9b-g), we get

(1.15,16) $$\underset{3}{\widetilde{R}}{}^{i}{}_{jmn} \neq \underset{3}{\widetilde{R}}{}^{i}{}_{jnm}\, , \quad \underset{jmn}{\mathrm{Cycl}}\, \underset{3}{\widetilde{R}}{}^{i}{}_{jmn} = 0 .$$

As in virtue of (2.14,5a-c) in [4] it is

(1.17) $$\underset{4}{\widetilde{R}}{}^{i}{}_{jmn} = R^{i}{}_{jmn} + \tfrac{1}{3}\left(-L^{i}_{\underset{\vee}{jm};n} + L^{i}_{\underset{\vee}{jn};m} - L^{p}_{\underset{\vee}{jm}} L^{i}_{\underset{\vee}{pn}} + L^{p}_{\underset{\vee}{jn}} L^{i}_{\underset{\vee}{pm}} - 2 L^{p}_{\underset{\vee}{mn}} L^{i}_{\underset{\vee}{pj}} \right),$$

where the covariant derivative with respect to $L^{i}_{\underline{jk}}$ is denoted by semicolon (;), it is evidently

(1.18) $$\underset{4}{\widetilde{R}}{}^{i}{}_{jmn} = -\underset{4}{\widetilde{R}}{}^{i}{}_{jnm}\, ,$$

while the relation of the form (1.16) for $\underset{4}{\widetilde{R}}$ is not in force.

Using the relations (0.9d-g) and the properties of the tensors $\underset{}{R}, \underset{1}{R}, \underset{2}{R}, \underset{2}{\widetilde{R}}$ it is easy to prove that the tensors $\underset{3}{\widetilde{R}}, \dots, \underset{8}{\widetilde{R}}$ do not possess the symmetry properties of the type (1.1,2).

2. The covariant curvature tensors of the space

2.0. In a generalized Riemannian space GR_M, which is a particular case of a space L_M, connexion coeficients Γ^i_{jk} are given by the basical metric tensor g_{ij}, which is non-symmetric. Accordingly ([6], eq.(1.1)):

(2.1) $$\Gamma_{ijk} = \frac{1}{2}(g_{ji,k} - g_{jk,i} + g_{ik,j}),$$

(2.2a,b) $$\Gamma^i_{jk} = g^{\underline{ip}}\Gamma_{pjk} = \frac{1}{2}g^{\underline{ip}}(g_{jp,k} - g_{jk,p} + g_{pk,j}),\ \Gamma_{ijk} = g_{\underline{ip}}\Gamma^p_{jk},$$

where $\underline{ij}$ signifies the symmetrisation on the i, j. From (2.1) it is

(2.3a,b) $$\Gamma_{ijk} + \Gamma_{jik} = g_{\underline{ij},k},\quad \Gamma_{ijk} + \Gamma_{kji} = g_{\underline{ik},j}.$$

One defines covariant curvature tensors in GR_M

(2.4a,b) $$\underset{p}{R}_{ijmn} = g_{\underline{is}}\underset{p}{R}^s{}_{jmn},\ \underset{q}{\widetilde{R}}_{ijmn} = g_{\underline{is}}\underset{q}{\widetilde{R}}^s{}_{jmn}\quad (p=1,\dots,4;\ q=1,\dots,8).$$

As it is known, in a usual Riemannian space (for which it is $g_{ij} = g_{ji}$) for covariant curvature tensor the following relations hold

(2.5a,b,c) $$R_{ijmn} = -R_{jimn} = -R_{ijnm};\ \underset{\alpha\beta\gamma}{Cycl}\, R_{ijmn} = 0,\ \{\alpha,\beta,\gamma\}\subset\{i,j,m,n\};\ R_{ijmn} = R_{mnij}.$$

Since the space GR_M is a generalization of the usual Riemannian space, we have to examine relations of the type (2.5) for covariant curvature tensors of the GR_M.

2.1. On the base of (2.1), (0.4), (2.2b) it is

(2.6) $$\underset{1}{R}_{ijmn} = g_{\underline{is}}\underset{1}{R}^s{}_{jmn} = g_{\underline{is}}\Gamma^s_{jm,n} - g_{\underline{is}}\Gamma^s_{jn,m} + \Gamma^p_{jm}\Gamma_{ipn} - \Gamma^p_{jn}\Gamma_{ipm}.$$

From (2.2b) we get

$$\Gamma_{ijm,n} = (g_{\underline{is}}\Gamma^s_{jm})_{,n} = g_{\underline{is},n}\Gamma^s_{jm} + g_{\underline{is}}\Gamma^s_{jm,n},$$

whence, by reason of (2.3) it follows

(2.7a,b) $$g_{\underline{is}}\Gamma^s_{jm,n} = \Gamma_{ijm,n} - \Gamma^s_{jm}(\Gamma_{isn} + \Gamma_{sin}),\ g_{\underline{is}}\Gamma^s_{jm,n} = \Gamma_{ijm,n} - \Gamma^s_{jm}(\Gamma_{ins} + \Gamma_{sni}).$$

In accordance with (2.7a) the equation (2.6) becomes

(2.8) $$\underset{1}{R}_{ijmn} = \Gamma_{ijm,n} - \Gamma_{ijn,m} + \Gamma_{pim}\Gamma^p_{jn} - \Gamma_{pjm}\Gamma^p_{in}.$$

Using (2.1), we can write the equation (2.8) in the form

(2.9) $$\underset{1}{R}_{ijmn} = \frac{1}{2}(g_{im,jn} - g_{jm,in} - g_{in,jm} + g_{jn,im}) + \Gamma_{pim}\Gamma^p_{jn} - \Gamma_{pjm}\Gamma^p_{in}.$$

From (2.9,8) we conclude

(2.10) $$\underset{1}{R}_{ijmn} = -\underset{1}{R}_{jimn} = -\underset{1}{R}_{ijnm},$$

that is, the same properties as in ordinary Riemannian space are valid.

In virtue of (2.8) one concludes that for $\underset{1}{R}_{ijmn}$ any of relations of the form (2.5b) is not in force.

Further, from (2.8) one gets

$$(2.11)\qquad \underset{1}{R}_{ijmn}-\underset{1}{R}_{mnij}=g_{\underset{\vee}{im},jn}-g_{\underset{\vee}{jm},in}-g_{\underset{\vee}{in},jm}+g_{\underset{\vee}{jn},im}+$$
$$+\Gamma_{pim}\Gamma^{p}_{jn}-\Gamma_{pmi}\Gamma^{p}_{nj}-\Gamma_{pjm}\Gamma^{p}_{in}+\Gamma_{pmj}\Gamma^{p}_{ni}\ ,$$

which is ingeneral $\neq 0$, that is a relation of the form (2.5c) does not hold.

2.2. As in the case of the tensor $\underset{1}{R}_{ijmn}$, one concludes that

$$(2.12)\qquad \underset{2}{R}_{ijmn}=-\underset{2}{R}_{jimn}=-\underset{2}{R}_{ijnm}\ ,$$

and estimating for $\underset{2}{R}$ the expression of the form (2.11), one concludes

$$(2.13)\qquad \underset{1}{R}_{ijmn}-\underset{1}{R}_{mnij}=\underset{2}{R}_{mnij}-\underset{2}{R}_{ijmn}\ ,$$

i.e. for $\underset{2}{R}$ a relation of the form (2.5c) does not hold.

2.3. In accordance with (2.4a), (0.7), (2.7b,a) it is

$$(2.14)\qquad \underset{3}{R}_{ijmn}=\Gamma_{ijm,n}-\Gamma_{inj,m}+\Gamma_{pim}\Gamma^{p}_{nj}-\Gamma_{pni}\Gamma^{p}_{jm}+2\Gamma^{p}_{nm}\Gamma_{i\underset{\vee}{pj}}\ .$$

From (2.1) one obtains

$$(2.15)\qquad \Gamma_{i\underset{\vee}{pj}}=-\Gamma_{j\underset{\vee}{pi}}=-\Gamma_{p\underset{\vee}{ij}}=-\Gamma_{i\underset{\vee}{jp}}\ ,$$

that is, the tensor $\Gamma_{i\underset{\vee}{pj}}$ is antisymmetric in relation to any pair of indices and (2.14) becomes

$$(2.16)\qquad \underset{3}{R}_{ijmn}=\tfrac{1}{2}(g_{ji,mn}-g_{ij,mn}+g_{im,jn}-g_{jm,in}+g_{nj,im}-g_{ni,jm})+$$
$$+g^{\underline{ps}}(\Gamma_{pim}\Gamma_{snj}-\Gamma_{pjm}\Gamma_{sni})+\tfrac{1}{2}\Gamma^{p}_{nm}(g_{ij,p}-g_{ji,p}+g_{jp,i}-g_{ip,j}+g_{pi,j}-g_{pj,i}),$$

wherefrom one concludes

$$(2.17)\qquad \underset{3}{R}_{ijmn}=-\underset{3}{R}_{jimn}\ ,$$

while in general $\underset{3}{R}_{ijmn}\neq\underset{3}{R}_{ijnm}$.

For the tensor $\underset{3}{R}_{ijmn}$ any symmetry of the forms (2.5b,c) does not hold.

2.4. For the tensor $\underset{4}{R}_{ijmn}$ is

$$(2.18)\qquad \underset{4}{R}_{ijmn}=-\underset{4}{R}_{jimn}\ ,$$

but in generally it is $\underset{4}{R}_{ijmn}\neq\underset{4}{R}_{ijnm}$. From (2.4) we have

$$\underset{jmn}{\mathrm{Cycl}}\,\underset{4}{R}_{ijmn}=\underset{jmn}{\mathrm{Cycl}}\left(g_{\underline{is}}\underset{4}{R}^{s}{}_{jmn}\right)=g_{\underline{is}}\underset{jmn}{\mathrm{Cycl}}\,\underset{4}{R}^{s}{}_{jmn}\ ,$$

and based on (1.9) it follows

$$(2.19a)\qquad \underset{jmn}{\mathrm{Cycl}}\,\underset{4}{R}_{ijmn}=0\ .$$

As from (2.18) it is $\underset{imn}{\mathrm{Cycl}}\,\underset{4}{R}_{ijmn}=-\underset{imn}{\mathrm{Cycl}}\,\underset{4}{R}_{jimn}$, from (2.19a) one gets

$$(2.19b)\qquad \underset{imn}{\mathrm{Cycl}}\,\underset{4}{R}_{ijmn}=0\ .$$

Similarly to (2.13), one obtains

$$(2.20)\qquad \underset{3}{R}_{ijmn}-\underset{3}{R}_{mnij}=\underset{4}{R}_{ijmn}-\underset{4}{R}_{mnij}\,.$$

2.5. Let us examine, further, the symmetry properties of the tensors $\underset{l}{\widetilde{R}}_{ijmn}\ (l=1,\dots,8)$.

From (0.9a) it is

$$(2.21)\qquad \underset{1}{\widetilde{R}}_{ijmn}=g_{\underline{is}}\underset{1}{R}{}^{s}{}_{jmn}=2R_{ijmn}-\frac{1}{2}\big(\underset{1}{R}_{ijmn}+\underset{2}{R}_{ijmn}\big),$$

and, using (2.5,10,12), we conclude

$$(2.22)\qquad \underset{1}{\widetilde{R}}_{ijmn}=-\underset{1}{\widetilde{R}}_{jimn}=-\underset{1}{\widetilde{R}}_{ijnm}\,.$$

The tensor $\underset{1}{\widetilde{R}}_{ijmn}$ does not possess any cyclic symmetry.

By means of (2.21,5c,13) one gets

$$(2.23)\qquad \underset{1}{\widetilde{R}}_{ijmn}=\underset{1}{\widetilde{R}}_{mnij}\,.$$

Using (1.12), we obtain

$$(2.24)\qquad \underset{2}{\widetilde{R}}_{ijmn}=R_{ijmn}+g^{\underline{ps}}\big(\Gamma_{ip\underset{\vee}{m}}\Gamma_{sj\underset{\vee}{n}}+\Gamma_{ip\underset{\vee}{n}}\Gamma_{sj\underset{\vee}{m}}\big),$$

wherefrom we see that the tensor $\underset{2}{\widetilde{R}}_{ijmn}$ is antisymmetric with respect neither to i, j nor to m, n.

From (1.14;2.5b,24,15) we have

$$(2.25)\qquad \underset{\alpha\beta\gamma}{\mathrm{Cycl}}\,\underset{2}{\widetilde{R}}_{ijmn}=0\,,\ \{\alpha,\beta,\gamma\}\subset\{i,j,m,n\}\,.$$

On the base (2.24) one obtains

$$\underset{2}{\widetilde{R}}_{ijmn}-\underset{2}{\widetilde{R}}_{mnij}=g^{\underline{ps}}\big(\Gamma_{ip\underset{\vee}{m}}\Gamma_{sj\underset{\vee}{n}}+\Gamma_{ip\underset{\vee}{n}}\Gamma_{sj\underset{\vee}{m}}-\Gamma_{mp\underset{\vee}{i}}\Gamma_{sn\underset{\vee}{j}}-\Gamma_{mp\underset{\vee}{j}}\Gamma_{sn\underset{\vee}{i}}\big),$$

and by reason of (2.15):

$$(2.26)\qquad \underset{2}{\widetilde{R}}_{ijmn}=\underset{2}{\widetilde{R}}_{mnij}\,.$$

Using (0.9b) and properties of the tensors $R,\underset{2}{\widetilde{R}}$, we conclude that

$$(2.27,28)\qquad \underset{\alpha\beta\gamma}{\mathrm{Cycl}}\,\underset{3}{\widetilde{R}}_{ijmn}=0\,,\ \{\alpha,\beta,\gamma\}\subset\{i,j,m,n\}\,;\quad \underset{3}{\widetilde{R}}_{ijmn}=\underset{3}{\widetilde{R}}_{mnij}\,,$$

while the tensor $\underset{3}{\widetilde{R}}_{ijmn}$ does not possess other symmetry properties.

From (0.9c,7,8) it is

$$(2.29)\qquad \underset{4}{\widetilde{R}}_{ijmn}=g_{\underline{is}}\underset{4}{\widetilde{R}}{}^{s}{}_{jmn}=\frac{1}{3}\big(4R_{ijmn}-\underset{1}{R}_{ijmn}+2\Gamma_{ip\underset{\vee}{j}}\Gamma^{p}_{\underset{\vee}{nm}}\big)\,,$$

and taking into consideration (2.15) and the properties of the tensors R, $\underset{1}{R}$, one concludes

$$(2.30)\qquad \underset{4}{\widetilde{R}}_{ijmn}=-\underset{4}{\widetilde{R}}_{jimn}=-\underset{4}{\widetilde{R}}_{ijnm}\,.$$

From (0.9d-g) and corresponding properties of the tensors R, $\underset{1}{R},\underset{2}{R},\underset{2}{\widetilde{R}}$, it follows that the tensors $\underset{5}{\widetilde{R}},\dots,\underset{8}{\widetilde{R}}$ do not possess any symmetry properties.

3. The curvature tensors of the space with respect to the subspace

Let L_N be a space of non-symmetric affine connexion $L^{\alpha}_{\beta\gamma}$. If y^{α} are coordinates in L_N, then the equations

$$(3.1)\qquad y^{\alpha}=y^{\alpha}(x^{1},\dots,x^{M}),\quad (M<N,\ \operatorname{rank}(y^{\alpha}_{,i})=M)$$

define a subspace L_M. We remark that in this section the Greek indices refer to the space L_N, while the Latin refer to the subspace L_M. In the work [7] for a tensor $a^{\rho_1\cdots\rho_\theta\, r_1\cdots r_u}_{\sigma_1\cdots\sigma_\omega\, t_1\cdots t_v}$ we have used the covariant derivatives $\underset{3}{|}m$, $\underset{4}{|}m$:

$$(3.2)\quad a^{\rho_1\cdots\rho_\theta\, r_1\cdots r_u}_{\sigma_1\cdots\sigma_\omega\, t_1\cdots t_v\underset{\substack{3\\4}}{|}m}=a^{\rho_1\cdots r_u}_{\sigma_1\cdots t_v,m}+\sum_{\ell=1}^{u}L^{r_\ell}_{\substack{pm\\mp}}\binom{p}{r_\ell}a^{\cdots}_{\cdots}-\sum_{f=1}^{v}L^{p}_{\substack{mt_f\\t_f m}}\binom{t_f}{p}a^{\cdots}_{\cdots}+$$

$$+\left[\sum_{\lambda=1}^{\theta}L^{\rho_\lambda}_{\substack{\pi\mu\\\mu\pi}}\binom{\pi}{\rho_\lambda}a^{\cdots}_{\cdots}-\sum_{\varphi=1}^{\omega}L^{\pi}_{\substack{\mu\sigma_\varphi\\\sigma_\varphi\mu}}\binom{\sigma_\varphi}{\pi}a^{\cdots}_{\cdots}\right]y^{\mu}_{,m}\ ,$$

and obtained next identity:

$$(3.3)\quad a^{\rho_1\cdots\rho_\theta\, r_1\cdots r_u}_{\sigma_1\cdots\sigma_\omega\, t_1\cdots t_v\underset{3}{|}m\underset{4}{|}n}-a^{\rho_1\cdots\rho_\theta\, r_1\cdots r_u}_{\sigma_1\cdots\sigma_\omega\, t_1\cdots t_v\underset{4}{|}n\underset{3}{|}m}=\sum_{\ell=1}^{u}\underset{4}{R}{}^{r_\ell}{}_{pmn}\binom{p}{r_\ell}a^{\cdots}_{\cdots}+$$

$$+\sum_{f=1}^{v}\underset{3}{R}{}^{p}{}_{t_f nm}\binom{t_f}{p}a^{\cdots}_{\cdots}+\sum_{\lambda=1}^{\theta}\underset{4}{R}{}^{\rho_\lambda}{}_{\pi mn}\binom{\pi}{\rho_\lambda}a^{\cdots}_{\cdots}+\sum_{\varphi=1}^{\omega}\underset{3}{R}{}^{\pi}{}_{\sigma_\varphi nm}\binom{\sigma_\varphi}{\pi}a^{\cdots}_{\cdots}\ ,$$

where the magnitudes

$$(3.4)\quad \underset{3}{R}{}^{\alpha}{}_{\beta mn}=(L^{\alpha}_{\beta\mu,\nu}-L^{\alpha}_{\nu\beta,\mu}+L^{\pi}_{\beta\mu}L^{\alpha}_{\nu\pi}-L^{\pi}_{\nu\beta}L^{\alpha}_{\pi\mu})y^{\mu}_{,m}y^{\nu}_{,n}+$$
$$+2L^{\alpha}_{\underset{\vee}{\beta\mu}}(y^{\mu}_{,mn}-L^{p}_{nm}y^{\mu}_{,p})$$

$$(3.5)\quad \underset{4}{R}{}^{\alpha}{}_{\beta mn}=(L^{\alpha}_{\beta\mu,\nu}-L^{\alpha}_{\nu\beta,\mu}+L^{\pi}_{\beta\mu}L^{\alpha}_{\nu\pi}-L^{\pi}_{\nu\beta}L^{\alpha}_{\pi\mu})y^{\mu}_{,m}y^{\nu}_{,n}+$$
$$+2L^{\alpha}_{\underset{\vee}{\beta\mu}}(y^{\mu}_{,mn}-L^{p}_{mn}y^{\mu}_{,p})$$

are tensors and we call them curvature tensors of the third respectively the fourth kind of the space L_N with respect to the subspace L_M. If the connexion is symmetric, that is $L^{\alpha}_{\beta\gamma}=L^{\alpha}_{\gamma\beta}$, the tensors $\underset{t}{R}{}^{\alpha}{}_{\beta mn}$ $(t=3,4)$ reduce to

$$(3.6)\qquad \underset{t}{R}{}^{\alpha}{}_{\beta mn}=R^{\alpha}{}_{\beta\mu\nu}\,y^{\mu}_{,m}y^{\nu}_{,n}\ ,$$

where $R^{\alpha}{}_{\beta\mu\nu}$ is the curvature tensor of the space with symmetric connexion. Because the tensor in the right side of (3.6) possesses only the antisymmetry with respect to m,n, we have to examine this property for the tensors (3.4,5). However, it is easy to see that such a property does not hold.

Let us observe, further, a generalized Riemannian space GR_N and his subspace GR_M. If $a_{\alpha\beta}$ and g_{ij} are basical metric tensors for GR_N and GR_M respectively, then the formulas (2.1,2) and similar formulas for $\Gamma_{\alpha\beta\gamma}$ by $a_{\alpha\beta}$ are in force. If we put

$$(3.7)\qquad \underset{t}{R}_{\alpha\beta mn} = a_{\underline{\alpha\delta}}\underset{t}{R}^{\delta}{}_{\beta mn} \qquad (t=3,4),$$

from (3.7,4),(2.7b,a) one gets

$$(3.8)\ \underset{3}{R}_{\alpha\beta mn} = a_{\underline{\alpha\delta}}\underset{3}{R}^{\delta}{}_{\beta mn} = \big(\Gamma_{\alpha\beta\mu,\nu} - \Gamma_{\alpha\nu\beta,\mu} + \Gamma_{\pi\alpha\mu}\Gamma^{\pi}_{\nu\beta} - \Gamma_{\pi\nu\alpha}\Gamma^{\pi}_{\beta\mu}\big)\gamma^{\mu}_{,m}\gamma^{\nu}_{,n} + 2\Gamma_{\alpha\underset{\vee}{\beta\mu}}\big(\gamma^{\mu}_{,mn} - \Gamma^{p}_{nm}\gamma^{\mu}_{,p}\big).$$

Calculating $\Gamma_{\alpha\beta\mu,\nu} - \Gamma_{\alpha\nu\beta,\mu}$ from formulas, which are similar to (2.1), we can write the previous equation in the form

$$(3.9)\ \underset{3}{R}_{\alpha\beta mn} = \frac{1}{2}\big(a_{\beta\alpha,\mu\nu} - a_{\alpha\beta,\mu\nu} + a_{\alpha\mu,\beta\nu} - a_{\beta\mu,\alpha\nu} + a_{\nu\beta,\alpha\mu} - a_{\nu\alpha,\beta\mu}\big)\gamma^{\mu}_{,m}\gamma^{\nu}_{,n} + a^{\underline{\varepsilon\delta}}\big(\Gamma_{\varepsilon\alpha\mu}\Gamma_{\delta\nu\beta} - \Gamma_{\varepsilon\beta\mu}\Gamma_{\delta\nu\alpha}\big)\gamma^{\mu}_{,m}\gamma^{\nu}_{,n} + 2\Gamma_{\alpha\underset{\vee}{\beta\mu}}\big(\gamma^{\mu}_{,mn} - \Gamma^{p}_{nm}\gamma^{\mu}_{,p}\big).$$

Since from (2.15) it is $\Gamma_{\alpha\underset{\vee}{\beta\mu}} = -\Gamma_{\beta\underset{\vee}{\alpha\mu}}$, in virtue of (3.9) one gets

$$(3.10)\qquad \underset{3}{R}_{\alpha\beta mn} = -\underset{3}{R}_{\beta\alpha mn}.$$

In the same way one concludes that we can write $\underset{4}{R}_{\alpha\beta mn}$ in the form, which differs from (3.8) only that in the last brackets instead of Γ^{p}_{nm} is Γ^{p}_{mn} and we have

$$(3.11)\qquad \underset{4}{R}_{\alpha\beta mn} = -\underset{4}{R}_{\beta\alpha mn}.$$

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OSOBINE SIMETRIJE TENZORA KRIVINE PROSTORA SA NESIMETRIČNOM AFINOM KONEKSIJOM I GENERALISANIH RIMANOVIH PROSTORA

U nekim prethodnim radovima smo dobili više tenzora krivine prostora L_M sa nesimetričnom afinom koneksijom i generalisanog Rimanovog prostora, kao specijalnog slučaja prostora L_M.

U ovom radu proučavamo osobine simetrije ovih tenzora (antisimetriju u odnosu na dva indeksa, cikličnu simetriju, simetriju u odnosu na dva para indeksa).

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