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SOME RELATIONS BETWEEN HYPERSPACES OF NEARLY COMPACT SPACES

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Abstract. In this paper we investigate further the results given in [7] . Let X be a topological space and $\mathcal{K}(X) \subset \exp(X)$ the space of all compact subsets of X with the finite topology . The main results in section 2 are: (a) In the class of spaces $\{X\}$ where $\mathcal{K}(X)$ is normal, the notions of hypercountably compactness and strongly countably compactness coincide. (b) In the class of spaces $\{X\}$ where exp(X) satisfies the first axiom of countability, the notions of compactness and hypercountably compactness coincide.

1. Introduction

The closure of A subset of a space X is denoted by $cl_X(A)$. In this paper we assume that all spaces are Hausdorff (T_2-spaces). For notations and definitions not given here see [2] .

Let X be a topological T_2-space. Then:

(1) exp(X) denotes the space of all non-empty closed subsets of X with finite topology. The finite topology on exp(X) is generated by open collection on the form $\langle U_1, U_2, \ldots, U_n\rangle = \{F \in \exp(X): F \subset \bigcup_{i=1}^{n} U_i$ and $F \cap U_i \neq \emptyset$ for $i=1,2,\ldots,n\}$ where $U_1, U_2, \ldots, U_n$ are open subsets of X.

(2) $\mathcal{K}(X)$ denotes the set of all non-empty compact subsets of X as a subspace of exp(X).

(3) $\mathcal{F}(X) = \{F \subset X : F$ is finite$\} \subset \mathcal{K}(X)$.

1.1 DEFINITION. A quasi-ordered and directed set $(S, \leq)$ is $\aleph_0$-directed if for every countable subset $S_0 = \{s_1, s_2, \ldots, s_n, \ldots\} \subseteq S$, there exists an $s \in S$ such that $s_n \leq s$, for all $n \in N$. (see [2])

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1.2. DEFINITION. A space X is strongly countably compact (scc) if every countable subset in X has a compact closure in X. (see [4]).

1.3. DEFINITION. A space X is hypercountably compact (hcc) if the union of every countable family of compact sets in X has a compact closure in X.(see [7]).

1.4. DEFINITION. A topological space X is called weakly compact if every locally finite family of non-empty open sets in X is finite.(see [1]).

1.5. DEFINITION. A space X is weakly normal provided that every pair of disjoint closed subsets, one of which is at most countably infinite, can be separated by open sets.(see [5]).

1.6. DEFINITION. A space X is pseudocompact if each real valued continuous function is bounded. A space is countably compact if every countable open cover of X has a finite subcover.(see [2]).

1.7. THEOREM. Let X be a T_2-space. The following are equivalent:

(1) X is hypercountably compact,

(2) $\mathcal{K}(X)$ is $\aleph_0$-directed by inclusion,

(3) $\mathcal{K}(X)$ is countably compact,

(4) $\mathcal{K}(X)$ is strongly countably compact,

(5) $\mathcal{K}(X)$ is hypercountably compact.(see [7]).

1.8. PROPOSITION. Each hypercountably compact(strongly countably compact) subset of a first countable T_2-space X is closed in X.(see [7]).

2. Normality and properties between compactness and pseudocompactness in hyperspaces

It is clear that every hypercountably compact space is a pseudocompact space. The following results give characterizations of hypercountably compact spaces by pseudocompactness.

2.1. PROPOSITION. Let X be a strongly countably compact T_2-

space. Then $\mathcal{K}(X)$ is a pseudocompact space.

PROOF: Suppose that $\mathcal{K}(X)$ is not pseudocompact. Then there exists $\mathcal{N} \subset \mathcal{K}(X)$ which is a closed discrete subset of cardinality $\aleph_0$. Let $F \subset \mathcal{N}$. According to the Proposition 4.13 of E. Michael [7] , F is a finite subset of X and each point $x \in F$ is isolated point in X. Then $|\mathcal{N}| = \cup\{F_n \in \mathcal{N} : n \in N\}$ is a countable subset of X. Let $A = cl_X(|\mathcal{N}|)$. Then A is compact by the strong countable compactness of X. Since A is compact, we have(see [6]) exp(A) is compact subset of $\mathcal{K}(X)$ and $\mathcal{N} \subset \exp(A)$. Furthermore, the set $\mathcal{N}$ is a closed subset of exp(A). Then $\mathcal{N}$ is a compact subset of $\mathcal{K}(X)$, which is impossible by the supposition that is a closed discrete subset of $\mathcal{K}(X)$. Hence $\mathcal{K}(X)$ is a pseudocompact space. This completes the proof.

2.2. PROPOSITION. If X completely regular and strongly countably compact, then $\mathcal{K}(X)$ is weakly compact.

PROOF: Since X is a completely regular and strongly countably compact we have: $\mathcal{K}(X)$ is a completely regular and pseudocompact. According to the Theorem 1 of S.Mardešić and P.Papić [5] $\mathcal{K}(X)$ is weakly compact. This completes the proof.

2.3. THEOREM. Let X be a completely regular strongly countably compact space. If $\mathcal{K}(X)$ is a weakly normal, the following assertion are mutually equivalent:

(1) X is strongly countably compact,

(2) X is hypercountably compact,

(3) $\mathcal{K}(X)$ is $\aleph_0$-directed by inclusion,

(4) $\mathcal{K}(X)$ is pseudocompact,

(5) $\mathcal{K}(X)$ is countably compact,

(6) $\mathcal{K}(X)$ is strongly countably compact,

(7) $\mathcal{K}(X)$ is hypercountably compact.

PROOF: It is clear that $(2) \to (1)$ and $(5) \to (4)$. By Theorem 1.7, $(2) \leftrightarrow (3) \leftrightarrow (5) \leftrightarrow (6) \leftrightarrow (7)$.

$(4) \leftrightarrow (5)$: Since $(5) \to (4)$ we shall prove the implication $(4) \to (5)$. By Proposition 2.2, $\mathcal{K}(X)$ is completely regular and weakly compact. Since $\mathcal{K}(X)$ is weakly normal and weakly compact, by results of S.Mardešić and P.Papić(see [5] Theorem 4 and Theorem 5), we have that $\mathcal{K}(X)$ is countably compact.

(1)⟷(4): By Proposition 2.1, (1)⟶(4). Since (4)⟷(5) (5)⟶(1) we have (4)⟶(1). This completes the proof.

2.4. THEOREM. If $\mathcal{K}(X)$ is a normal T_2-space, the following are equivalent:

(1) X is strongly countably compact,

(2) X is hypercountably compact,

(3) $\mathcal{K}(X)$ is pseudocompact,

(4) $\mathcal{K}(X)$ is countably compact.

PROOF: That (2) and (4) are equivalent follows immediately from Theorem 1.7. Hewitt [3] has shown the equivalence of (3) and (4). By Proposition 2.1, (1)⟶(3). Since (3)⟷(2) and (2) ⟶(1) we have (3)⟶(1). This completes the proof.

2.5. THEOREM. Let exp(X) be a first countable T_2-space. The following are equivalent:

(1) X is compact,

(2) X is hypercountably compact.

PROOF: It is clear that every compact space is a hypercountably compact.

(2)⟶(1): Let X be hypercountably compact space. Then, by Theorem 1.7, $\mathcal{K}(X)$ is hypercountably compact suspace of exp(X). Since exp(X) satisfies the first axiom of countability, by Proposition 1.8, $\mathcal{K}(X)$ is closed in exp(X). Furthermore, $cl_{exp(X)}(\mathcal{K}(X))=exp(X)$ ($\mathcal{F}(X)\subset\mathcal{K}(X)$, the set $\mathcal{F}(X)$ of finite subsets of X is dense in exp(X)). Then exp(X)= $\mathcal{K}(X)$ and every closed subset of X is a compact subset of X. Hence X is a compact space. This completes the proof.

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NEKI ODNOSI IZMEĐU HIPERPROSTORA
SKORO KOMPAKTNIH PROSTORA

U ovom radu nastavlja se sa proširivanjem rezultata koji su dobijeni u [7] . Osnovni rezultati su sledeći: (a) U klasi prostora {X} kod kojih je hiper-prostor na kolekciji kompaktnih skupova normalan koincidiraju strogo prebrojiva i hiper-prebrojiva kompaktnost. (b) U klasi prostora {X} kod kojih je hiper-prostor exp(X) prve prebrojivosti koincidiraju kompaktnost i hiper-prebrojiva kompaktnost.

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