Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 1(11)(1987), 41-50

Vlajko Lj. Kocić

ON THE GENERALIZED RICCATI EQUATION

(Recieved 10.05.1987.)

Abstract. For the operator Riccati equation (0.1), it is shown that it has similar properties as the well known Riccati differential equation. The obtained results are applied to different classes of equations: differential, difference, partial differenial, functional, partial difference, etc. The results are illustrated by a number of examples.

0. Introduction.

Let V be a commutative algebra over R (or C) an let L be a linear operator on V, which belongs to the class $D_\alpha(V)$, $\alpha \in \ker L$ (see [1] - [4]), i.e.

$$L(uv) = uLv + vLu + \alpha LuLv, \text{ for every } u,v \in V.$$

The properties of linear equations which involve the operator L are investigated in detail in [1] - [4].

In this paper we will consider the generalized Riccati equation:

$$(0.1) \qquad Lu + a(u + \alpha Lu)u + bu + c = 0, \ (a \neq 0, b,c \in V, \alpha \in \ker L)$$

The above equation, as special cases, contains the well known Riccati differential equation:

$$(0.2) \qquad y' + a(x)y^2 + b(x)y + c(x) = 0,$$

Riccati difference equation

$$(0.3) \qquad y(x+1) + a(x)y(x+1)y(x) + b(x)y(x) + c(x) = 0$$

AMS Subject Classification (1980): 39B70
UDK 517.9

(see, for example [6] - [7]), as well as certain well-known functional equations.

In this paper we will investigate the properties of the equation (0.1). It will be shown, in section 1, that (0.1) has a number of similar properties as a standard Riccati differential equation. Also, in section 2, the results from section 1 are applied to different classes of equations of the type (0.1): differential, difference, functional, partial differential, partial difference, etc. A large number of examples illustrate the obtained results.

REMARK 0.1. (i) In further text we will suppose that $(V,\cdot)$ be a group. This additional supposition is introduced only for the technical reasons. Without this supposition the results have a similar form. In all examples this is satisfied.

(ii) Instead of (0.1) we can consider the equation

$$pLu + q(u + \alpha Lu)u + ru + s = 0 \quad (p,q,r,s \in V),$$

which reduces to (0.1).

1. General properties.

In this section we will investigate the general properties of (0.1). Certain properties will be given without a proof, and in other cases only an outline of proof will be noted.

THEOREM 1.1. Substitution

$$u = Lv/av \tag{1.1}$$

reduces (0.1) to the linear second order equation:

$$L^2v + (b(1 - \alpha La/a) - La/a)Lv + c(a + \alpha La)v = 0. \tag{1.2}$$

PROOF. Since $a \neq 0$ and

$$Lu = L(1/a)Lv/v + (vL^2v - (Lv)^2)/v(v+\alpha Lv)(a+\alpha La),$$

then (0.1) becomes (1.2).

THEOREM 1.2. Let u_1 be a particular solution of (0.1). Then the substitution

$$u = 1/z + u_1 \tag{1.3}$$

reduces (0.1) to the linear first order equation:

(1.4) $(\alpha a(u_1+\alpha Lu_1)+\alpha b-1)Lz + (a(u_1+\alpha Lu_1) + au_1+b)z + a = 0$

The proof is omitted.

THEOREM 1.3. Let u_1, u_2, u_3, be different particular solutions of (0.1). Then the general solution of (0.1) is given by

(1.5) $1/(u - u_1) - 1/(u_2 - u_1) = C(1/(u_2 - u_1) - 1/(u_3 - u_1))$,

where $C \in \ker L$ is arbitrary.

PROOF. Since u_1 satisfies (0.1), then $1/(u_2-u_1)$, $1/(u_3-u_1)$ are particular solutions of (1.4) wherefrom follows (1.5).

THEOREM 1.4. The transformation

(1.6) $$u = (Pv + Q)/(Rv + S), \quad P,Q,R,S \in V,$$

reduces (0.1) to the Riccati equation

(1.7) $$Lv + A(v + \alpha Lv)v + Bv + Cv = 0,$$

where

(1.8) $$D = SP - RQ + \alpha(SLP-QLR) + \alpha(aQ(P + \alpha LP) + (bQ + cS)(R + \alpha LR)) = 0.$$

In this case $A,B,C \in V$ are given by:

(1.9) $$\begin{cases} A = (RLP - PLR + aP(P+\alpha LP) + (bP+cR)(R+\alpha LR))/D, \\ B = (RLQ - QLR + SLP - PLS + a(Q(P+\alpha LP) + P(Q+\alpha LQ)) \\ \quad + b(Q(R+\alpha LR) + P(S+\alpha LS)) + c(S(R+\alpha LR) + R(S+\alpha LS)))/D \\ C = (SLQ - QLS + aQ(Q+\alpha LQ) + (bQ+cS)(S+\alpha LS))/D. \end{cases}$$

PROOF. Substituting (1.6) into (0.1) it follows that v satisfies (1.7) with (1.8) and (1.9).

REMARK 1.1. It was shown that the Riccati operator equation has the similar properties as Riccati differential equation (see for example [5], [6], [8] - [11]).

THEOREM 1.5. Let there exist $d \in V$ such that

(1.10) $(1 + \alpha(d-b)/2)L((d-b)/2a) - (d-b)^2/4a + d(d-b)/2a + c = 0$,

then the equation (0.1) reduces to a linear first order equation.

PROOF. Since $u_1 = (d - b)/2a$ is a particular solution of (0.1), then from theorem 1.2 the above result follows.

Specially, if $d = 0$, we get the condition:

$$c = L(b/2a) + b^2/4a - bL(b/a)/4.$$

The condition (1.10) is analogous to well-known Mitrinović's condition for the integrability of Riccati differenatial equation (see [5], [6], [8] - [11]).

The canonical form of operator Riccati equations is

$$(1.11) \qquad LU + U(U + \alpha LU) + F = 0, \; (F \in V).$$

In further text we will give two transformations under which equation (0.1) reduces to the canonical form.

THEOREM 1.6. The transformations

$$(1.12) \qquad u = PU + Q,$$

$$(1.13) \qquad P = 1/a + \alpha Q, \; Q + \alpha LQ = -(L(1/a) + b/a)/2$$

and

$$(1.14) \qquad u = P/U + Q,$$

$$(1.15) \; P = (\alpha b - 1)/c + \alpha Q, \; Q + \alpha LQ = (L(1/c) - b/c - \alpha L(b/c))/2,$$

reduce (0.1) to the canonical form (1.11) with

$$(1.16) \quad F = \frac{LQ + Q(-aL(1/a) + b)/2 + c}{(1/a + \alpha L(1/a)/2 - \alpha b/2a)(1 + \alpha aQ)},$$

$$(1.17) \quad F = \frac{(\alpha b-1)LQ + Q(b - cL((\alpha b-1)/c)/2 + c}{((\alpha b-1)/c + \alpha L((\alpha b-1)/c)/2 - \alpha b/2c)(\alpha b - 1 + \alpha Qc)},$$

respectively.

The proof is omitted.

THEOREM 1.7. The canonical Riccati equation (1.11), under the transformation

$$(1.18) \qquad V = U/M(P - U) - N,$$

where

$$(1.19) \quad \begin{cases} N + \alpha LN = -Lp/2 + q, \quad M = p + \alpha N, \\ p = (P + \alpha LP + \alpha F)/(LP + P(P + \alpha LP) + F), \\ q = (LP + 2F)/(LP + P(P + \alpha LP) + F), \end{cases}$$

reduces to the canonical equation

$$(1.20) \qquad LV + V(V + \alpha LV) + G = 0,$$

where

$$(1.21) \quad G = \frac{(LN + N(-Lp/2p + q) + F/(P + \alpha LP + \alpha F))}{(p + \alpha Lp/2 - \alpha q/2)(1 + \alpha N/p)}.$$

PROOF. After the transformation $U = Pv/(v+1)$, the equation (1.11) becomes

$$(1.22) \qquad Lv + A_1 v(v + \alpha Lv) + A_2 v + A_3 = 0,$$

where

$$A_1 = (LP + P(P + \alpha LP) + F)/(P + \alpha LP + \alpha F),$$

$$A_2 = (LP + 2F)/(P + \alpha LP + \alpha F), \; A_3 = F/(P + \alpha LP + \alpha F).$$

Furthermore the equation (1.22) reduces to the canonical form (1.20) by the transformation $v = MV + N$, where (1.19) and (1.21) hold.

REMARK 1.2. Properties analogous to theorems 1.6 and 1.7 and their applications to the Riccati differential equation can be found in [9] - [11]. For example, starting with the canonical equation (1.11) which is reducible to the linear first order equation, then by using theorem 1.7 it is possible to obtain the infinite sequence of the canonical equaitions with the same property.

2. Examples and remarks.

In this section we will consider certain special cases of the operator L and various special equations of the Riccati type.

(i) Let $L = d/dx$ and V is a set of differentiable real valued functions. Then $\alpha = 0$, $\ker L = R$, and the equation (0.1) becomes the standard Riccati equation (0.2). Then the theorems 1.1 - 1.7 are nothing else but the well-known properties of Riccati equation (see, e.g. [5], [6], [8] - [11]).

(ii) Let $L = A(x,y)\partial/\partial x + B(x,y)\partial/\partial y$, and V is the set of differentiable real functions of two variables. In this case $\alpha = 0$, $\ker L$ is the set of functions of the form $F(w(x,y))$, where F is an arbitrary differentiable function, $w(x,y)=C$ is the general solution of $y = -B(x,y)/A(x,y)$, and (0.1) has the form

(2.1) $A(x,y)u_x + B(x,y)u_y + a(x,y)u^2 + b(x,y)u + c(x,y) = 0.$

(iii) Let $L = D$, D is the Kolosoff's operator defined by $Dw = (u_x - v_y + i(v_x + u_y))/2$, V is the set of complex functions $w = u + iv$, such that u and v are differentiable functions. In this case (0.1) becomes an elliptic system:

$$(2.2)\begin{cases} (u_x - v_y)/2 + a_1(u^2 - v^2) - 2a_2uv + b_1u - b_2v + c_1 = 0 \\ (v_x + u_y)/2 + a_2(u^2 - v^2) - 2a_1uv + b_2u + b_1v + c_2 = 0 \end{cases}$$

or in the complex form:

(2.3) $Dw + a(z,\bar{z})w^2 + b(z,\bar{z})w + c(z,\bar{z}) = 0$

($a = a_1 + ia_2$, $b = b_1 + ib_2$, $c = c_1 + ic_2$).

REMARK 2.1. The facts that the equation (2.1) and system (2.2), or (2.3) have the same properties as Riccati equation (0.2) also follow from the Kečkić's result [7].

(iv) Let V be the set of all real sequences, $L = \Delta$ ($\Delta x_n = x_{n+1} - x_n$). Then $\alpha = 1$, $\ker L = R$, and (0.1) becomes the Riccati difference equation (see, e.g. [6],[7])

(2.4) $x_{n+1} + a_nx_nx_{n+1} + (b_n - 1)x_n + c_n = 0.$

For example, the transformation $x_n = p_ny_n + q_n$, where

$$q_n = -(1/a_n - 1/a_{n-1} + b_{n-1}/a_{n-1})/2,$$

$$p_n = (1/a_n + 1/a_{n-1} - b_{n-1}/a_{n-1})/2,$$

reduces (2.4) to the canonical form:

(2.5) $y_{n+1} - y_n + y_ny_{n+1} + f_n = 0.$

We note, in passing, that in [7], the solution of (2.4) is expressed by using continued fractions.

(v) Let V be the set of real valued functions and $L = \Delta$, ($\Delta u = u(x+1) - u(x)$). In this case, also, $\alpha = 1$, and $\ker L$ is the set of all periodic functions with period 1. Then the equation (0.3) is of the form (0.1).

The canonical equation is

$$u(x+1) - u(x) + u(x+1)u(x) + F(x) = 0.$$

For example, the equation

$$(2.6)\quad u(x+1) - u(x) + u(x+1)u(x) - x^2 + x + 1 = 0,$$

has the particular solution $u_1 = 1/x$, and by the substitution $u = 1/z + u_1$, (2.6) becomes $x\Delta z + (2x + 1)z + 1 = 0$.

The general solution of (2.6) is given by:

$$u = (-1)^{[x]}x^{-1}\Big(P(x) + \sum_{k=o}^{[x]-1} (-1)^k (x-[x]-k)^{-1}(x+1-[x]-k)^{-1}\Big)^{-1} + 1/x$$

where P is an arbitrary periodic function with period 1.

(vi) Let V be the set of real valued functions and L be defined by $Lf(x) = f(wx) - f(x)$, where w is given real function and $wx = w(x)$. Then $\alpha = 1$ and the following functional equation is of the Riccati type:

$$(2.7)\quad f(wx) + a(x)f(wx)f(x) + (b(x) - 1)f(x) + c(x) = 0,$$

(a,b,c are given functions).

Specially, the equation

$$(2.8)\quad f(wx) + f(x) + (2 - x - wx)f(wx)f(x)/(xwx - 1) + (x + wx - 2xwx)/(xwx - 1) = 0,$$

where $w^2 = I$ ($I(x) = x$, i.e. $w(w(x)) = x$), $xwx - 1 = 0$, has the particular solutions $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = wx$, and the general solution of (2.8) is

$$1/(f(x)-1) = 1/(x-1) + (F(x,wx)+F(wx,x))(1/(x-1) - 1/(wx-1)),$$

(F is an arbitrary function).

(vii) Let L be the q-difference operator, i.e. $Lu = u(qx) - u(x)$ ($q = \text{const}$, $q \neq 1$, $q > 0$), and V be the set of real valued functions. Then $\ker L = \{ P(\log_q x) \mid P(t+1) = P(t),\ t \in R \}$ and $= 1$. Then we have the q-difference Riccati equation:

$$(2.9)\quad u(qx) + a(x)u(qx)u(x) + (b(x) - 1)u(x) + c(x) = 0.$$

(viii) Let V be the set of real valued functions of two variables and L be defined by: $Lf(x,y) = f(y,x) - f(x,y)$. Then $\alpha = 1$ and $\ker L = \{ F(x,y) + F(y,x) \mid F \text{ is arbitrary} \}$.

Riccati equation has the form:

$$(2.10)\quad f(y,x) + a(x,y)f(y,x)f(x,y) + (b(x,y)-1)f(x,y) + c(x,y)=0$$

For example, the equation

$$f(x,y) + f(y,x) + (x + y)f(y,x)f(x,y)/(xy - x^2 - y^2) - (x - y)^2(x + y)/(x^2 - xy + y^2) = 0,$$

has the general solution:

$$f(x,y) = (x-y) + y(2y-x)/((2y-x) + (F(y,x) + F(x,y))(y-x)$$

(F is an arbitrary function).

REMARK 2.3. Specially, if a, b, c = const. the equation (2.10) reduces to a particular case of the equation which is treated in [13].

(ix) The well-known equations related to gamma function (see, for example, [14])

$$f(x+1)f(x) = 1/x, \tag{2.11}$$

$$f(x)f(1-x) = x/\sin\pi x, \tag{2.12}$$

are also of the Riccati's type.

We note that particular solutions of (2.11) and (2.12) are given by:

$$f_1(x) = \Gamma(x/2)/\sqrt{2}\,\Gamma((x+1)/2),\ f_2(x) = \Gamma(x),$$

respectively. This gives the possibility for obtaining general solutions of these equations which have the form:

$$f(x) = f_1(x)((P(x)(-1)^{[x]} - 1/2)^{-1} + 1)\ ,$$

$$f(x) = f_2(x)(((F(x) + F(1-x))(-1)^{[x]} - 1/2)^{-1} + 1)$$

respectively, where P is arbitrary periodic function with period 1, F is an arbitrary function.

R e f e r e n c e s

[1] J. D. KEČKIĆ, On some classes of linear equations, Publ. Inst. Math. (Beograd). 24(38)(1978), 89-97.

[2] J. D. KEČKIĆ, On some classes of linear equations, II, Ibid. 26(40)(1979), 135-144

[3] J. D. KEČKIĆ and M. S. STANKOVIĆ, On some classes of linear equations, III, Ibid. 31(45)(1982), 83-85.

[4] J. D. KEČKIĆ, On some classes of linear equations, IV, Ibid. 29(43)(1981), 89-96.

[5] E. KAMKE, Differentialgleichungen. Losungmethoden und Losungen, Leipzig 1942.

[6] L. BRAND, Differential and difference equations, John Wiley & sons. New York - London - Sydney 1966.

[7] J. D. KEČKIĆ, Riccati's difference equation and the solution of linear homogenous second order difference equation, Math. Balkanica 8,17(1978), 145-146.

[8] D. S. MITRINOVIĆ, Theorems relatives a l'equation differentielle de Riccati, C. R. Paris 206 (1938), 411-413.

[9] D. S. MITRINOVIĆ, Quelques propositions relatives a l'equation differentielle de Riccati, Bull. Acad. Sci. Math. Nat. Belgrade. 6(1938), 121-156.

[10] D. S. MITRINOVIĆ and P. M. VASIĆ, Complements au Traite de Kamke XII. Des criteres d'integrabilite de l'equation differentielle de Riccati, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No.175 - No.179 (1967), 15-21.

[11] D. S. MITRINOVIĆ and P. M. VASIĆ, Complements au Traite de Kamke XIII. O kriterijumima integrabilnosti Riccatieve jednačine. Ibid. No.210 - No.228 (1968), 43-48.

[12] J. D. KEČKIĆ, A differential operator and its application to partial differential equations and nonanalytic functions, Ibid. No.329 (1970), 1-47.

[13] D. S. MITRINOVIĆ and P. M. VASIĆ, O jednoj kvadratnoj funkcionalnoj jednačini. Ibid. No.210 - No. 228 (1968), 1-9.

[14] M. KUCZMA, Functional Equations in a Single Variable. Monografie Matematyczne No. 46. Warszawa 1968, 384pp.

Vlajko Lj. Kocić

O UOPŠTENOJ RICCATIJEVOJ JEDNAČINI

Za operatorsku Riccatijevu jednačinu (0.1) je pokazano da ima osobine koje su slične Riccatijevoj diferencionalnoj jednačini. Dobijeni rezultati su primenjeni na različite klase jednačina: diferencijalne, diferencne, funkcionalne, parcijalne diferencijalne, parcijalne diferencne itd. Rezultati su ilustrovani brojnim primerima.

Department of Mathematics
Faculty of Electrical Engineering
University of Belgrade
Bulevar Revolucije 73
11000 Belgrade, Yugoslavia