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Janković Svetlana

ONE GENERAL ITERATIVE METHOD TREATING STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS

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Abstract. In this paper a general iterative method for solving one type of stochastic integrodifferential equation is presented. Sufficient conditions for almost sure convergence of a sequence of iterations to the strong solution of the original equation are given.

1. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which all random variables and random processes are defined. Let $W = (W_t, \mathcal{F}_t)$, $t \in [o,T]$, $T = \text{const} > o$, be a standard k-dimenzional Wiener process adapted to a filtration satisfying the usual conditions.

Let us introduce some notions and conditions which are needed in our discussion. First of all, real functions

$$a : [o,T] \times R_k \times R_m \to R_k \qquad a_n : [o,T] \times R_k \times R_m \to R_k$$

$$b : [o,T] \times R_k \times R_m \to R_k \otimes R_k \qquad b_n : [o,T] \times R_k \times R_m \to R_k \otimes R_k$$

$$\varphi : J \times R_k \to R_m \qquad \varphi_n : J \times R_k \to R_m$$

$$\psi : J \times R_k \to R_m \qquad \psi_n : J \times R_k \to R_m$$

$$J = \{(t,s) : (t,s) \in [o,T] \times [o,T],\ s \leq t\}, \quad n \in N,$$

are assumed to be Borel-measurable with respect to their arguments and they are continuous in t and in (t,s). The matrices b and b_n, $n \in N$, are uniformly nonsingular with the norm $\|\cdot\|$,

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$\|b\| = \sum_{i,j=1}^{k} |b_{ij}|^2$, and $|\cdot|$ is an usual Euclid norm. All functions satisfy the Lipschitz condition and the condition of the restriction on growth, ie. there exist a constant $K > o$ and a positive function $\mu(t,s)$ with $\int_o^T\int_o^T \mu^2(t,s)dsdt < \infty$, such that for all $x',x'' \in R_k$, $y',y'' \in R_m$, $t \in [o,T]$ and $(t,s) \in J$, we have

$$\begin{aligned} &|a(t,x',y')-a(t,x'',y'')| + \|b(t,x',y')-b(t,x'',y'')\| \leq K(|x'-x''| + |y'-y''|), \\ &|\varphi(t,s,x')-\varphi(t,s,x'')| + |\Psi(t,s,x')-\Psi(t,s,x'')| \leq \mu^2(t,s)\cdot|x'-x''|, \\ &|a(t,x,y)| + \|b(t,x,y)\| \leq K(1+|x|+|y|), \\ &|\varphi(t,s,x)| + |\Psi(t,s,x)| \leq K(1+|x|), \end{aligned} \tag{1.1}$$

and analogously for a_n, b_n, φ_n and Ψ_n, $n \in N$.

In the paper [4] is proved that under these conditions there exists an unique strong, almost surely continuous solution of the stochastic integrodifferential equation (shortly SIDE)

$$\begin{aligned} X(t) = X_o + \int_o^t a(s,X(s),\int_o^s \varphi(s,u,X(u))du)ds + \\ + \int_o^t b(s,X(s),\int_o^s \Psi(s,u,X(u))du)dW(s), \quad X(o) = X_o , \end{aligned} \tag{1.2}$$

where X is a random variable measurable with respect to $\mathcal{F}_o$ and $E\{|X_o|^2\} < \infty$. Also, $\sup_t E\{|X(t)|^2\} < \infty$.

2. Main results and proofs

In the present paper one iterative method for solving the SIDE (1.2) will be described. The idea of this investigation goes back to the papers [3] and [5] treating an analogous iterative procedure for solving stochastic differential equation of Ito type.

Now the sequence of iterations, ie. the sequence of stochastic processes $\{X_n, n \in N\} = \{(X_n(t), t \in [o,T]), n \in N\}$ will be formed as follows :

- $X_1 = (X_1(t), t \in [o,T])$ is arbitary stochastic process with $X_1(o) = X_o$ and $\sup_t E\{|X_1(t)|^2\} < \infty$;
- $X_{n+1} = (X_{n+1}(t), t \in [o,T])$ is an unique strong solution of the SIDE

$$(2.1)\qquad X_{n+1}(t) = X_o + \int_0^t a_n(s,X_{n+1}(s),\int_0^s \varphi_n(s,u,X_{n+1}(u))du)ds +$$

$$+ \int_0^t b_n(s,X_{n+1}(s),\int_0^s \psi_n(s,u,X_{n+1}(u))du)dW(s), \quad X_{n+1}(o) = X_o .$$

The main problem is to give some conditions of a closeness of the functions a_n, b_n, φ_n and ψ_n, $n \in N$, with the functions a, b, φ and ψ respectively, such that the sequence of solutions $\{X_n, n \in N\}$ converges almost surely to the solution X as $n \to \infty$.

THEOREM. Let the functions a, b, φ, ψ, a_n, b_n, φ_n and ψ_n, $n \in N$, be defined as the above and let the conditions

$$(2.2)\qquad \sum_{n=1}^{\infty} \sup_{t,x,y} \{ |a(t,x,y)-a_n(t,x,y)| + \| b(t,x,y)-b_n(t,x,y) \| \} < \infty$$

and

$$(2.3)\qquad \sum_{n=1}^{\infty} \sup_{t,s,x} \{ |\varphi(t,s,x)-\varphi_n(t,s,x)| + |\psi(t,s,x)-\psi_n(t,s,x)| \} < \infty$$

be satisfied. Then the sequence of solutions $\{X_n, n \in N\}$ of the equations (2.1) converges almost surely, uniformly in t, $t \in [o,T]$, to the solution X of the equation (1.2) as $n \to \infty$.

PROOF. By subtraction of the equations (1.2) and (2.1), we have

$$(2.4)\qquad X(t) - X_{n+1}(t) = \int_0^t A_n(s)ds + \int_0^t B_n(s)dW(s),$$

where

$$A_n(s) = a(s,X(s),\int_0^s \varphi(s,u,X(u))du) - a_n(s,X_{n+1}(s),\int_0^s \varphi_n(s,u,X_{n+1}(u))du),$$

$$B_n(s) = b(s,X(s),\int_0^s \psi(s,u,X(u))du) - b_n(s,X_{n+1}(s),\int_0^s \psi_n(s,u,X_{n+1}(u))du).$$

Applying the Cauchy-Schwartz inequality and one of the basic properties of stochastic integrals of Ito type (see [2]), we obtain

$$(2.5)\qquad E\{|X(t)-X_{n+1}(t)|^2\} \le 2t\int_0^t E\{|A_n(s)|^2\}ds + 2\int_0^t E\{|B_n(s)|^2\}ds.$$

In order to estimate each of these integrals, we will describe them as

$$A_n(s) = \sum_{i=1}^{4} A_{ni}(s), \qquad B_n(s) = \sum_{i=1}^{4} B_{ni}(s),$$

where

$$A_{n1}(s) = a(s,X(s),\int_0^s \varphi(s,u,X(u))du) - a(s,X_n(s),\int_0^s \varphi_n(s,u,X_n(u))du),$$

$$A_{n2}(s) = a(s,X_n(s),\int_0^s \varphi_n(s,u,X_n(u))du) - a_n(s,X_n(s),\int_0^s \varphi_n(s,u,X_n(u))du),$$

$$A_{n3}(s) = a_n(s,X_n(s),\int_0^s \varphi_n(s,u,X_n(u))du) - a_n(s,X(s),\int_0^s \varphi_n(s,u,X(u))du),$$

$$A_{n4}(s) = a_n(s,X(s),\int_0^s \varphi_n(s,u,X(u))du) - a_n(s,X_{n+1}(s),\int_0^s \varphi_n(s,u,X_{n+1}(u))du),$$

and similarly for $B_n(s)$. Clearly, each of these integrals must be estimated. Denote

$$\varepsilon_n = E\Big\{\sup_t\Big[\Big|a(t,X_n(t),\int_0^t \varphi_n(t,s,X_n(s))ds) - a_n(t,X_n(t),\int_0^t \varphi_n(t,s,X_n(s))ds\Big|^2 + \Big\|b(t,X_n(t),\int_0^t \psi_n(t,s,X_n(s))ds) - b_n(t,X_n(t),\int_0^t \psi_n(t,s,X_n(s))ds)\Big\|^2\Big]\Big\} \tag{2.6}$$

and

$$\delta_n = E\Big\{\sup_{t,s}\Big[|\varphi(t,s,X_n(s)) - \varphi_n(t,s,X_n(s))|^2 + |\psi(t,s,X_n(s)) - \psi_n(t,s,X_n(s))|^2\Big]\Big\}. \tag{2.7}$$

From (1.1) and (2.7), adding some terms, we have

$$|A_{n1}(s)| \leq K\Big[|X(s)-X_n(s)| + \int_0^s \mu(s,u)\cdot|X(u)-X_n(u)|\,du + \delta_n s\Big].$$

Since $\rho_n(t) = E\{|X(t)-X_n(t)|^2\}$ is nonnegative, continuous and integrable function on $[o,T]$, we obtain finally

$$\int_0^t E\{|A_{n1}(s)|^2\}ds \leq 3K^2\int_0^t\Big[\rho_n(s) + s\int_0^s \mu^2(s,u)\cdot\rho_n(u)du\Big]ds + \delta_n^2\frac{K^2t^2}{2}$$

From (2.6) it follows

$$\int_0^t E\{|A_{n2}(s)|^2\}ds \leq \varepsilon_n^2 t.$$

Similarly,

$$\int_0^t E\{|A_{n3}(s)|^2\}ds \leq 2K^2\int_0^t\Big[\rho_n(s) + s\int_0^s \mu^2(s,u)\cdot\rho_n(u)du\Big]ds$$

and

$$\int_0^t E\{|A_{n4}(s)|^2\}ds \leq 2K^2\int_0^t[\rho_{n+1}(s) + s\int_0^s \mu^2(s,u)\rho_{n+1}(u)du]ds.$$

From the preceding relations we get

$$(2.8) \qquad \int_0^t E\{|A_n(s)|^2\}ds \leq 4\sum_{i=1}^{4} \int_0^t E\{|A_{ni}(s)|^2\}ds \leq$$

$$\leq 20K^2\int_0^t[\rho_n(s) + s\int_0^s \mu^2(s,u)\rho_n(u)du]ds +$$

$$8K^2\int_0^t[\rho_{n+1}(s) + s\int_0^s \mu^2(s,u)\rho_{n+1}(u)du]ds + 2\delta_n^2K^2t^3 + 4\varepsilon_n^2 t.$$

It is not difficult to conclude that $\int_0^t E\{|B_n(s)|^2\}ds$ can be estimated as $\int_0^t E\{|A_n(s)|^2\}ds$, because the functions b, ψ, b_n and ψ_n satisfy the same initial conditions as a, φ, a_n and φ_n respectively.

From (2.4) we get

$$\rho_{n+1}(t) \leq 2(t+1)\int_0^t E\{|A_n(s)|^2\}ds.$$

If $\alpha = 2(T+1)20K^2\max\{T,1\}$, $\beta = 2(T+1)4\max\{K^2T^3,T\}$, it follows

$$(2.9) \qquad \rho_{n+1}(t) \leq \alpha\int_0^t[\rho_n(s) + \int_0^s \mu^2(s,u)\rho_n(u)du]ds +$$

$$+ \alpha\int_0^t[\rho_{n+1}(s) + \int_0^s \mu^2(s,u)\rho_{n+1}(u)du]ds + \beta(\delta_n^2 + \varepsilon_n^2).$$

From the last inequality it is not easy to obtain some estimation for $\rho_{n+1}(t)$. Because of that, an estimation for the sum $S_p(t) = \sum_{n=2}^{p}\rho_n(t)$, $p \in N$, will be determined. Since $\sup_t E\{|X(t)|^2\} < \infty$ and $\sup_t E\{|X_1(t)|^2\} < \infty$, it follows that

$$\alpha\int_0^t[\rho_1(t) + \int_0^s \mu^2(t,s)\rho_1(s)ds]dt = c, \quad c=\text{const}<\infty.$$

From (2.9), summing up the terms on the right side and on the left side, adding

$$\alpha\int_0^t[\rho_{p+1}(s) + \int_0^s \mu^2(s,u)\rho_{p+1}(u)du]ds$$

on the right side, we get

$$S_{p+1}(t) \le 2\alpha\int_0^t [S_{p+1}(s) + \int_0^s \mu^2(s,u)S_{p+1}(u)du]ds + \beta\sum_{n=1}^{p}(\delta_n^2 + \varepsilon_n^2) + c$$

If we apply one version of the well-known Gronwall-Bellman inequality, we come to the estimation

$$S_{p+1}(t) \le [\beta\sum_{n=2}^{p}(\delta_n^2 + \varepsilon_n^2) + c]\cdot\exp(2\alpha\int_0^t [1 + \int_0^s \mu^2(s,u)du]ds).$$

Therefore, in accordance with the conditions for the function $\mu(t,s)$ on the interval $[o,T]$, one upper bound for $S_{p+1}(t)$ can be found as

$$S_{p+1}(t) \le c_1\cdot[\beta\sum_{n=1}^{p}(\delta_n^2 + \varepsilon_n^2) + c], \tag{2.1o}$$

where $c_1 > o$ is the corresponding constant.

From (2.4) it follows

$$\sup_t |X(t)-X_{n+1}(t)| \le \int_0^T |A_n(s)|ds + \sup_t |\int_0^t B_n(s)dW(s)| .$$

Using the Cauchy-Schwartz inequality, (2.8), (2.9) and one well-known inequality for stochastic integrals of Ito type (see [2]), we come to the conclusion that

$$E\{\sup_t |X(t)-X_{n+1}(t)|^2\} \le 2[T\int_0^T E\{|A_n(s)|^2\}ds +$$
$$+ 4\int_0^T E\{|B_n(s)|^2\}ds \le \alpha'\int_0^T [\rho_n(s) + \int_0^s \mu^2(s,u)\rho_n(u)du]ds +$$
$$+ \alpha'\int_0^T [\rho_{n+1}(s) + \int_0^s \mu^2(s,u)\rho_{n+1}(u)du]ds + \beta'(\delta_n^2 + \varepsilon_n^2),$$

where α' and β' are some constants similar to α and β. Hence, samming up the terms on the right side and on the left side of the last relation, and from the estimation (2.1o) for $S_{p+1}(t)$, we obtain

$$\sum_{n=1}^{p} E\{\sup_t |X(t)-X_{n+1}(t)|^2\} \le$$
$$\le 2\alpha'\int_0^T [S_{p+1}(s) + \int_0^s \mu^2(s,u)S_{p+1}(u)du]ds + \alpha'\int_0^T [\rho_1(s) +$$
$$+ \int_0^s \mu^2(s,u)\rho_1(u)du]ds + \beta'\sum_{n=1}^{p}(\delta_n^2 + \varepsilon_n^2) \le c_2 + \beta''\sum_{n=1}^{p}(\delta_n^2 + \varepsilon_n^2),$$

where c_2 and β'' are the corresponding constants. According to the Chebyshev's inequality, we have

$$\sum_{n=1}^{\infty} P\{\sup_t |X(t)-X_{n+1}(t)| \geq \varepsilon\} \leq$$

$$\leq \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty} E\{\sup_t |X(t)-X_{n+1}(t)|^2\} = \frac{1}{\varepsilon^2} \lim_{p\to\infty}\sum_{n=1}^{p} E\{\sup_t |X(t)-X_{n+1}(t)|^2\} \leq$$

$$\leq \frac{1}{\varepsilon^2}[c_2 + \beta''\sum_{n=1}^{\infty}(\delta_n^2 + \varepsilon_n^2)] .$$

From the conditions (2.2) and (2.3) and from the notions (2.6) and (2.7) for δ_n and ε_n, it follows that the series $\sum_{n=1}^{\infty}\delta_n^2$ and $\sum_{n=1}^{\infty}\varepsilon_n^2$ are convergent, and hence $\sum_{n=1}^{\infty} P\{\sup_t |X(t)-X_{n+1}(t)| \geq \varepsilon\} < \infty$. By the Borel-Cantelli's lemma and the Weierstrass' uniform convergence theorem, it follows that the sequence of stochastic processes $\{X_n, n \in N\}$ converges almost surely, uniformly in t, $t\in[0,T]$, to the solution X of the SIDE (1.2) as $n\to\infty$, as it was required.

QED

Notice that the proof of the Theorem is different as the proof of the analogous theorem in the paper [3]. The main difference is in finding an estimation for $\rho_{n+1}(t)$ from the reccurence inequation (2.9). It seems that it is not possible to obtain it directly and for this reason the sums are used.

The conditions of the Theorem could be weakened. All functions could be random. Then the conditions (1.1), (2.2) and (2.3) have to be valid almost surely. Moreover, in the iterative procedure, (2.2) and (2.3) could be replaced by

$$(2.11) \qquad \sum_{n=1}^{\infty} E\{\sup_t [|a(t,X_n(t),\int_0^t \varphi(t,s,X_n(s))ds) - a_n(t,X_n(t),\int_0^t \varphi_n(t,s,X_n(s))ds|^2 + \|b(t,X_n(t),\int_0^t \psi(t,s,X_n(s))ds) - b_n(t,X_n(t),\int_0^t \psi_n(t,s,X_n(s))ds\|^2]\} < \infty .$$

Analogously to the consideration in [3], moreover in [5], we use the notion Z-algorithm for this iterative procedure. Since the solution X_{n+1} of the equation (2.1) with the functions a_n, b_n, φ_n and ψ_n is determined, the sequence

$$\{(a_n(t,x,y), b_n(t,x,y), \varphi_n(t,s,x), \Psi_n(t,s,x)), n \in N\}$$

<u>determined sequence for the Z-algorithm</u> is called. Of course, this iterative procedure depends on the choice of the starting process and determined sequence. Because of the fact that the SIDE (1.2) is very complicated for solving, our main problem is to find some determined sequence in order that the SIDE-s (2.1) can be solved.

Now one simple form of determined sequence will be described approximating only the functions a and b with linear functions a_n and b_n, $n \in N$. If a,b, φ and Ψ satisfy the conditions from the Preliminaries, and if $\{\alpha_n(t), n \in N\}$ and $\{\beta_n(t), n \in N\}$ are sequences of continuous, uniformly bounded functions on $[o,T]$, then the sequence of the functions

$$\{(a_n(t,x), b_n(t,x), n \in N\} =$$

$$= \{(\alpha_n(t)\cdot(x-X_n(t)) + a(t,X_n(t),\int_o^t \varphi(t,s,X_n(s))ds),$$

$$\beta_n(t)\cdot(x-X_n(t)) + b(t,X_n(t),\int_o^t \Psi(t,s,X_n(s))ds)), n \in N\}$$

is the determined sequence for the Z-algorithm. The proof is similar as in [3] and it will be omit here. Note that the condition (2.11) is satisfied, because the sum of the series is equal to zero. Also, note that this approximation is one linearisation of the original equation (1.2) and, at least theorethicaly, it is possible to express the solutions of the SIDE-s (2.1) in the known way. It is clear that the usual Picard-Lindelöf method of succesive approximations is a special case of the Z-algorithm, if $\alpha_n(t) = \beta_n(t) \equiv o$, $n \in N$.

It would be very interesting to form some other determined sequences, such that the equations (2.1) could be effectively solved. Of course, it is a very strict requirement and it will be difficult to make such algorithm. Finally, let us note that this iterative method could be applyed to other types of SIDE-s, for example using the results in the paper [1]. Also, it could be extended to SIDE-s involving stochastic integrals with respect to any continuous martingale and martingale measure.

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Janković Svetlana

JEDNA OPŠTA ITERATIVNA METODA ZA REŠAVANJE SLUČAJNIH INTEGRODIFERENCIJALNIH JEDNAČINA

U ovom radu razmatrana je jedna opšta iterativna metoda za rešavanje slučajne integrodiferencijalne jednačine (1.2) pomoću niza slučajnih jednačina (2.1). Dati su dovoljni uslovi za skoro izvesnu konvergenciju niza iteracija ka strogom rešenju jednačine (1.2). Opštost ove metode treba shvatiti u smislu da se za različit izbor koeficijenata jednačina (2.1) dobijaju posebne iterativne metode. Pokazano je da je Picard-Lindelöfova metoda sukcesivnih aproksimacija jedan takav poseban algoritam.

Filozofski fakultet
Ćirila i Metodija 2
18 ooo Niš
Jugoslavija