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A SIMPLIFICATION OF BOHM FORMULA

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Abstract: A simplification of Bohm formula, giving number of different symmetry group categories of E^n space, is realized.

The classification of symmetry group categories of the E^n space and corresponding $G_{n\ldots}$ Bohm symbolism, introduced in the work [1], is extended to the multidimensional case ($n \geq 4$) [2] in accordance with definition:

Let as define the n-dimensional Fedorov(n-space) groups as G_n, their subgroups infinite in m dimensions(m-plane) with the invariant m-dimensional plane as G_{nm} (where $n > m$) and their subgroups with invariant m-dimensional ... and k-dimensional planes inserted into one another in succession (a stright line is considered as a one-dimensional plane, a point is a zero-dimensional plane) as $G_{nm\ldots k}$ (where $n > m > \ldots > k$).

In the same article [2], by use of the n-dimensional Euclidean geometry arguments, the identity relations of the G_{nml} (1) and G_{nmlk} (2,3) categories are derived:

(1) $$G_{nml} = G_{n(n-m+l)l}$$

(2) $$G_{nmlk} = G_{n(n-m+l)lk} \quad , \quad G_{nmlk} = G_{nm(m-l+k)k}$$

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$$(3) \qquad G_{nmlk} = \left\{ \begin{array}{l} G_{nm(m-l+k)k} = G_{n(n-l+k)(m-l+k)k} \\ G_{n(n-m+l)lk} = G_{n(n-m+l)(n-m+k)k} \end{array} \right\} = G_{n(n-l+k)(n-m+k)k}$$

These relations make possible the complete symmetry group categorization, listed for the E^n spaces ($0 \leq n \leq 6$) in the works [2, 3, 4].

In the dissertation of A. F. Palistrant [3], the Bohm formula, allowing the computing of the Z(n) number of different crystallographic symmetry group $G_{n\ldots}$ categories of the E^n space, is stated:

$$Z(n) = \sum_{t=0}^{n} p(n-t) \quad ,$$

where the $n = \dim E^n$ and $p(n-t)$ is the number of different decompositions of the (n-t) number ($0 \leq t \leq n$) in the sum of natural numbers, forming non-increasing sequence, and $p(0) = 1$.

By the application of this formula, the results:

$p(0)=1$	$p(1)=1$	$p(2)=2$	$p(3)=3$	$p(4)=5$	$p(5)=7$	$p(6)=11$
$Z(0)=1$	$Z(1)=2$	$Z(2)=4$	$Z(3)=7$	$Z(4)=12$	$Z(5)=19$	$Z(6)=30$

are obtained [2,3,4].

For example, for different values of the n ($0 \leq n \leq 6$), there are the following categories [2,3,4] :

n=0 $\quad G_0$

n=1 $\quad G_1, G_{10}$

n=2 $\quad G_2, G_{21}, G_{20}, G_{210}$

n=3 $\quad G_3, G_{32}, G_{31}, G_{30}, G_{321}, G_{320} = G_{310}, G_{3210}$

n=4 $\quad G_4, G_{43}, G_{42}, G_{41}, G_{40}, G_{432}, G_{431} = G_{421}, G_{420}, G_{430} = G_{410},$
$G_{4320} = G_{4310} = G_{4210}, G_{4321}, G_{43210}$

n=5 $\quad G_5, G_{54}, G_{53}, G_{52}, G_{51}, G_{50}, G_{510} = G_{540}, G_{520} = G_{530}, G_{521} = G_{541},$
$G_{531}, G_{532} = G_{542}, G_{543}, G_{5210} = G_{5410} = G_{5430}, G_{5310} = G_{5320} = G_{5420},$
$G_{5321} = G_{5421} = G_{5431}, G_{5432}, G_{53210} = G_{54210} = G_{54320}, G_{54321},$
G_{543210}

n= 6 G_{6}, G_{60}, G_{61}, G_{62}, G_{63}, G_{64}, G_{65}, $G_{610} = G_{650}$, $G_{620} = G_{640}$, G_{630}, $G_{621} = G_{651}$, $G_{631} = G_{641}$, $G_{632} = G_{652}$, G_{642}, $G_{643} = G_{653}$, G_{654}, $G_{6210} = G_{6510} = G_{6540}$, $G_{6310} = G_{6320} = G_{6410} = G_{6430} = G_{6520} = G_{6530}$, G_{6420}, $G_{6321} = G_{6521} = G_{6541}$, $G_{6421} = G_{6431} = G_{6531}$, $G_{6432} = G_{6532} = G_{6542}$, G_{6543}, $G_{63210} = G_{65210} = G_{65410} = G_{65430}$, $G_{64210} = G_{64310} = G_{64320} = G_{65310} = G_{65320} = G_{65420}$, $G_{64321} = G_{65321} = G_{65421} = G_{65431}$, G_{65432}, $G_{643210} = G_{653210} = G_{654210} = G_{654310} = G_{654320}$, G_{654321}, $G_{6543210}$

In this work, in order to simplify, the Euler relation (i) [10] is used:

(i) $$p(n)=\sum_{m=1}^{\infty} (-1)^{m+1} (p(n-\frac{1}{2}m(3m-1))+ p(n-\frac{1}{2}m(3m+1))),\ p(0)=1,\ n\in N \text{ and}$$

$p(M)=0$ if $M < 0$, by use of which , the Bohm formula is obtained in the new form :

(ii) $$Z(n)=\sum_{m=1}^{\infty} (-1)^{m+1} (Z(n-\frac{1}{2}m(3m-1))+ Z(n-\frac{1}{2}m(3m+1))) + Z(0)$$

$Z(0)= p(0)=1$, $n\in N$ and $Z(M)=0$ if $M<0$.

Proof:

In accordance with relation (i):

$$Z(n) = \sum_{t=0}^{n} p(n-t) = \sum_{k=0}^{n} p(k) =$$

$$=\sum_{k=1}^{n}\sum_{m=1}^{\infty} (-1)^{m+1} (p(k-\frac{1}{2}m(3m-1)) + p(k-\frac{1}{2}m(3m+1))) + p(0) =$$

$$=\sum_{m=1}^{\infty}\sum_{k=1}^{n} (-1)^{m+1} (p(k-\frac{1}{2}m(3m-1)) + p(k-\frac{1}{2}m(3m+1))) + Z(0) =$$

$$=\sum_{m=1}^{\infty} (-1)^{m+1} (Z(n-\frac{1}{2}m(3m-1)) + Z(n-\frac{1}{2}m(3m+1))) + Z(0)$$

the relation (ii) is obtained ■

The resulting formula (ii) makes possible the simple calculation of the Z(n) numbers, without any use of the p(n) numbers. By the application of this formula, the Z(n) numbers read as follows:

$Z(0)=1$

$Z(1)=Z(0)+Z(0)=2$

$Z(2)=Z(1)+Z(0)+Z(0)=4$

$Z(3)=Z(2)+Z(1)+Z(0)=7$

$Z(4)=Z(3)+Z(2)+Z(0)=12$

$Z(5)=Z(4)+Z(3)=19$

$Z(6)=Z(5)+Z(4)-Z(1)+Z(0)=30$

$Z(7)=Z(6)+Z(5)-Z(2)=45$

$Z(8)=Z(7)+Z(6)-Z(3)-Z(1)+Z(0)=67$

$Z(9)=Z(8)+Z(7)-Z(4)-Z(2)+Z(0)=97$

. . . .

$Z(100)=Z(99)+Z(98)-Z(95)-Z(93)+Z(88)+Z(85)-Z(78)-Z(74)+Z(65)+Z(60)-Z(49)-Z(43)+$
$+Z(30)+Z(23)-Z(8)=1\ 642\ 992\ 568$

As the final result of the mentioned calculation, the list of the Z(n) numbers ($1 \leq n \leq 100$) is completed:

n	Z(n)	n	Z(n)	n	Z(n)	n	Z(n)
1	2	26	11732	51	1535914	76	70826486
2	4	27	14742	52	1817503	77	81446349
3	7	28	18460	53	2147434	78	93578513
4	12	29	23025	54	2533589	79	107427163
5	19	30	28629	55	2984865	80	123223639
6	30	31	35471	56	3511688	81	141227966
7	45	32	43820	57	4125842	82	161734221
8	67	33	53963	58	4841062	83	185072690
9	97	34	66273	59	5672882	84	211616350
10	139	35	81156	60	6639349	85	241783707
11	195	36	99133	61	7760854	86	276046669
12	272	37	120770	62	9061010	87	314934342
13	373	38	146785	63	10566509	88	359042451
14	508	39	177970	64	12308139	89	409038376
15	684	40	215308	65	14320697	90	465672549
16	915	41	259891	66	16644217	91	529784908
17	1212	42	313065	67	19323906	92	602318715
18	1597	43	376326	68	22411641	93	684328892
19	2087	44	451501	69	25965986	94	776998612
20	2714	45	540635	70	30053954	95	881650031

21	3506	46	646193	71	34751159	96	999764335
22	4508	47	770947	72	40143942	97	1132995265
23	5763	48	918220	73	46329631	98	1283193401
24	7338	49	1091745	74	53419131	99	1452423276
25	9296	50	1295971	75	61537395	100	1642992568

The (1), (2), (3) relations induce a large field of problems, concerning the characteristics and distribution of the $G_{n\ldots}$ categories. For example, a natural induced questions are the problem of algorithm, giving exclusively the different $G_{n\ldots}$ categories, without generating repeted categories, as well as the question on the number of different categories, belonging to the $G_{(n,p)}$ category class, consisting of the $G_{n\ldots}$ categories, which except the n-dimensional invariant space mentain invariant the sequence of the p inserted subspaces, i.e. on the number of the categories, with category symbol expressed using the (p+1) index.

As the introductory problems, the two consequences of the (1), (2), (3) relations, giving the criterion of category equality, are discussed.

DEFINITION: All the categories, obtained from the $G_{n\ldots}$ category applying the (1), (2), (3) relations are the equivalents of the $G_{n\ldots}$ category. The $G_{(n,p)}$ category class is a set of the $G_{n\ldots}$ categories, with category symbol expressed using the (p+1) index.

THEOREM: Every $G_{(n,p)}$ category class ($p \geq 0$)*) contains exactly $\left[\frac{n}{p}\right]$ categories, which posses no equivalents.

Proof: The $G_{n\ldots}$ category has no equivalents iff the (1), (2), (3) relations transforme the $G_{n\ldots}$ category into itself. In accordance with (1), (2), (3), that happens iff the relations:

(1') $$m=\frac{n+1}{2}$$

(2') $$l=\frac{m+k}{2}$$

(3') $$n+k = m+1$$

*) For the p = 0, there is only one category G_n.

are satisfied for the every successive three or four indexes, nml and nmlk respectively. The simultaneous validity of the (1'), (2'), (3') relations for the every three and four successive indexes is equivalent with the condition that the n... forme the arithmetic sequence. Since the $\left[\frac{n}{p}\right]$ is the number of decreasing arithmetic sequences, starting with the n index and containing the (p+1) member, the $G_{(n,p)}$ category class contains exactly $\left[\frac{n}{p}\right]$ categories, which posses no equivalents ■

CONSEQUENCE: The $G_{(n,1)}$ category class contains the n different categories, as well as the $G_{(n,2)}$ class the $\frac{\binom{n}{2}+\left[\frac{n}{2}\right]}{2}$ different categories.

The first Consequence statement results directly from the Theorem, and the second one deserves a simple explanation. The $G_{(n,2)}$ category class consists of the $\binom{n}{2}$ categories, containing the $\left[\frac{n}{2}\right]$ categories which posses no equivalents. The remaining $\binom{n}{2}-\left[\frac{n}{2}\right]$ categories forme the $\frac{\binom{n}{2}-\left[\frac{n}{2}\right]}{2}$ pairs of the equivalent categories. Consequently, the $\frac{\binom{n}{2}+\left[\frac{n}{2}\right]}{2}$ is the number of different categories, forming the $G_{(n,2)}$ class.

The next possible application domain of the Bohm formula is the antisymmetry theory, more precisely, it's connection with the Mackay compound groups. There is a mention in dissertation [3], that the same formula Z(n) determines the number of different kinds of compound groups, introduced by Mackay [5] and discussed in the monograph [6]. The compound groups correspond to the generating, simple and multiple antisymmetry groups of the M^m type, without differing the order of the m antisymmetry kinds, generated by antiidentities $e_1, e_2, \dots, e_m$ [6, 7]. In the same way, as in the case of simple and multiple antisymmetry, the compound groups allow the identification of the antiidentity transformation e_i ($1 \leq i \leq m$) with the corresponding (hyper)plane reflection T_i [8, 9], mentaining invariant the E^{m-1} subspace of the E^m space [2]. Because the different categories of $G_{n\dots}$ groups are discussed in the "non-coordinate" E^n space, i.e. without differing the order of coordinats, but only with regard to the invariant E^n space and the sequence of maximal invariant subspaces of the E^n space, connected with incusion relation, the obtained identity of the Z(n) number formulas is a natural consequence of the mentioned identification of the e_i with T_i, i.e. the consequence of the geometric treatment of antisymmetry [2].

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POJEDNOSTAVLJENJE BOHM-OVE FORMULE

Ostvareno je pojednostavljenje Bohm-ove formule, koja daje broj različitih kategorija grupa simetrije prostora E^n.

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