Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 1(11)(1987), 17-22

R. Dimitrijević, Lj. Kočinac

SOME CLASSES OF CONNECTED NEIGHBOURHOOD SPACES

(Received 4. 3. 1987.)

Abstract. In this paper some results about W- and V-topological spaces [1] are extended to a class of neighbourhood spaces.

0. Introduction

Throughout this paper notation and terminology concerning neighbourhood spaces are the same as in the books [4] and [5]. By a neighbourhood space we understand a pair (X,τ) consisting of a non-empty set X and a mapping τ assigning to each set $A\subset X$ a set $\tau A\subset X$ such that the following conditions are satisfied for every $A,B\subset X$:

(i) $\tau\emptyset = \emptyset$; (ii) $A\subset\tau A$; (iii) $\tau(A\cup B) = \tau A\cup\tau B$.

Let (X,τ) be a neighbourhood space. Then

1) Two subsets A and B of X are called τ-separated (or simply, separated) if they are non-empty and $(\tau A\cap B)\cup(A\cap\tau B) = \emptyset$. X is connected if it cannot be represented as the union of two τ-separated subsets.

2) When X is represented as the union of two separated sets A and B, we shall write $X = A + B$. If $X = A + B$ and A contains a set M and B contains a set N, we write $X = A(M) + B(N)$.

3) We shall say that a set $E\subset X$ separates the sets $M,N\subset X$ if there exist two sets $A,B\subset X$ such that $X\setminus E = A(M) + B(N)$.

4) X is called a T_0-space if for any two distinct points x,y in X at least one of the relation $x\notin\tau y$ and $y\notin\tau x$ is satisfied.

AMS Subject Classification (1980): 54A05, 54D05
UDK 513.83

X is a T_1-space if for each $x \in X$ one has $\tau x = x$.

The following two lemmas are well known (see [2]).

0.1. LEMMA. Let C be a connected subspace of a connected neighbourhood space (X,τ). If $X \setminus C = A + B$, then the sets $A \cup C$ and $B \cup C$ are connected.

0.2. LEMMA. Let C be a connected subspace of a connected neighbourhood space (X,τ). If K is a component of the subspace $X \setminus C$, then $X \setminus K$ is connected.

1. Neighbourhood W-spaces

1.1. DEFINITION. A neighbourhood space (X,τ) is called a (neighbourhood) W-space if it is connected and for any two disjoint connected sets $A, B \subset X$ the set $\tau A \cap \tau B$ has at most one point.

The proofs of the following two propositions are easy, and so we omit them.

1.2. PROPOSITION. Every neighbourhood W-space is a T_0-space.

1.3. PROPOSITION. If (X,τ) is a neighbourhood W-space and (Y,σ) is a connected subspace of X, then Y is a W-space.

1.4. THEOREM. A connected neighbourhood space (X,τ) is a W-space if and only if the boundary of each component in the complement of any non-empty connected proper subset of X is a single point.

PROOF. Let X be a W-space. Take an arbitrary non-empty connected proper subset C of X and let K be a component in $X \setminus C$. By Lemma 0.2 the set $X \setminus K$ is connected. Therefore, for disjoint connected sets $X \setminus K$ and K we have $|\tau K \cap \tau(X \setminus K)| = |bd(K)| \leq 1$. If $bd(K) = \emptyset$, then would follow that $X = K + (X \setminus K)$ which contradicts the fact that X is connected. So, $|bd(K)| = 1$.

To prove the converse, suppose, on the contrary, that there exist two disjoint connected sets $A, B \subset X$ with $|\tau A \cap \tau B| \geq 2$. Let $x, y \in \tau A \cap \tau B$, $x \neq y$. If K is the component of $X \setminus A$ containing B, then from $\tau B \subset \tau K$ we have $x, y \in \tau K$. On the other hand, $A \subset X \setminus K$ implies $x, y \in \tau(X \setminus K)$. Hence $x, y \in \tau K \cap \tau(X \setminus K) = bd(K)$ which contradicts our assumption. This means that X is a W-space. The theorem is proved.

The following three theorems prove that the class of W-spaces is wide enough. Recall that a connected neighbourhood space X is called <u>treelike</u> if for any two distinct points $x, y \in X$ there is a point $z \in X$ that separates x and y (see [3]). A connected neighbourhood space X is said to be <u>biconnected</u> if it is not the union of two disjoint nondegenerate connected subspaces (a connected set is called degenerate if it consists of a single point).

1.5. THEOREM. Every biconnected T_1 neighbourhood space is a W-space.

PROOF. Let (X,τ) be a biconnected neighbourhood space and let A and B be disjoint connected subsets of X. If K is a component of $X \setminus A$, then, according to Lemma 0.2, the set $X \setminus K$ is connected. As X is biconnected, then either $|K| = 1$ or $|X \setminus K| = 1$. In the first case $|\tau B| \leq 1$ and in the second case $|\tau A| \leq 1$, i.e. $\tau A \cap \tau B$ has at most one point. Hence X is a W-space.

1.6. THEOREM. Every treelike neighbourhood space (X,τ) is a W-space.

PROOF. Suppose contrary to the statement of the theorem that there are two disjoint connected subsets A and B of X such that $|\tau A \cap \tau B| \geq 2$. Let $x, y \in \tau A \cap \tau B$, $x \neq y$. Since X is treelike we can choose a point $z \in X$ which separates x and y: $X \setminus \{z\} = M(x) + N(y)$. We have two possibilities:

(i) $z \notin \tau A \cup \tau B$. In this case $(\tau A \cup \tau B) \cap M$ and $(\tau A \cup \tau B) \cap N$ make a disconnection of the (connected) set $\tau A \cup \tau B$ which is impossible.

(ii) $z \in \tau A \cup \tau B$. If $z \in \tau A \setminus \tau B$, then $\tau B \cap M$ and $\tau B \cap N$ is a disconnection of the set τB which is a contradiction since the last set is connected. If $z \in \tau A \cap \tau B$, then either $\tau A \setminus \{z\}$ or $\tau B \setminus \{z\}$ would not be connected, which is impossible [5].

In both cases (i) and (ii) we have a contradiction and thus the set $\tau A \cap \tau B$ has at most one point, i.e. X is a W-space.

1.7. THEOREM. If (X,τ) is a connected T_1 neighbourhood space in which the intersection of any two connected subspace is connected, then X is a W-space.

PROOF. Let A and B be disjoint connected subsets of X. Suppose that there are distinct points x and y in the set $\tau A \cap \tau B$.

Then from $A \subset A \cup \{x,y\} \subset \tau A$ and $B \subset B \cup \{x,y\} \subset \tau B$ it follows that the sets $A \cup \{x,y\}$ and $B \cup \{x,y\}$ are connected [5]. By hypothesis, then $(A \cup \{x,y\}) \cap (B \cup \{x,y\}) = \{x,y\}$ is a connected subset of X, which is impossible since X is a T_1-space. So $\tau A \cap \tau B$ has at most one point and the theorem is proved.

2. Neighbourhood V-spaces

Let C be a connected subspace of a connected neighbourhood space (X,τ). A point $x \in C$ is called an end point of C if $C \setminus \{x\}$ is a connected set.

2.1. DEFINITION. A connected T_1 neighbourhood space (X,τ) is called a (neighbourhood) V-space if every connected subset of X has at most one end point.

The simple proof of the following proposition will be omitted.

2.2. PROPOSITION. A connected subspace of a neighbourhood V-space is a V-space.

2.3. THEOREM. A connected T_1 neighbourhood space (X,τ) is a V-space if and only if every connected proper subspace of X has at most one end point.

PROOF. The necessity is obvious, and so we only need to prove the sufficiency. Suppose to the contrary that X has two distinct end points x and y. We shall first prove that in this case the set $X \setminus \{x,y\}$ must be connected. Assume $X \setminus \{x,y\} = A + B$. Since A is both open and closed in the sets $X \setminus \{x\}$ and $X \setminus \{y\}$, by Lemma 0.1 $A \cup \{x\}$ and $A \cup \{y\}$ are connected. Thus $A \cup \{x,y\}$ is a connected proper subset of X with two distinct end points (x and y) which contradicts the assumption. So, $X \setminus \{x,y\}$ is connected. Second, we prove that for any $z \in X \setminus \{x,y\}$ the set $X \setminus \{z\}$ is not connected. Indeed, if $X \setminus \{z\}$ is connected, then $X \setminus \{x,z\}$ and $X \setminus \{y,z\}$ are also connected and therefore $X \setminus \{z\}$ would be a connected proper subset of X with two distinct end points (x and y) which is again a contradiction. The assertion $X \setminus \{z\}$ is not connected for all $z \in X \setminus \{x,y\}$ is proved. We have to consider two cases:

(i) $X \setminus \{z\} = M(x) + N(y)$; (ii) $X \setminus \{z\} = M(x,y) + N$.

(i) The set $M \cup \{z\}$ is connected according to Lemma 0.1. Moreover,

it is a proper subset of X with two distinct end points x and z. Indeed, connectedness of $X\setminus\{x\}$ implies that the set $((M\cup\{z\})\setminus\{x\})\cap(X\setminus\{x\}) = (M\cup\{z\})\setminus\{x\}$ is connected, so that x is an end point of $M\cup\{z\}$. On the other hand, z is an end point of $M\cup\{z\}$ because M is a component in $X\setminus\{z\}$. We have a contradiction.
(ii) In the same way as above one can easily check that the set $M\cup\{z\}$ is a connected proper subset of X having two end points x and y which is again a contradiction.

Hence, in both cases we obtain a contradiction and thus X must have at most one end point. This completes the proof of the theorem.

Now, we shall see that on a neighbourhood V-space (X,τ), in a natural manner, one can introduce a partial order as follows:

$x < y$ iff y is contained in some open component of $X\setminus\{x\}$.

2.4. THEOREM. $<$ is a partial order on (X,τ).

PROOF. (a) $<$ is antisymmetric. Suppose that $x<y$ and $y<x$ hold. Then we have $X\setminus\{x\} = U(y) + A$ and $X\setminus\{y\} = V(x) + B$, where U and V are open components of the sets $X\setminus\{x\}$ and $X\setminus\{y\}$, respectively. According to Lemma 0.1, $A\cup\{x\}$ is connected and since it is contained in $X\setminus\{y\}$ it follows $A\cup\{x\}\subset V$. In a similar way we deduce $B\cup\{y\}\subset U$. Hence, $U\setminus\{y\} = B\cup(U\cap V)$ and $V\setminus\{x\} = A\cup(U\cap V)$. But $\tau B\cap(U\cap V) = (\tau B\cap V)\cap U = \emptyset$ and $B\cap\tau(U\cap V)\subset B\cap\tau V=\emptyset$, so that $U\setminus\{y\} = B + (U\cap V)$. Similarly, $V\setminus\{x\} = A + (U\cap V)$. The sets $(U\cap V)\cup\{x\}$ and $(U\cap V)\cup\{y\}$ are connected by Lemma 0.1, and consequently $(U\cap V)\cup\{x,y\}$ is a connected subset of X with two end points x and y. This contradiction shows that $x = y$.

(b) $<$ is transitive. Let $x<y$ and $y<z$. It is clear that $x\neq z$, because otherwise $x = y$ by (a). There are open connected sets $U,V\subset X$ such that $X\setminus\{x\} = U(y) + A$ and $X\setminus\{y\} = V(z) + B$. Now $x<z$ will follow from the fact $z\in U$. To prove this, suppose, on the contrary, $z\in A$. $A\cup\{x\}$ is connected in X (by Lemma 0.1), and therefore it is connected in $X\setminus\{y\}$ [5]. But then $A\cup\{x\}\subset V$ because V is a component in $X\setminus\{y\}$. Thus $x\in V$. This means that $y<x$ holds, which is a contradiction. So, $z\in U$ and the proof of (b) is complete. The theorem is proved.

2.5. THEOREM. Let (X,τ) be a neighbourhood V-space. Then the set $\{y \in X: y \prec x\}$ is linearly ordered for every $x \in X$.

PROOF. Let $a, b \in \{y \in X: y \prec x\}$. We shall prove that a and b are comparable. Suppose that this is not true. Then there exist open connected sets U and V such that $X \setminus \{a\} = A(b) + U(x)$ and $X \setminus \{b\} = B(a) + V(x)$. The connected set $U \cup \{a\}$ is contained in $X \setminus \{b\}$ and therefore we have $U \cup \{a\} \subset B$ which implies $x \in U$. This contradiction shows that a and b are comparable and the theorem is proved.

REFERENCES

[1] A.E. BROUWER, Treelike spaces and related connected topological spaces, Mathematical Centre Tracts 75, Mathematisch Centrum, Amsterdam, 1977.

[2] E. ČECH Topological Spaces (revised by Z. Frolik and M. Katetov), Czech. Academy of Sciences, Prague, 1966.

[3] R. DIMITRIJEVIĆ and Lj. KOČINAC, On treelike neighbourhood spaces, Matem. vesnik 39(1987)

[4] Z.P. MAMUZIĆ, Introduction to general topology, Noordhoff, Gronningen, 1963.

[5] Z.P. MAMUZIĆ, Koneksni prostori, Matematički institut, knjiga 11, Beograd, 1974.

R. Dimitrijević, Lj. Kočinac

NEKE KLASE POVEZANIH OKOLINSKIH PROSTORA

U radu su neki rezultati u vezi sa topološkim W- i V-prostorima (detaljno izloženi u [1]) prošireni na klasu okolinskih prostora koji zadovoljavaju aksiom distributivnosti (prema terminologiji iz [4]).

Filozofski fakultet
18000 Niš, Jugoslavija