Zbornik radova Filozofskog fakulteta u Nišu
Serija Matematika 1(11)(1987), 11-16

R.S.Dimitrijević

BASE AND SUBBASE OF R-PROXIMITY

(Received 1. 4. 1987.)

Abstract. Two definitions of a proximity base and subbase have been given, one by A. Császar and S. Mrówka [1] and the other by O. Njastad [7], but neither of these definitions is perfectly satisfactory. In [8], P.L.Sharma introduced a new definition of proximity base and subbase removing the fault that the previous definitions had had. In this paper we shall show that the notion of proximity base and subbase can be introduced in the spaces of R-proximity in a similar way.

0. Introduction

A relation δ on the family of all subsets of a set X is called an **R-proximity** on the set X if δ satisfies the following axioms:

(R.1) $A\delta B$ implies $B\delta A$

(R.2) $A\delta(B\cup C)$ if and only if $A\delta B$ or $A\delta C$

(R.3) $A\delta B$ implies $A\neq\emptyset$, $B\neq\emptyset$

(R.4) If $A\cap B\neq\emptyset$ then $A\delta B$

(R.5) If $x\bar{\delta}B$ then there exists a set $E\subset X$ such that $x\bar{\delta}E$ and $X-E\,\bar{\delta}B$.

An R-proximity δ on X is **separated** if $x\delta y$ implies $x=y$. A pair (X,δ) consisting of a set X and an R-proximity δ on the set X is called a **space of R-proximity**.

Every R-proximity δ on a set X induces a topology τ on X. More exactly, the formula

$$\bar{A}=\{x\in X:\ x\delta A\}$$

defines a closure operator on the set X (see [4]). We shall denote the topology induced by an R-proximity δ by $\tau(\delta)$.

AMS Subject Classification (1980): 54E05
UDK 513.83

If δ_1 and δ_2 are two R-proximities on a set X and for all A, $B\subset X$ the relation $A\delta_1 B$ implies the relation $A\delta_2 B$, then we say that the R-proximity δ_1 is **finer** than the proximity δ_2, or that δ_2 is **coarser** than δ_1 and we write $\delta_2 \leqslant \delta_1$.

Let (X,δ_1) and (Y,δ_2) be two spaces of an R-proximity. A mapping f of the set X to the set Y is called proximally continuous mapping with respect to the proximities δ_1 and δ_2 if for any sets $A,B\subset X$ close with respect to δ_1, the images $f(A), f(B)\subset Y$ are close with respect to δ_2.

We shall denote the set of the first n natural numbers by I_n.

1. Base and subbase of R-proximity

Let X be a nonempty set. A binary relation $\mathcal{B}$ defined on the power set of the set X is called an **R-proximity base**, if the following axioms are satisfied:

(RB.1) $A\mathcal{B}B$ implies $B\mathcal{B}A$

(RB.2) If $A\cap B\neq\emptyset$ then $A\delta B$

(RB.3) $A\delta B$ implies $A\neq\emptyset$, $B\neq\emptyset$

(RB.4) If $A\mathcal{B}B$ and $A\subset A^*$, $B\subset B^*$ then $A^*\mathcal{B}B^*$

(RB.5) If $A\bar{\mathcal{B}}B$ then there exists a set $E\subset X$ such that $A\bar{\mathcal{B}}E$ and $X-E\bar{\mathcal{B}}B$.

An R-proximity base is **separated** if $x\mathcal{B}y$ implies $x=y$ for all $x,y\in X$.

1.1. THEOREM. Let $\mathcal{B}$ be an R-proximity base on a set X and let $\delta_{\mathcal{B}}$ be a relation defined on the power set of the set X as follows:

$A\delta_{\mathcal{B}}B$ if and only if for any two finite covers $\{A_i: i\in I_m\}$ and $\{B_j: j\in I_n\}$ of the sets A and B respectively there exists a pair $(i,j)\in I_m\times I_n$ such that $A_i\mathcal{B}B_j$.

Then $\delta_{\mathcal{B}}$ is the coarsest R-proximity on X finer than the relation $\mathcal{B}$. Moreover, the R-proximity $\delta_{\mathcal{B}}$ is separated if and only if the R-proximity base $\mathcal{B}$ is separated.

PROOF: First we shall prove that the relation $\delta_{\mathcal{B}}$ satisfies the axioms (RB.1)-(RB.2). Let $A\delta_{\mathcal{B}}B$ and let $\{A_i: i\in I_m\}$ and $\{B_j: j\in I_n\}$ be the covers of the sets A and B respectively. Then there exists a pair $(i,j)\in I_m\times I_n$ such that $A_i\mathcal{B}B_j$. But then $B_j\mathcal{B}A_i$ and therefore $B\delta_{\mathcal{B}}A$.

To prove the axiom (R.2), let us first assume that $A\bar{\delta}_{\mathcal{B}}(B\cup C)$. Then there exist covers $\{A_i: i\in I_m\}$ and $\{B_j: j\in I_n\}$ of the sets A and $B\cup C$ respectively such that $A_i \bar{\mathfrak{B}} B_J$ for any $(i,j)\in I_m x I_n$. Since $\{B_j: j\in I_n\}$ is the cover of the sets B and C, then $A\bar{\delta}_{\mathfrak{B}}B$ and $A\bar{\delta}_{\mathfrak{B}}C$. To prove the converse, let us suppose that $A\bar{\delta}_{\mathfrak{B}}B$ and $A\bar{\delta}_{\mathfrak{B}}C$. Then there exist covers $\{A_i: i\in I_m\}$, $\{B_j: j\in I_n\}$, $\{P_k: k\in I_r\}$ and $\{Q_l: l\in I_s\}$ of the sets A, B, A and C respectively, such that $A_i \bar{\mathfrak{B}} B_j$ for any $(i,j)\in I_m x I_n$ and $P_k \bar{\mathfrak{B}} Q_l$ for any $(k,l)\in I_r x I_s$. Let $M_{(i,k)} = A_i \cap P_k$, $(i,k)\in I_m x I_r$ and let $\{N_p: p\in I_{n+s}\}$ be a family of the sets defined by

$$N_p = \begin{cases} B_p, & \text{for } 1\leq p\leq n \\ Q_p, & \text{for } n+1\leq p\leq n+s \end{cases}$$

It is obvious that $\{M_{(i,k)}: (i,k)\in I_m x I_n\}$ is a cover of the set A and $\{N_p: p\in I_{n+s}\}$ is a cover of the set $B\cup C$. Then $M_{(i,k)} \bar{\mathfrak{B}} N_p$ holds for any $(i,k)\in I_m x I_n$ and any $p\in I_{n+s}$. This immediately follows from the construction of this covers and the axiom (RB.4). This proves that the axiom (R.2) holds true.

If $A\delta_{\mathfrak{B}}B$, then for any two covers $\{A_i: i\in I_m\}$ and $\{B_j: j\in I_n\}$ of the sets A and B there exist the sets A_i and B_j such that $A_i \mathfrak{B} B_j$. But then $A_i\neq\emptyset$ and $B_j\neq\emptyset$ according to axiom (RB.3). Therefore $A\neq\emptyset$ and $B\neq\emptyset$.

Let us suppose that $A\cap B\neq\emptyset$ and let $\{A_i: i\in I_m\}$ and $\{B_j: j\in I_n\}$ be any two covers of the sets A and B respectively. Then $A_i\cap B_j\neq\emptyset$ for a pair $(i,j)\in I_m x I_n$. But then $A_i \mathfrak{B} B_j$, which proves that $A\delta_{\mathfrak{B}}B$.

To prove the axiom (R.5), let us suppose that $x\bar{\delta}_{\mathfrak{B}}B$. It is obvious that there exists a cover $\{B_j: j\in I_n\}$ of the set B such that $x\bar{\mathfrak{B}}B_j$ for any $j\in I_n$. But then according to axiom (RB.5) for each $j\in I_n$ there exists a set E_j such that $x\bar{\mathfrak{B}}E_j$ and $X-E_j\bar{\mathfrak{B}}B_j$. Let $E=\cup\{E_j: j\in I_n\}$. Since $x\bar{\mathfrak{B}}E_j$, then $x\bar{\delta}_{\mathfrak{B}}E_j$ for any $j\in I_n$. Since we proved that the axiom (R.2) holds for relation $\delta_{\mathfrak{B}}$, then $x\bar{\delta}_{\mathfrak{B}}E$. Moreover, $X-E=\cap(X-E_j)\subset X-E_j$, $j\in I_n$, and therefore $X-E\bar{\mathfrak{B}}B_j$ for any $j\in I_n$ by (RB.4). Hence $X-E\bar{\delta}_{\mathfrak{B}}B_j$ for any $j\in I_n$. From this we get that $X-E\bar{\delta}_{\mathfrak{B}}B$ by axiom (R.2). In this manner we proved that $\delta_{\mathfrak{B}}$ is an R-proximity on X.

Let δ be any R-proximity on X which is finer than $\mathfrak{B}$. If $A\delta B$ then for any two covers $\{A_i: i\in I_m\}$ and $\{B_j: j\in I_n\}$ there exist the sets A_i and B_j such that $A_i\delta B_j$. But then $A_i\mathfrak{B}B_j$. This implies $A\delta_{\mathfrak{B}}B$ by definition of relation $\delta_{\mathfrak{B}}$. This proves that $\delta\geq\delta_{\mathfrak{B}}$. Since $\delta_{\mathfrak{B}}\geq\mathfrak{B}$, we proved that $\delta_{\mathfrak{B}}$ is the coarsest R-proximity finer than the relation $\mathfrak{B}$.

It is obvious that R-proximity $\delta_{\mathfrak{B}}$ is separated if and only if R-proximity base is separated.

Let X be a nonempty set. A binary relation s defined on the power set of the set X is called **R-proximity subbase**, if the following axioms are satisfied:

(RS.1) If $A \cap B \neq \emptyset$ then AsB

(RS.2) If $x \bar{s} B$ then there exists a set $E \subset X$ such that $x \bar{s} E$ and $X-E \bar{s} B$.

An R-proximity subbase is **separated** if the following axiom is satisfied:

(RS.3) If $x \neq y$ and $x \bar{s} y$, then there exist two subset P and Q such that $x \in P$, $y \in Q$ and either $P \bar{s} Q$ or $Q \bar{s} P$.

1.2. **THEOREM.** If s is an R-proximity subbase on a set X, then there exists the coarsest R-proximity δ_s on X finer than the relation s. It is separated if and only if s is separated.

PROOF: Let us define a binary relation $\mathfrak{B}_s$ on the power set of the set X in the following way:

$A \mathfrak{B}_s B$ if and only if $A \neq \emptyset$, $B \neq \emptyset$ and $A^* s B^*$ and $B^* s B^*$ for any two sets $A^* \supset A$, $B^* \supset B$.

It is obvious that $\mathfrak{B}_s \geq s$. Let us prove that $\mathfrak{B}_s$ is an R-proximity base on X. The axiom (RB.1) is obviously satisfied. If $A \cap B \neq \emptyset$, then $A^* \cap B^* \neq \emptyset$ for any $A^* \supset A$ and $B^* \supset B$. Therefore $A^* s B^*$ by (RS.1) which implies $A \mathfrak{B}_s B$. The axiom (RB.3) obviously holds true. To prove the axiom (RB.4), let us suppose that $A \mathfrak{B}_s B$, $A \subset A^*$, $B \subset B^*$ and let us prove that $A^* \mathfrak{B}_s B^*$. Let $A^* \subset A^{**}$ and $B^* \subset B^{**}$. Since $A \mathfrak{B}_s E$, $A \subset A^{**}$ and $B \subset B^{**}$, it follows that either $A^{**} s B^{**}$ or $B^{**} s A^{**}$. This proves that $A^* \mathfrak{B}_s B^*$. To prove the axiom (RB.5), let us suppose that $x \bar{\mathfrak{B}}_s B$. We shall distinguish two cases.

First we shall suppose that $B = \emptyset$. It is obvious that in this case the set $E = \emptyset$ satisfies the condition of the axiom (RB.5).Let us now suppose that $B \neq \emptyset$. Since $x \bar{\mathfrak{B}}_s B$, there exist the sets $A^* \supset \{x\}$ and $B^* \supset B$ such that $A^* \bar{s} B^*$ or $B^* \bar{s} A^*$. If $A^* \bar{s} B^*$, then by the axiom (RS.2) there exists a set $E \subset X$ such that $A^* \bar{s} E$ and $X-E \bar{s} B^*$. This proves that $x \bar{\mathfrak{B}}_s E$ and $X-E \bar{\mathfrak{B}}_s B$ holds true. If $B^* \bar{s} A^*$, in a similar way can be proved that there exists a set $E \subset X$ such that $x \bar{\mathfrak{B}}_s E$ and $X-E \bar{\mathfrak{B}}_s B$. In this way it has been proved that $\mathfrak{B}_s$ is an R-proximity base.

Let δ_s be an R-proximity generated by the R-proximity base $\mathfrak{B}_s$. Then $\delta_s \geq \mathfrak{B}_s \geq s$. Let δ be any R-proximity on X finer than the relation s. According to the definition of $\mathfrak{B}_s$ it is obvious that $\delta \geq \mathfrak{B}_s$. But then $\delta \geq \delta_s$ according to above theorem.

It is easy to see that s is separated if and only if ${}_s$ is separated.

The R-proximity base which is constructed in the proof of the above theorem is the coarsest R-proximity base on X finer than the relation s. Indeed, let $\mathfrak{B} \geqslant s$ be any R-proximity base on X and let us suppose that $A\mathfrak{B}B$. According to axiom (RB.4) $A^{*}\mathfrak{B}B^{*}$ holds for any two sets $A^{*} \supset A$ and $B^{*} \supset B$. Moreover, $B^{*}\mathfrak{B}A^{*}$ by the axiom (RB.1). But then $A^{*}\, s\, B^{*}$ and $B^{*}\, s\, A^{*}$ implies $A\mathfrak{B}_s B$. This proves that $\mathfrak{B} \geqslant \mathfrak{B}_s$. We shall say that R-proximity is generated by the R-proximity subbase s.

2. Some properties of R-proximity spaces

The proofs of several theorems in spaces of R-proximity can be simplified by using the R-proximity base and subbase. Let us mention some properties of R-proximity spaces which are usual for proximity spaces. The proofs of these theorems will be omitted because they are identical to the proofs of the coresponding theorems in [8].

2.1. THEOREM. Let $\{\delta_\alpha : \alpha \in I\}$ be a collection of R-proximities on the set X. Then there exists the coarsest R-proximity on X finer than δ_α for each $\alpha \in I$.

2.2. THEOREM. Let $\{\delta_\alpha : \alpha \in I\}$ be a collection of R-proximities on a set X. Then there exists the finest R-proximity δ on X which is coarser than δ_α for each $\alpha \in I$.

2.3. THEOREM. The family of all R-proximities on a set X is a complete lattice with respect to the order $\leqslant$.

2.4. THEOREM. Let $\{(X_\alpha, \delta_\alpha) : \alpha \in I\}$ be a family of R-proximity spaces and let $X = \prod\{X_\alpha : \alpha \in I\}$. A binary relation $\mathfrak{B}$ defined on the power set of the set X with

$$A\mathfrak{B}B \text{ if and only if } p_\alpha(A)\,\delta_\alpha\, p_\alpha(B) \text{ for each } \alpha \in I$$

is an R-proximity base on the product of R-proximity spaces X .

2.5. THEOREM. Let $\{(X_\alpha, \delta_\alpha) : \alpha \in I\}$ be a collection of R-proximity spaces and let (X, δ_n) be the product of these spaces. A function f from R-proximity space (Y, δ) to the producat (X, δ_Π) is proximally continuous if and only if the composition $p_\alpha \circ f$ is proximally continuous for each $\alpha \in I$.

R E F E R E N C E S

[1] A. Császar et S. Mrówka, Sur la compactification des espaces

de proximité, Fund.Math. 46(1959), 195-207.

[2] E. ČECH, Topological spaces (rev. by Z.Frolik and M.Katetov), Czech.Academy of Sciences, Prague, 1966.

[3] R. ENGELKING, General topology, PWN-Polish Scientific Publishers, Warszawa, 1977.

[4] D. HARRIS, Regular-closed spaces and proximities, Pacif.J.of Math. Vol.34, No.3(1970), 675-685.

[5] Z.P.MAMUZIĆ, Introduction to general topology, Noordhoff, Gronningen, 1963.

[6] S.A.NAIMPALLY and B.D.WARRACK, Proximity spaces, Cambridge Tract no.59, Camb.Univ.Press, 1970.

[7] O. NJASTAD, On Wallman type compactifications, Math.Z., 91, (1966), 267-276.

[8] P.L. SHARMA, Proximity bases and subbases, Pacif. J. of Math. Vol. 37, No. 2, 1971.

R.S.Dimitrijević

BAZA I PREDBAZA R-BLISKOSTI

U prostorima R-bliskosti uvedeni su pojmovi baze i predbaze. Dokazano je da svaka baza (predbaza) generiše jedinstvenu R-bliskost. Navedena su neka svojstva ovih prostora koja se dokazuju primenom uvedenih definicija baze i predbaze.

Filozofski fakultet
ul.Ćirila i Metodija 2
18 000 Niš
Jugoslavija