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INFLATIONS OF SEMIGROUPS AND SEMIRINGS

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Abstract. Inflations of semigroups are studied in [2]. In the present paper another construction for n-inflation of a semigroup will be given. Also, we describe an n-inflation of a semiring.

1.

Let S and T be two disjoint semigroups and suppose that T has a zero element. A semigroup V is said to be an (ideal) extension of S by T if it contains S as an ideal and the Rees factor - semigroup V/S is isomorphic to T. V is a retract extension of S if there exists a homomorphism Φ of V onto S and $\Phi(x)=x$ for all $x \in S$. In this case we call Φ a retraction.

For undefined notions and notaticns we refer to [3].

DEFINITION. A semigroup S is an n-inflation of a semigroup T if $S^{n+1} \subseteq T$ and S is a retract exitension of T.

M.S.Putcha and J.Weissglass, [4] considered an n-inflation of a semigroup, but in another sense.

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THEOREM 1. *Let* T *be a semigroup. To each* $a \in T$ *we associate a family of sets* X_i^a, $i=1,2,\dots,n$ *such that*

$$\text{(1.1)} \qquad a \in X_1^a, \quad X_i^a \cap X_j^b = \emptyset \ \text{ if } \ i \neq j$$

$$X_i^a \cap X_j^b = \emptyset \ \text{ if } \ a \neq b.$$

Let

$$\text{(1.2)} \qquad \Phi_{(i,j)}^{(a,b)} : X_i^a \times X_j^b \to \bigcup_{\nu=1}^{i-1} X_\nu^{ab}, \quad 2 \leq i \leq n$$

$$\Phi_{(1,j)}^{(a,b)}(x,y) = \Phi_{(i,1)}^{(a,b)}(x,b) = ab$$

be functions for which

$$\text{(1.3)} \qquad (\forall s \leq i-1)(\forall t \leq j-1)\, \Phi_{(s,k)}^{(ab,c)}\left(\Phi_{(i,j)}^{(a,b)}(x,y), z\right) = \Phi_{(i,t)}^{(a,bc)}\left(x, \Phi_{(j,k)}^{(b,c)}(y,z)\right)$$

for all $a,b,c \in T$, $2 \leq i \leq n$; $1 \leq j \leq n$.

Let $Y_a = \bigcup_{i=1}^{n} X_i^a$ *and define a multiplication on* $S = \bigcup_{a \in T} Y_a$ *by: for* $x \in Y_a$, $y \in Y_b$,

$$x * y = \Phi_{(i,j)}^{(a,b)}(x,y) \ \text{ if } \ x \in X_i^a, \ y \in X_j^b, \ 1 \leq i,j \leq n.$$

Then $(S,*)$ *is an* n-*inflation of a semigroup* T.

Conversely, every n-*inflation can be so constructed.*

PROOF. Let $x,y,z \in S$. Then there exist $a,b,c \in T$ such that $x \in Y_a$, $y \in Y_b$, $z \in Y_c$, i.e. $x \in X_i^a$, $y \in X_j^b$, $z \in X_k^c$ for some $1 \leq i,j,k \leq n$. Assume that $i,j \neq 1$. Then

$$(x * y) * z = \Phi_{(i,j)}^{(a,b)}(x,y) * z, \quad \Phi_{(i,j)}^{(a,b)}(x,y) \in X_s^{ab}, \ 1 \leq s \leq i-1$$

$$= \Phi^{(ab,c)}_{(s,k)} (\Phi^{(a,b)}_{(i,j)} (x,y),z)$$

$$x*(y*z)=x* \Phi^{(b,c)}_{(j,k)}(y,z) \ , \quad \Phi^{(b,c)}_{(j,k)}(y,z)\in X^{bc}, \quad 1\leqq t\leqq j-1$$

$$= \Phi^{(a,bc)}_{(i,t)} (x, \Phi^{(b,c)}_{(j,k)}(y,z))$$

and by (1.3) we have associativity. If at least of i and j is equal 1, for example $2\leqq i$, $j=1$, then by (1.2) we have that

$$(x*y)*z = \Phi^{(a,b)}_{(i,1)} (x,y)*z, \quad \Phi^{(a,b)}_{(i,1)}(x,y)\in X^{ab}_s, \ 1\leqq s\leqq i-1$$

$$= \Phi^{(ab,c)}_{(s,k)} (\Phi^{(a,b)}_{(i,1)}(x,y),z)$$

$$x*(y*z) = x*(bc) = abc$$

and by (1,3) we have associativity.

Therefore, (S,*) is a semigroup.

Assume $u\in S^{n+1}$, i.e. $u = s_1*s_2*\ldots*s_{n+1}$, $s_r\notin T$, $r=1,2,\ldots,n+1$. Let $s_r\in X^{a_r}_n$, where $a_r\in T$. Then

$$u=s_1*s_2*\ldots*s_{n+1}$$

$$= \Phi^{(a_1,a_2)}_{(n,n)} (s_1,s_2)*s_3*\ldots*s_{n+1}, \quad \Phi^{(a_1,a_2)}_{(n,n)} (s_1,s_2)=u_1\in X^{a_1a_2}_{t_1}, 1\leqq t_1\leqq n-1$$

$$= \Phi^{(a_1a_2,a_3)}_{(t_1,n)} (u_1,s_3)*s_4*\ldots*s_{n+1}, \quad \Phi^{(a_1a_2,a_3)}_{(t_1,n)} (u_1,s_3)=u_2\in X^{a_1a_2a_3}_{t_2}, \ 1\leqq t_2\leqq n-2$$

$$\vdots$$

$$= \Phi^{(a_1a_2\ldots a_{n-1},a_n)}_{(t_{n-2},n)} (u_{n-2},s_n)*s_{n+1}, \quad \Phi^{(a_1a_2\ldots a_{n-1},a_n)}_{(t_{n-2},n)} (u_{n-2},s_n) =$$

$$u_{n-1}\in X^{a_1a_2\ldots a_n}_{t_{n-1}}, \quad 1\leqq t_{n-1}\leqq 1$$

$$= \Phi^{(a_1a_2\dots a_n, a_{n+1})}_{(1,n)} (u_{n-1}, s_{n+1})$$

$$= a_1a_2\dots a_{n+1} \in T.$$

In other cases ($s_r \in X^{a_r}_{k_r}$, $1 \leq k_r < n$) it is clear that $u \in T$. Hence, $S^{n+1} \subseteq T$.

Define a mapping $\Phi : S = \bigcup_{a \in T} Y_a \to T$ by $\Phi(Y_a) = a$. For any $x, y \in S$ there exist $a, b \in T$ such that $x \in Y_a$, $y \in Y_a$, i. e. $x \in X^a_i$, $y \in X^b_j$ for some $1 \leq i, j \leq n$. Now

$$(x*y) = \Phi(\Phi^{(a,b)}_{(i,j)}(x,y)), \quad \Phi^{(a,b)}_{(i,j)}(x,y) \in X^{ab}_k \subseteq Y_{ab}$$

for some $1 \leq k \leq i-1$. If $i=1$, then by (1.2) we have that $\Phi^{(a,b)}_{(i,j)}(x,y) = ab \in Y_{ab}$. Now by the definition of Φ we obtain

$$\Phi(x*y) = ab = \Phi(x)\Phi(y).$$

It is clear that $\Phi(x) = x$ for every $x \in T$ and that T is an ideal of S. Therefore, S is a retract extension of T.

Conversely, let n be the smalest positive integer such that $S^{n+1} \subseteq T$ and let Φ be a retraction of S onto T. Assume the following sets (ideals) of S:

$$A_i = \{x \in S : xS^i \subseteq T\}, \quad i = 1, 2, \dots, n-1$$

It is clear that $T \subseteq A_1 \subseteq A_2 \subseteq \dots \subseteq A_{n-1}$. For $a \in T$ we define the sets: $Y_a = \Phi^{-1}(a)$,

$$X^a_1 = Y_a \cap A_1$$

$$X^a_2 = Y_a \cap (A_2 - A_1)$$

$$\vdots$$

$$X^a_{n-1} = Y_a \cap (A_{n-1} - A_{n-2})$$

$$X^a_n = Y_a \cap (S - A_{n-1}).$$

It is clear that the conditions (1,1) hold for every X^a_i and X^b_j: $1 \leq i, j \leq n$.

If $a \in T$, then $Y_a = \bigcup_{i=1}^{n} X_i^a$ and $S = \bigcup_{a \in T} Y_a$. For $x, y \in S$ There exist $a, b \in T$ such that $x \in A_y$, $y \in Y_b$. By Proposition 1.1. [2] we have that

$$(1.4) \qquad Y_a Y_b \subseteq Y_{ab}$$

Let $x \in X_i^a$, $y \in X_j^b$, $2 \leq i \leq n$, $1 \leq i \leq n$. Then

$$xy\, S^{i-1} \subseteq xS^i \subseteq T$$

and by (1.4) we have that $xy \in X_k^{ab} \subseteq Y_{ab}$ for some $1 \leq k \leq i-1$. Therefore,

$$x \in X_i\ , \quad y \in X_j \ \Rightarrow\ xy \in \bigcup_{\nu-1}^{i-1} X_\nu^{ab}\ .$$

If $i=1$ or $y=b$, then $xy \in Y_{ab} \cap T = \{ab\}$. In this way the functions $\Phi_{(i,j)}^{(a,b)}$ are defined and the conditions (1.3) hold.

2.

A non empty set S with two binary associative operations "+" and "." for which the distributive law holds:

$$x(y+z)=xy+xz, \quad (x+y)z=xz+yz$$

for every $x, y, z \in S$ is a semiring. A subsemiring A of a semiring S is an ideal of S if A is an aditive and multiplicative ideal of S. On S we define a relation ρ_A with: $x \rho_A y \Leftrightarrow x=y$ or $x, y \in A$. Then S/A (or S/ρ_A) is the Rees factor semiring modA . ‾ . It is clear that S/A is a semiring with a zero element and for every $a \in S/A$:

$$(2.1) \qquad a=0=0+a=0=a0=0a.$$

Let H and T be the disjoint semirings and T has a zero ((2.1) holds). A semiring S is an ideal extension of a semiring H by a semiring T if H is an ideal of S and $S/H=T$. Let H be a subsemiring of semirings S and S' and let f be a homomorphism of S into S' leaving every element of H fixed, then f is a H-homomorphism. H is a retract of S if there exists a H-homomorphism f of S onto H, in this case we call f a retraction. If, in addition, H is and ideal of S, then H is a retract ideal of S and S is a retract extension of H.

DEFINITION. A semiring S is an n-inflation of a semiring H if $(n+1)S \subseteq H$, $S^{n+1} \subseteq H$ and S is a retract extension of H.

THEOREM 2. Let $(H,+,.)$ be a semiring. To each $a \in T$ we associate a family of sets X_i^a, $i=1;2,\ldots,n$ such that

(2.2) $$a \in X_1^a\ ,\quad X_i^a \cap X_j^b = \emptyset \ \text{if}\ i \neq j$$

(2.3) $$X_i^a \cap X_j^b = \emptyset \ \text{if}\ a \neq b.$$

Let.

(2.4) $$\Phi_{(i,j)}^{(a,b)} : X_i \times X_j \to \bigcup_{\nu=1}^{i-1} X_\nu^{a+b}\ ,\quad i \geqq 2$$

(2.5) $$\Phi_{(1,j)}^{(a,b)}(x,y) = \Phi_{(i,1)}^{(a,b)}(x,b) = a+b$$

(2.6) $$\Psi_{(i,j)}^{(a,b)} : X_i \times X_j \to \bigcup_{\nu=1}^{i-1} X_\nu^{ab}\ ,\quad i \geqq 2$$

(2.7) $$\Psi_{(1,j)}^{(a,b)}(x,y) = \Psi_{(i,1)}^{(a,b)}(x,b) = ab$$

be functions for which

(2.8) $$(\forall s \leqq i-1)(\forall t \leqq j-1)\ \Phi_{(s,k)}^{(a+b,c)}(\Phi_{(i,j)}^{(a,b)}(x,y),z) = \Phi_{(i,t)}^{(a,b+c)}(x, \Phi_{(j,k)}^{(b,c)}(y,z))$$

(2.9) $$(\forall s \leqq i-1)(\forall t \leqq j-1)\ \Psi_{(s,k)}^{(ab,c)}(\Psi_{(i,j)}^{(a,b)}(x,y),z) = \Psi_{(i,t)}^{(a,bc)}(x, \Psi_{(j,k)}^{(b,c)}(y,z))$$

(2.10) $$(\forall s \leqq j-1)(\forall p,q \leqq i-1)\ \Psi_{(i,s)}^{(a,b+c)}(x, \Phi_{(j,k)}^{(b,c)}(y,z)) =$$

$$\Phi^{(ab,ac)}_{(p,q)}(\Psi^{(a,b)}_{(i,j)}(x,y),\Psi^{(a,c)}_{(i,k)}(x,z))$$

(2.11) $$(\forall s,p\leq i-1)(\forall q\leq j-1)\Psi^{(a+b,c)}_{(s,k)}(\Phi^{(a+b,c)}_{(i,j)}(x,y),z)=$$

$$\Psi^{(a+c),b+c)}_{(p,q)}(\Phi^{(a,c)}_{(i,k)}(x,z),\ \Phi^{(b,c)}_{(j,k)}(y,z))$$

for all $a,b,c\in H$, where $2\leq i\leq n;\ 1\leq j\leq n$

Let $Y_a=\bigcup_{i=1}^{n}A_i^a$ and define two operations $\perp$ and $\circ$ on $S=\bigcup_{a\in H}Y_a$ by :

$$x\perp y=\Phi^{(a,b)}_{(i,j)}(x,y)$$

$$x\in X_i^a,\ y\in X_j^b,\quad 1\leq i,j\leq n$$

$$x\circ y=\Psi^{(a,b)}_{(i,j)}(x,y)$$

Then $(S,\perp,\circ)$ is an n-inflation of a semiring H.

Conversely, every n-inflation of a semiring can be so constructed.

PROOF. Let $x,y,z\in S$. Then there exist $a,b,c\in H$ such that $x\in Y_a$, $y\in Y_b$, $z\in Y_c$, i.e. $x\in X_i^a$, $y\in X_j^b$, $z\in X_k^c$, for some $1\leq i,j,k\leq n$. So

$$x\circ(y\perp z)=x\circ\Phi^{(b,c)}_{(j,k)}(y,z),\quad \Phi^{(b,c)}_{(j,k)}(y,z)\in X_s^{b+c},\ 1\leq s\leq i-1$$

$$=\Psi^{(a,b+c)}_{(i,s)}(x,\Phi^{(b,c)}_{(j,k)}(y,z))$$

$$(x\circ y)\perp(x\circ z)=\Psi^{(a,b)}_{(i,j)}(x,y)\perp\Psi^{(a,c)}_{(i,k)}(x,z),\ \Psi^{(a,b)}_{(i,j)}(x,y)\in X_p^{(ab},$$

$$\Psi^{(a,c)}_{(i,k)}(x,z)\in X_q^{ac},\ 1\leq p,q\leq i-1$$

$$=\Phi^{(ab,ac)}_{(p,q)}(\Psi^{(a,b)}_{(i,j)}(x,y),\Psi^{(a,c)}_{(i,k)}(x,z))$$

and by (2.10) we have that $x\circ(y\perp z)=(x\circ y)\perp(x\circ z)$. In a similar way it can be proved that $(x\perp y)\circ z=(x\circ z)\perp(y\circ z)$. Now by Theorem 1 we have that $(S,\perp,\circ)$ is an n-inflation of a semiring $(H,+,.)$.

Conversely, let n be the smalest positive integer such that $(n+1)S\subseteq H$, $S^{n+1}\subseteq H$ and let Φ be a retraction of S onto H. Assume the following subsemigroups of S:

$$A_i=\{x\in S:\ x+iS\subseteq H\}\ ,\quad B_i=\{x\in S:\ xS^i\subseteq H\},$$

$i=1,\dots,n-1$. It is clear that $H\subseteq A_1\subseteq A_2\subseteq\dots\subseteq A_{n-1}$ and $H\subseteq B_1\subseteq B_2\subseteq\dots\subseteq B_{n-1}$. Let

$$C_i=A_i\cap B_i\ ,\quad i=1,2,\dots,n-1$$

and for any $a\in H$ we define the sets: $Y_a=\Phi^{-1}(a)$.

$X^a_1=Y_a\cap C_1$

$X^a_2=Y_a\cap(C_2-C_1)$

$\vdots$

$X_{n-1}=Y_a\cap(C_{n-1}-C_{n-2})$

$X_n=Y_a\cap(S-C_{n-1})$.

It is clear that the conditions (2.2) and (2.3) hold for every X^a_i and X^b_j, $1\leq i,j\leq n$.

If $a\in H$, then $Y_a=\bigcup_{i=1}^{n}X^a_i$ and so $S=\bigcup_{a\in H}Y_a$. For $x,y\in S$ there exist $a,b\in H$ such that $x\in Y_a$, $y\in Y_b$. So by Theorem 1. [1] we have that

$$(2.12)\qquad Y_a+Y_b\subseteq Y_{a+b}\ ,\quad Y_aY_b\subseteq Y_{ab}.$$

Let $x\in X^a_i$, $y\in X^b_j$, $2\leq i,j\leq n$. Then

$$x+y+(i-1)S\subseteq x+iS\subseteq H,\quad xyS^{i-1}\subseteq xS^i\subseteq H.$$

So by (2.12) $x+y\in X^{a+b}_k\subseteq Y_{a+b}$, $xy\in X^{ab}_k\subseteq Y_{ab}$ for some $1\leq k\leq i-1$.

Therefore,

$$x+y \in \bigcup_{\nu=1}^{i-1} X^{a+b}, \quad xy \in \bigcup_{\nu=1}^{i-1} X_{\nu}^{ab}.$$

If $i=1$ or $y=b$, then

$$x+y \in Y_{a+b} \cap H = \{a+b\}, \quad xy \in Y_{ab} \cap H = \{ab\}.$$

In this way functions $\Phi_{(i,j)}^{(a,b)}$ and $\Psi_{(i,j)}^{(a,b)}$ from (2.4)-(2.7) are defined and the conditions (2.8)-(2.11) hold.

R E F E R E N C E S

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INFLACIJE POLUGRUPA I POLUPRSTENA

U ovom radu daje se jedna nova konstrukcija za inflacije polugrupa različita od one iz [2]. Data je i konstrukcija za n-inflacije poluprstena.

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