Performance Analyses of Recurrent Neural Network Models Exploited for Online Time-Varying Nonlinear Optimization

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Abstract. In this paper, a special recurrent neural network (RNN), i.e., the Zhang neural network (ZNN), is presented and investigated for online time-varying nonlinear optimization (OTVNO). Compared with the research work done previously by others, this paper analyzes continuous-time and discrete-time ZNN models theoretically via rigorous proof. Theoretical results show that the residual errors of the continuous-time ZNN model possesses a global exponential convergence property and that the maximal steady-state residual errors of any method designed intrinsically for solving the static optimization problem and employed for the online solution of OTVNO is $O(\tau)$, where τ denotes the sampling gap. In the presence of noises, the residual errors of the continuous-time ZNN model can be arbitrarily small for constant noises and random noises. Moreover, an optimal sampling gap formula is proposed for discrete-time ZNN model in the noisy environments. Finally, computer-simulation results further substantiate the performance analyses of ZNN models exploited for online time-varying nonlinear optimization.

Keywords: performance analysis, Zhang neural network (ZNN), online time-varying nonlinear optimization (OTVNO), Newton conjugate gradient model.

1. Introduction

Viewed as an essential step of many solutions, the online optimization often arises in mathematics and control theory, and finds its applications in numerical analysis [1], traffic control [2] machine learning [3], robotics [4–6] and animal migration analysis[7]. Because of its fundamental roles, much effort has been devoted to the fast and high accuracy solution of nonlinear optimization problem, and subsequently a great deal of models have been proposed and investigated for solving it [8]. Recursive (or iterative) methods and direct methods are two main techniques for nonlinear optimization, of which the latter ones are prohibitively expensive (or even impossible) for the problem with a large number of variables [9]. Thus, in modern scientific and engineering applications, the solving of nonlinear optimization problem often has no choice but to seek the recursive methods due to the limitation of direct methods [9].

The recurrent neural network (RNN) has received considerable investigation in many scientific and engineering fields, which has several potential advantages in real-time applications (e.g., parallel processing, distributed storage, self adaptation) [4, 10–17]. Therefore, the RNN is generally taken into account as one of the powerful parallel-computational schemes for online solution of various challenging problems [13, 15, 17]. As a novel type of RNN specifically designed for solving time-varying problems, Zhang neural network (ZNN) is able to perfectly track time-varying solution by exploiting the time derivative of time-varying parameters [4, 11, 17–19]. Different ZNN models have been proposed in [4] for solving online time-varying nonlinear optimization (OTVNO) problem in the presence of zero noise.

In implementations of an RNN model, we usually assume that it is free of all kinds of noises or external errors [9]. However, there always exist some realization errors in hardware implementations or disturbances in applications of RNN, which can be deemed as constant noises. Moreover, the environmental interference as well as other external errors can be viewed as the random noises. Sometimes these noises have significant impacts on the accuracy of the RNN for solving time-varying problems, and in some cases, they may cause failure of the solving task. Therefore, it is worth investigating the performance of ZNN models from the control perspective for solving time-varying nonlinear optimization problem with rigorous proof.

The rest of this paper is organized as follows. Section 2 introduces the problem formulation and presents the ZNN models for online solution of time-varying optimization problem. In addition, Section 3 provides the related work done by others and the corresponding analysis. The convergence and robustness analyses of the ZNN models are presented in Section 4. In Section 5, illustrative simulative results are shown to verify the convergence and robustness of the ZNN model for solving time-varying nonlinear optimization problem, which further substantiate the theoretical analysis. Finally, conclusions are drawn in Section 6. Before ending this introductory section, the main contributions of the paper are pointed out below.

– The online time-varying nonlinear optimization problem is investigated. Its solution is obtained using the continuous-time and discrete-time ZNN models with satisfactory performance.
– Theoretical analyses and results for continuous-time ZNN model are presented, which guarantee that its residual error possesses a global exponential convergence property. Moreover, in the presence of noises, its residual error can be arbitrarily small for constant noises and random noises.
– Theoretical analyses and results also show that the maximal steady-state residual errors of any method designed intrinsically for solving the static optimization problem and employed for the online solution of OTVNO is $O(\tau)$. In addition, the optimal sampling gap formula for discrete-time ZNN model in noisy environments is proposed.
– Computer simulation and numerical experiment results are illustrated, which further substantiate the performance analyses of continuous-time and discrete-time ZNN models exploited for online time-varying nonlinear optimization.

2. Problem formulation and ZNN solution

As a basis for further discussion, the problem formulation for online time-varying optimization and continuous-time and discrete-time ZNN models are presented in this section.

2.1. Problem formulation

Let us consider the following time-varying optimization problem in a continuous form, which is the same task problem presented in [4]:

$$\min_{\mathbf{x}(t)\in\mathbb{R}^n} f\big(\mathbf{x}(t),t\big) \in \mathbb{R}, \ \ t\in[0,+\infty), \tag{1}$$

where the second-order differentiable $f(\cdot,\cdot) : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$ denotes a nonlinear mapping function. In addition, the gradient of (1) can be formulated as $\mathbf{g}(\mathbf{x}(t),t) = \partial f(\mathbf{x}(t),t)/\partial\mathbf{x}(t)$.

2.2. ZNN Solution

Defining the error function $\mathbf{e}(t) = \mathbf{g}(\mathbf{x}(t),t)$, we show how to design the corresponding continuous-time ZNN model via the ZNN design formula. Let us define the following evolution for $\mathbf{e}(t)$:

$$\dot{\mathbf{e}}(t) = -\gamma\mathbf{e}(t), \tag{2}$$

where $\gamma > 0$ is a scaling factor. ZNN design formula (2) indicates that $\dot{\mathbf{e}}(t)$ is evaluated as the negative direction of $\mathbf{e}(t)$ such that $\mathbf{e}(t)$ converges to zero, which means that $\mathbf{x}(t)$ converges to the zero point $\mathbf{x}^*(t)$ of $\mathbf{e}(t)$. By expanding the ZNN design formula, the following differential equation of a ZNN model is obtained:

$$H(\mathbf{x}(t),t)\dot{\mathbf{x}}(t) = -\gamma\mathbf{g}(\mathbf{x}(t),t) - g_t'(\mathbf{x}(t),t), \tag{3}$$

where Hessian matrix $H(\mathbf{x}(t),t)$ and time-derivative vector $g_t'(\mathbf{x}(t),t)$ are defined respectively as [4]:

$$H(\mathbf{x}(t),t) = \frac{\partial\mathbf{g}(\mathbf{x}(t),t)}{\partial\mathbf{x}^{\mathrm{T}}(t)} = \frac{\partial^2 f(\mathbf{x}(t),t)}{\partial\mathbf{x}(t)\partial\mathbf{x}^{\mathrm{T}}(t)} \in \mathbb{R}^{n\times n},$$

$$g_t'(\mathbf{x}(t),t) = \frac{\partial\mathbf{g}(\mathbf{x}(t),t)}{\partial t} = \frac{\partial^2 f(\mathbf{x}(t),t)}{\partial\mathbf{x}(t)\partial t} \in \mathbb{R}^n,$$

with superscript $^{\mathrm{T}}$ denoting the transpose of a vector or matrix argument. For the interesting nonsingular situation of $H(\mathbf{x}(t),t)$ that we consider in this paper, (3) is rewritten as

$$\dot{\mathbf{x}}(t) = -H^{-1}(\mathbf{x}(t),t)\left(\gamma\mathbf{g}(\mathbf{x}(t),t) + \frac{\partial\mathbf{g}(\mathbf{x}(t),t)}{\partial t}\right) \tag{4}$$

where $\mathbf{x}(t)$, starting from a randomly-generated initial condition $\mathbf{x}(0) \in \mathbb{R}^n$, denotes the neural state corresponding to the zero point $\mathbf{x}^*(t) \in \mathbb{R}^n$ of OTVNO (1). If Hessian matrix $H(\mathbf{x}(t),t)$ is positive definite [4], then $\mathbf{x}(t)$ is the solution of OTVNO (1). Note
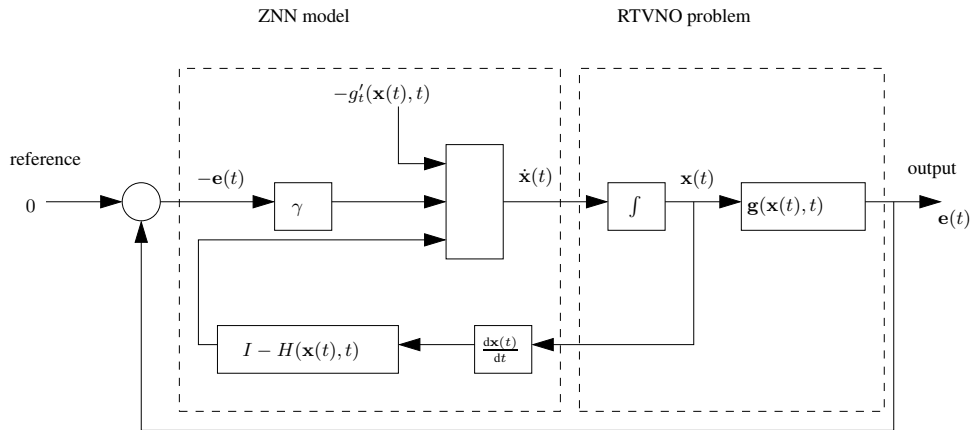
Fig. 1. Realization of continuous-time ZNN model (4) for solving OTVNO problem (1) represented as a feedback control system.

that, to focus on the performance analyses of ZNN models, we consider the situation that $H(\mathbf{x}(t), t)$ is positive definite in this paper.

In terms of OTVNO problem (1), continuous-time ZNN model (4) is an equivalent expansion of ZNN design formula (2). For a better understanding of ZNN design formula (2) as well as continuous-time ZNN model (4), the role of each term in ZNN design formula (2) can be interpreted from the viewpoint of control with its realization represented as a control system shown in Fig. 1. From the figure, we can find that continuous-time ZNN model (4) can be deemed as a generalized proportional-derivative controller with the control input for the derivative part being $\dot{\mathbf{x}}(t)$ and that for the proportional part being $\mathbf{e}(t)$. For such a control system, it will be proven in the ensuing Theorem 2 that $\mathbf{e}(t)$ globally and exponentially converges to zero, which means that the presented continuous-time ZNN model (4) possesses the property of globally exponential stability.

The corresponding discrete-time ZNN model [4] based on Euler forward difference formula can be directly given as

$$\mathbf{x}_{k+1} = \mathbf{x}_k - H^{-1}(\mathbf{x}_k, t_k) \left(hg(\mathbf{x}_k, t_k) + \tau g'_t(\mathbf{x}_k, t_k) \right), \tag{5}$$

where step-size $h = \tau\gamma > 0$, with τ denoting the sampling gap.

3.   Related work

Newton iteration and its various modified models [1] are the classical computational method for solving nonlinear optimization problems. Among them, Newton conjugate gradient (Newton-CG) method is a variant of Newton method and frequently used in scientific and engineering areas, of which the pseudocode is shown in Table 1. Newton-CG method is well suited for high dimensional problems, but it has a weakness. That is, when Hessian matrix is nearly singular, the Newton-CG direction can be long and of poor quality, requiring many function evaluations in the line search and giving only a small reduction in the function. In addition, we have the following theorem to guarantee that the

Table 1. Pseudocode of Newton-CG method

```
Given initial point x₀;
    for k = 0, 1, 2, · · ·
        Define tolerance
         εₖ = min ( 0.5, √||(∂f/∂x)ₖ|| ) ||(∂f/∂x)ₖ||;
        Set z₀ = 0, r₀ = (∂f/∂x)ₖ,
            d₀ = −r₀ = −(∂f/∂x)ₖ;
        for j = 0, 1, 2, · · ·
            if dⱼᵀHₖdⱼ ≤ 0
                if j = 0
                    Return pₖ = −(∂f/∂x)ₖ;
                else
                    Return pₖ = zⱼ;
            Set aⱼ = rⱼᵀrⱼ/dⱼᵀHₖdⱼ;
            Set zⱼ₊₁ = zⱼ + aⱼdⱼ;
            Set rⱼ₊₁ = rⱼ + aⱼHₖdⱼ;
            If ||rⱼ₊₁|| < εₖ
                return pₖ = zⱼ₊₁;
            Set βⱼ₊₁ = rⱼ₊₁ᵀrⱼ₊₁/rⱼᵀrⱼ;
            Set dⱼ₊₁ = −rⱼ₊₁ + βⱼ₊₁dⱼ;
        end  (for)
        Set xₖ₊₁ = xₖ + aₖpₖ, where aₖ satisfies the Wolfe, Goldstein, or Armijo
        backtracking conditions [1] (using aₖ = 1 if possible);
    end
```

maximal steady-state residual error of Newton-CG method for solving OTVNO problem (1) is $O(\tau)$.

THEOREM 1. *Suppose that the Newton-CG method converges to the optimal solution to a static optimization problem within computational time τ. If the Newton-CG method is employed for OTVNO (1), then the maximal steady-state residual error $\|g(\mathbf{x}_{k+1}, t_{k+1})\|_2$ of Newton-CG method is $O(\tau)$.*

Proof. As assumed, the time derivative of $\mathbf{x}_i(t)$ exists, i.e., $\mathrm{d}\mathbf{x}_i(t_k)/\mathrm{d}t = p_i$ at time instant t_k with p_i being a constant. It can be readily derived that $\lim_{\tau \to 0} \Delta\mathbf{x}_i(t_k)/\tau = \mathrm{d}\mathbf{x}_i(t_k)/\mathrm{d}t = p_i$ and $\Delta\mathbf{x}_i(t_k) \approx p_i\tau$. Therefore, $\Delta\mathbf{x}_i(t_k)$ changes in an $O(\tau)$ pattern, i.e., $\Delta\mathbf{x}_i(t_k) = O(\tau)$. Note that, at computational time interval $[k\tau, (k+1)\tau)$, the Newton-CG method converges to the optimal solution $\mathbf{x}_i^*(t_k)$ to the time-varying optimization problem at time instant t_k and $\mathbf{x}_i^*(t_{k+1}) = \mathbf{x}_i^*(t_k) + \Delta\mathbf{x}_i(t_k)$. Thus, at time instant t_{k+1}, the difference between the solution generated by the Newton-CG method and the optimal solution is $\Delta\mathbf{x}(t_k)$, i.e., $\mathbf{x}^*(t_{k+1}) = \mathbf{x}^*(t_k) + \mathbf{O}(\tau) = \mathbf{x}(t_{k+1}) + \mathbf{O}(\tau)$, where $\mathbf{O}(\tau)$ denotes a vector with each element being $O(\tau)$. Then, adopting Taylor expansion

theorem, we obtain

$$
\begin{aligned}
g(\mathbf{x}_{k+1}, t_{k+1}) &= g\left(\mathbf{x}_{k+1}^* + \mathbf{O}(\tau), t_{k+1}\right) \\
&= g(\mathbf{x}_{k+1}^*, t_{k+1}) + H(\mathbf{x}_{k+1}^*, t_{k+1})\mathbf{O}(\tau) + \mathbf{O}(\tau^2) \\
&= H(\mathbf{x}_{k+1}^*, t_{k+1})\mathbf{O}(\tau) + \mathbf{O}(\tau^2) \\
&= H(\mathbf{x}_{k+1}^*, t_{k+1})\mathbf{O}(\tau).
\end{aligned}
$$

Consequently, we further have

$$
\|g(\mathbf{x}_{k+1}, t_{k+1})\|_2 = \|H(\mathbf{x}_{k+1}^*, t_{k+1})\mathbf{O}(\tau)\|_2 = O(\tau).
$$

The proof is thus completed.

As proven in [4], the maximal steady-state residual error of discrete-time ZNN model (5) possesses an $O(\tau^2)$ pattern. Actually, it can be generalized from Theorem 1 that the maximal steady-state residual errors of any method designed intrinsically for solving the static optimization problem and employed for the online solution of OTVNO is $O(\tau)$. The above analysis further demonstrates the superiority of discrete-time ZNN model (5) for OTVNO solving (compared with the conventional methods).

4.   Theoretical analyses

In this section, we prove that $\mathbf{e}(t)$ of continuous-time ZNN model (4) globally and exponentially converges to zero. In addition, in the presence of noises, continuous-time ZNN model (4) is proven to have a satisfactory robust performance. Moreover, an optimal sampling gap formula is proposed for discrete-time ZNN model in the noisy environments.

THEOREM 2. *Continuous-time ZNN model (4), starting with randomly generated initial state* $\mathbf{x}(0) \in \mathbb{R}^n$, *globally and exponentially converges to the theoretical solution to OTVNO (1).*

Proof. ZNN design formula (2) is a compact vector-form equations with its ith element being $\dot{e}_i(t) = -\gamma e_i(t)$. By defining the Lyapunov function candidate $v_i(t) = e_i^2(t)$ with $\dot{v}_i(t) = -2\gamma e_i^2(t)$, it can be readily generalized that continuous-time ZNN model (4) globally converges to theoretical solution $\mathbf{x}^*(t)$ to OTVNO (1).

In addition, from ZNN design formula (2), one can readily conclude that $\mathbf{e}(t) = \mathbf{e}(0)\exp(-\gamma t)$ with $\mathbf{e}(0)$ denoting the initial error of continuous-time ZNN model (4). Therefore, it can be readily generalized that continuous-time ZNN model (4) exponentially converges to theoretical solution $\mathbf{x}^*(t)$ to OTVNO (1).

The proof is thus completed.

In the online solving process of OTVNO (1), noises are the external errors and undesired disturbances, which may misdirect the computational model to evolve along a wrong direction. For any noise, it may be decomposed into the following two parts: the constant part and the rest. Note that ZNN design formula (2) is a linear system, which satisfies the principle of superposition. Therefore, for a non-zero mean random noise, an aggressive output error bound can be obtained by separately considering the constant part and the rest.

To investigate the performance of continuous-time ZNN model (4) in the presence of noises, two theorems on the constant noise and the random noise are presented as follows.

THEOREM 3. Consider that continuous-time ZNN model (4) is polluted with constant noise $\eta(t) = \bar{\eta} \in \mathbb{R}^n$. Continuous-time ZNN model (4) converges towards theoretical solution $\mathbf{x}^*(t)$ to OTVNO (1) with the upper bound of the steady-state residual error $\lim_{t\to\infty} \|\mathbf{e}(t)\|_2$ being $\|\bar{\eta}\|_2/\gamma$. Furthermore, the steady-state residual error $\lim_{t\to\infty} \|\mathbf{e}(t)\|_2$ decreases to zero as γ tends to positive infinity.

Proof. Using Laplace transform to the ith subsystem of the noise-polluted continuous-time ZNN model (4) leads to

$$se_i(s) - e_i(0) = -\gamma e_i(s) + \eta_i(s), \tag{6}$$

i.e.,

$$e_i(s) = \frac{e_i(0) + \eta_i(s)}{s + \gamma}, \tag{7}$$

with the transfer function being $1/(s + \gamma)$, where the pole is $s = -\gamma$. For $\gamma > 0$, it can be readily concluded that this pole locates on the left half-plane, which implies that this system is stable and that the final value theorem applies. Notice that $\eta_i(s) = \bar{\eta}_i/s$ as $\eta_i(t) = \bar{\eta}_i$ amounts to a step signal for constant vector $\bar{\eta}$. Using the final value theorem to (7), we have

$$\lim_{t\to\infty} e_i(t) = \lim_{s\to 0} se_i(s) = \lim_{s\to 0} \frac{s(e_i(0) + \bar{\eta}_i/s)}{s + \gamma} = \frac{\bar{\eta}_i}{\gamma}.$$

Therefore, it can be concluded that $\lim_{t\to\infty} \|\mathbf{e}(t)\|_2 = \|\bar{\eta}\|_2/\gamma$. Furthermore, the steady-state residual error $\lim_{t\to\infty} \|\mathbf{e}(t)\|_2$ decreases to zero as γ tends to positive infinity. The proof is thus completed.

Note that the nonlinear time-varying noise can be deemed as a random noise in the time-varying nonlinear optimization problem solving process, and we have the following theorem for the performance of continuous-time ZNN model (4) in the presence of unknown random noise.

THEOREM 4. Consider that continuous-time ZNN model (4) is polluted with bounded random noise $\eta(t) = \sigma(t) \in \mathbb{R}^n$. Continuous-time ZNN model (4) converges towards theoretical solution $\mathbf{x}^*(t)$ to OTVNO (1) with the upper bound of the steady-state residual error $\lim_{t\to\infty} \|\mathbf{e}(t)\|_2$ being $(\xi\sqrt{n})/\gamma$ with $\xi = \max_{1\le i\le n}\{\max_{0\le\tau\le t} |\sigma_i(\tau)|\}$. Furthermore, the steady-state residual error $\lim_{t\to\infty} \|\mathbf{e}(t)\|_2$ decreases to zero as γ tends to positive infinity.

Proof. Rewrite random-noise-polluted continuous-time ZNN model (4) as

$$\dot{\mathbf{e}}(t) = -\gamma\mathbf{e}(t) + \sigma(t),$$

of which the ith ($\forall i \in 1, 2, \cdots, n$) subsystem can be written as

$$\dot{e}_i(t) = -\gamma e_i(t) + \sigma_i(t). \tag{8}$$

The solution to subsystem (8) can be obtained as

$$e_i(t) = e_i(0)\exp(-\gamma t) + \int_0^t \exp(-\gamma(t-\tau))\sigma_i(\tau)\mathrm{d}\tau.$$

From the triangle inequality, we have

$$|e_i(t)| \leq |e_i(0)\exp(-\gamma t)| + \int_0^t |\exp(-\gamma(t-\tau))||\sigma_i(\tau)|\mathrm{d}\tau.$$

We further have

$$|e_i(t)| \leq |e_i(0)\exp(-\gamma t)| + \frac{1}{\gamma}\max_{0\leq\tau\leq t}|\sigma_i(\tau)|$$

Finally, we have

$$\lim_{t\to\infty}\sup\|e(t)\|_2 \leq \frac{\xi\sqrt{n}}{\gamma},$$

with $\xi = \max_{1\leq i\leq n}\{\max_{0\leq\tau\leq t}|\sigma_i(\tau)|\}$. The proof is thus completed.

It is worth noting that, for discrete-time ZNN model (5), the sampling gap τ, the noise corruption, the round-off errors as well as the truncation errors have influences on the total error, and thus a smaller sampling gap τ does not necessarily generate a smaller total error. Therefore, it is worth investigating the performance of discrete-time ZNN model (5) in the presence of noise and finding the optimal sampling gap.

THEOREM 5. The optimal sampling gap of discrete-time ZNN model (5) is $\tau_{optimal} = 2((\varepsilon+\sigma)/M)^{1/2}$, where ε denotes the maximum absolute value of round-off errors of $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$ in the numerical computations, σ denotes the upper bound of the noise, and M denotes the maximum absolute value of $\dddot{\mathbf{x}}_{k+c}$ with c lying between 0 and 1.

Proof. Based on the Taylor expansion, we have the following rule:

$$\mathbf{x}_{k+1} = \mathbf{x}(k\tau + \tau) = \mathbf{x}_k + \tau\dot{\mathbf{x}}_k - \frac{\tau^2}{2!}\ddot{\mathbf{x}}_{k+c}, \tag{9}$$

where c lies between 0 and 1. Then, with M denoting the maximum absolute value of $\ddot{\mathbf{x}}_{k+c}$, we have the following Euler forward difference formula with truncation error:

$$\dot{\mathbf{x}}_k = \frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{\tau} + \frac{\tau}{2}M, \tag{10}$$

The round-off errors and the noises in the numerical computations can be simplified as following equations:

$$\mathbf{x}_{k+1} = \mathbf{y}_{k+1} + \varepsilon_{k+1} + \xi_{k+1},$$

$$\mathbf{x}_k = \mathbf{y}_k + \varepsilon_k + \xi_k,$$

where $\mathbf{x}_{k+1}$ and $\mathbf{x}_0$ are approximated by numerical values $\mathbf{y}_{k+1}$ and $\mathbf{y}_k$, respectively. In addition, ε_{k+1} and ε_k are the corresponding round-off errors. Besides, ξ_{k+1} and ξ_0 are the corresponding noises with the upper bound being σ.

According to (10), we can obtain

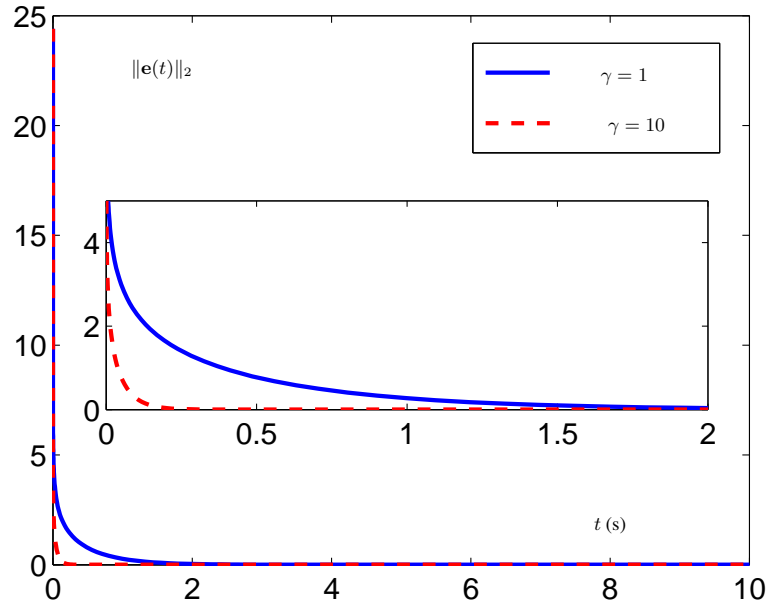$$\dot{\mathbf{x}}_k = \frac{\mathbf{y}_{k+1} - \mathbf{y}_k}{\tau} + E(\mathbf{x}, \tau),$$

Fig. 2. Residual errors of continuous-time ZNN model (4) for solving OTVNO (13) with initial state $\mathbf{x}(0) = [0, 4, -8, -6]^{\mathrm{T}}$.

in which

$$E(\mathbf{x}, \tau) = \frac{\varepsilon_{k+1} + \xi_{k+1} - \varepsilon_k - \xi_k}{\tau} + \frac{\tau M}{2}.$$

Evidently, the total error term $E(\mathbf{x}, \tau)$ contains two parts, i.e., a part due to round-off errors as well as the noise corruption, and a part due to truncation errors.

Considering that ε denotes the maximum absolute value of round-off errors of $\mathbf{x}_k$ and $\mathbf{x}_{k+1}$ and that the upper bound value of ξ_k and ξ_{k+1} is σ, we have

$$|E(\mathbf{x}, \tau)| \leq \frac{2(\varepsilon + \sigma)}{\tau} + \frac{\tau M}{2}. \tag{11}$$

Thus, the value of τ that minimizes the right-hand side of formula (11) is

$$\tau_{\text{optimal}} = 2 \left(\frac{\varepsilon + \sigma}{M} \right)^{1/2}. \tag{12}$$

The proof is thus completed.

5. Illustrative examples

The previous sections have presented the performance analyses of continuous-time ZNN model (4) and discrete-time ZNN model (5) for online solution of OTVNO problem (1).
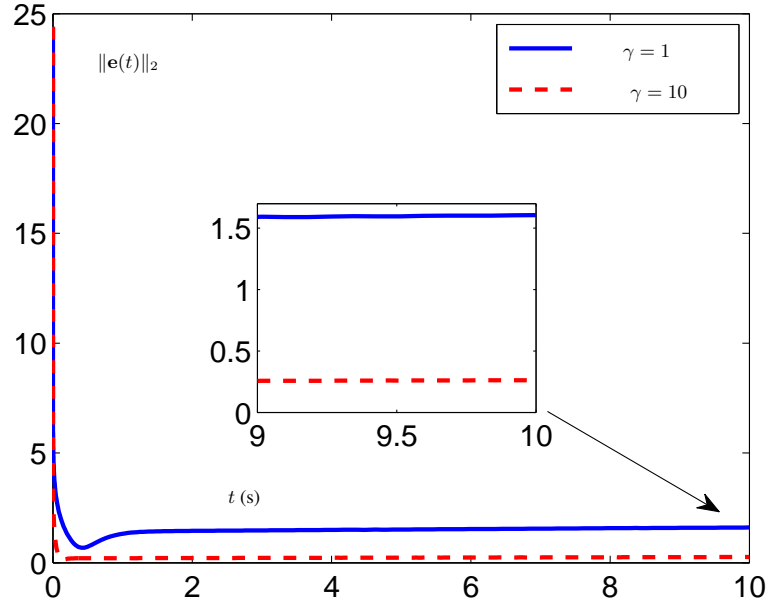
Fig. 3. Residual errors of continuous-time ZNN model (4) for solving OTVNO (13) with initial state $\mathbf{x}(0) = [0, 4, -8, -6]^{\mathrm{T}}$ and constant noise $[1, 1, 1, 1]^{\mathrm{T}}$.

In this section, computer-simulation results and observations are provided to verify the characteristics of these models.

5.1.   Continuous-time ZNN model

Example 1. For illustration and comparison, let us consider the following time-varying nonlinear optimization problem, which is the same problem as in [4]:

$$\min_{\mathbf{x}(t) \in \mathbb{R}^4} f(\mathbf{x}(t), t) = (x_1(t) + t)^2 + (x_2(t) + t)^2 + (x_3(t) - \exp(-t))^2$$
$$+ 0.1(t - 1)x_3(t)x_4(t) - (x_1(t) + \ln(0.1t + 1))(x_2(t) + \sin(t))$$
$$+ (x_1(t) + \sin(t))x_3(t) + (x_4(t) + \exp(-t))^2. \tag{13}$$

Fig. 2 illustrates the residual errors of continuous-time ZNN model (4). As shown in the figure, the residual errors of continuous-time ZNN model (4) with different γ converge rapidly to zero. Particulary, the residual error with $\gamma = 1$ converges to zero within around 2 s and that with $\gamma = 10$ converges to zero within around 0.2 s, which verifies the exponential convergence property proven in Theorem 2. In the implementation of RNN, the corresponding model-implementation error is hard to avoid and can be viewed as the constant bias noise added to the RNN model. It is worth noting that, the constant bias noise degrades the performance of some models and sometimes they fail to solve the problem
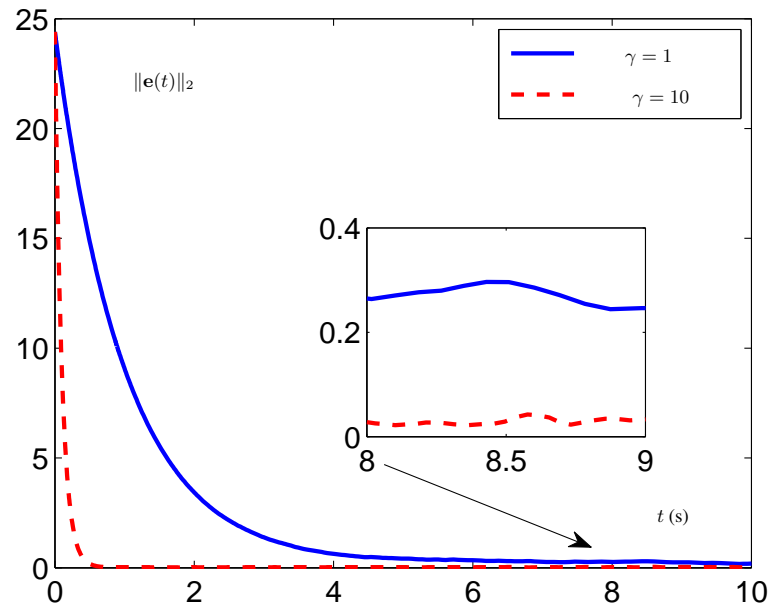
Fig. 4. Residual errors of continuous-time ZNN model (4) for solving OTVNO (13) with initial state $\mathbf{x}(0) = [0, 4, -8, -6]^{\mathrm{T}}$ and constant noise $\sigma(t) \in [-0.5, 0.5]^{4 \times 1}$.

for a large constant bias noise. Therefore, it is worth investigating the performance of continuous-time ZNN model (4) in the presence of constant noise.

As visualized in Fig. 3(a), the residual error of continuous-time ZNN model (4) with $\gamma = 1$ rapidly converges towards zero and remains stable around 1.7. In addition, the residual error of continuous-time ZNN model (4) with $\gamma = 10$ also rapidly converges towards zero and remains stable at the order of 10^{-1}. In summary, these results verify Theorem 3.

In the solving process of OTVNO problem (1), noise is an external error or undesired disturbance, which misdirects the conventional model to evolve along a wrong direction. Numerous methods have been presented and investigated for denoising, such as Wiener filtering and Kalman filtering as well as their extensions. However, by considering the facts that many types of noises may not satisfy the requirements of the denoising method, and that any preprocessing for noise reduction may consume extra time, possibly violating the requirement of real-time computation, conventional denoising methods may be not available for OTVNO problem (1). In addition, the nonlinear time-varying noises can be deemed as random noises. Therefore, it is worth investigating the performance of continuous-time ZNN model (4) in the presence of random noises. The corresponding simulation results are illustrated in Fig. 4.

It can be seen from Fig. 4(a) that the residual error of continuous-time ZNN model (4) with $\gamma = 1$ converges to near zero in around 6 s and remains at an order of 10^{-1}. In
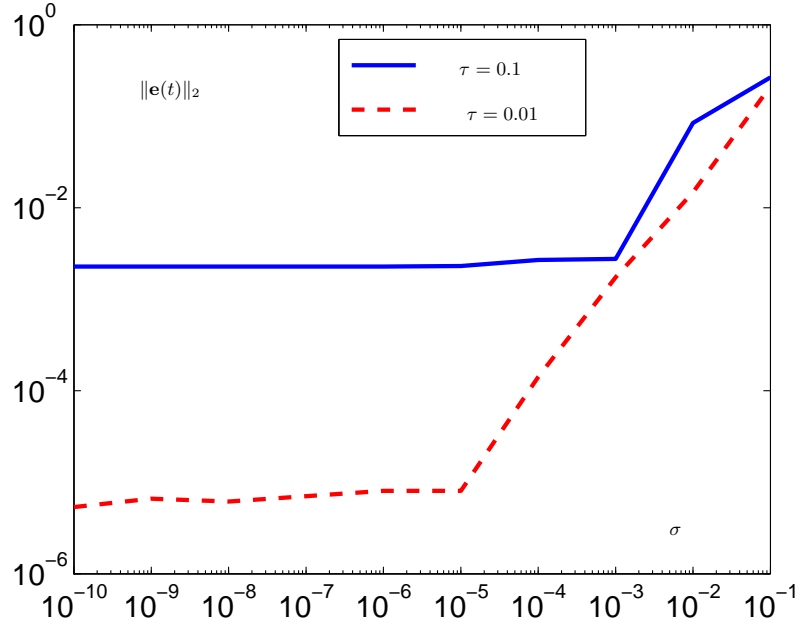
Fig. 5. Means of residual errors of discrete-time ZNN model (5) with $h = 1$ in noisy environments for 10 trials.

addition, the residual error of continuous-time ZNN model (4) with $\gamma = 10$ also rapidly converges towards zero and remains stable at the order of 10^{-2}.

In summary, the above simulation results, i.e., Figs. 2 through 4, have verified the correctness of the presented Theorem 2 through Theorem 4.

5.2.   Discrete-time ZNN model

It is worth investigating the performance of discrete-time ZNN model (5) in noisy environments.

Example 2. Consider the following time-varying nonlinear optimization problem:

$$\min_{\mathbf{x}(t) \in \mathbb{R}^2} f(\mathbf{x}(t), t) = x_1^3(t) - \sin(t)x_1^2(t) + \sin^2(t)x_1(t) + x_2^3(t)$$
$$- \cos(t)x_2^2(t) + \cos^2(t)x_2(t). \tag{14}$$

From the optimal sampling-gap rule $\tau_{\text{optimal}} = 2((\varepsilon + \sigma)/M)^{1/2}$, the zero-mean noises can be counted as having no significant impact on the residual error of discrete-time ZNN model (5) for the upper bound $\sigma < (\tau^2 M - 4\varepsilon)/4$. For OTVNO problem (14), the maximum absolute value of the 3th time-derivative of the element is 1 (e.g., for $\sin(t)$, its 3th time-derivative $|-\cos(t)| \le 1$). In addition, floating-point numbers have limited precision in computer, e.g., the minimum precision of floating-point number eps in MATLAB

environment is of order 10^{-16} (i.e., 2^{-52}). Thus, we have $M = 1$ and $\varepsilon = 2.2 \times 10^{-16}$. The values of σ that do not influence the performance of discrete-time ZNN model (5) are 2.5×10^{-3} and 2.5×10^{-5} corresponding to using sampling gap $\tau = 0.1$ and 0.01, respectively. Means of residual errors of discrete-time ZNN model (5) with zero-mean noises for 10 trials are shown in Fig. 5. As seen from the figure, starting with the randomly-generated initial state, the performance of discrete-time ZNN model (5) does not reduce for $\sigma < 10^{-3}$ with $\tau = 0.1$, and for $\sigma < 10^{-5}$ with $\tau = 0.01$. In addition, for the upper bound $\sigma > (\tau^2 M - 4\varepsilon)/4$, it can be observed from the figure that the noises have remarkable impacts on the performance of discrete-time ZNN model (5), which means that some noise reduction technologies can be considered for better performance. For example, as often done in real-world implementation of the proportional-integral-derivative controller, it is preferable to pass the signals through the low-pass filter before they go to discrete-time ZNN model (5) to reduce the sensitivity of discrete-time ZNN model (5) to noise.

6.   Conclusions

In this paper, the Zhang neural network (ZNN) has been presented and investigated for online time-varying nonlinear optimization (OTVNO). Compared with the research work done previously by others, this paper has analyzed continuous-time and discrete-time ZNN models theoretically via rigorous proof. Theoretical results have shown that the maximal steady-state residual errors of the continuous-time ZNN model possesses a global exponential convergence property and that the maximal steady-state residual errors of any method designed intrinsically for solving the static optimization problem and employed for the online solution of OTVNO is $O(\tau)$. Moreover, in the presence of noises, the residual errors of the continuous-time ZNN model can be arbitrarily small for constant noises and random noises. Finally, computer-simulation results have further substantiated the performance analyses of ZNN models exploited for online time-varying nonlinear optimization.

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