Induced intuitionistic fuzzy ordered weighted averaging - weighted average operator and its application to business decision-making

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Abstract. We present the induced intuitionistic fuzzy ordered weighted averaging-weighted average (I-IFOWAWA) operator. It is a new aggregation operator that uses the intuitionistic fuzzy weighted average (IFWA) and the induced intuitionistic fuzzy ordered weighted averaging (I-IFOWA) operator in the same formulation. We study some of its main properties and we have seen that it has a lot of particular cases such as the IFWA and the intuitionistic fuzzy ordered weighted averaging (IFOWA) operator. We also study its applicability in a decision-making problem concerning strategic selection of investments. We see that depending on the particular type of I-IFOWAWA operator used, the results may lead to different decisions.

Keywords: Intuitionistic fuzzy sets, weighted average, IOWA operator, decision-making.

1.   Introduction

Different types of aggregation operators are found in the literature for aggregating numerical data information [1-4].The weighted average (WA) is one of the most common aggregation operators found in the literature. It can be used in a wide range of different problems including statistics, economics and engineering. Another interesting aggregation operator is the ordered weighted averaging (OWA) operator introduced by Yager [5], whose prominent characteristic is the reordering step. The OWA operator provides a parameterized family of aggregation operators that includes as special cases the maximum, the

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minimum and the average criteria. Since its appearance, the OWA operator has been used in a wide range of applications [6-16].

A very practical extension of the OWA operator is the induced OWA (IOWA) operator [17]. The IOWA operator differs in that the reordering step is not developed with the values of the arguments but can be induced by another mechanism such that the ordered position of the arguments depends upon the values of their associated order-inducing variables. The IOWA operator has been studied by different authors in recent years [18-27].

Recently, some authors have tried to unify the OWA operator with the WA in the same formulation. It is worth mentioning the work developed by Torra [9] with the introduction of the weighted OWA (WOWA) operator and the work of Xu and Da [4] about the hybrid averaging (HA) operator. Both models arrived to an unification between the OWA and the WA because both concepts were included in the formulation as particular cases. However, as it has been studied by Merigó [20], these models seem to be a partial unification but not a real one because they can unify them but they cannot consider how relevant these concepts are in the specific problem considered. For example, in some problems we may prefer to give more importance to the OWA operator because we believe that it is more relevant and vice versa. To overcome this issue, Merigó [20] has developed the ordered weighted averaging weighted averaging (OWAWA) operator that unifies the OWA and the WA in the same formulation considering the degree of importance that each concept may have in the problem. Merigó [20] also presented a new operator that unifies the OWA operator with the WA when we assess the information with induced aggregation operators called the induced ordered weighted averaging–weighted average (IOWAWA) operator. The main advantage of this approach is that it unifies the IOWA and the WA taking into account the degree of importance that each concept has in the formulation. Thus, we are able to consider situations where we give more or less importance to the IOWA and the WA depending on our interests on the problem analyzed. Note that by using the IOWA we are considering complex reordering processes affected by different factors such as the degree of optimism, time pressure and psychological aspects. By using the WA we are considering the subjective probability of the decision-maker against the possibility that each state of nature will occur.

Usually, when using the IOWAWA and the above aggregation operators, it is assumed that the available information is clearly known and can be assessed with exact numbers. However, in the real-life world, due to the increasing complexity of the socio-economic environment and the lack of knowledge or data about the problem domain, exact numerical is sometimes unavailable. Thus, the input arguments may be vague or fuzzy in nature. Atanassov [28] defined the notion of an intuitionistic fuzzy set (IFS), whose basic elements are intuitionistic fuzzy numbers (IFNs) [10, 24, 29], each of which is composed of a membership degree and a non-membership degree. In many practical situations, particularly in the process of group decision making under uncertainty, the experts may come from different research areas and thus have different backgrounds and levels of knowledge, skills, experience, and personality. The experts may not have enough expertise or possess a sufficient level of knowledge to precisely express their preferences over the objects, and then, they usually have some uncertainty in providing their preferences, which makes the results of cognitive performance

exhibit the characteristics of affirmation, negation, and hesitation. In such cases, the data or preferences given by the experts may be appropriately expressed in IFNs. For example, in multi-criteria decision-making problems, such as personnel evaluations, medical diagnosis, project investment analysis, etc., each IFN provided by the expert can be used to express both the degree that an alternative should satisfy a criterion and the degree that the alternative should not satisfy the criterion. The IFN is highly useful in depicting uncertainty and vagueness of an object, and thus can be used as a powerful tool to express data information under various different fuzzy environments which has attracted great attentions [29-39].

The aim of this paper is to extend the IOWAWA operator to accommodate the intuitionistic fuzzy situations. For doing so, we present a new intuitionistic fuzzy aggregation operator called the induced intuitionistic fuzzy ordered weighted averaging weighted average (I-IFOWAWA) operator, which unifies the IOWA operator with the WA when the available information is uncertain and can be assessed with IFNs. The main advantage of this approach is that it unifies the OWA and the WA taking into account the degree of importance of each case in the formulation and considering that the information is given with IFN. Thus, we are able to consider situations where we give more or less importance to the IFOWA and the IFWA depending on our interests and the problem analyzed. Furthermore, by using the I-IFOWAWA, we are able to use a complex reordering process in the OWA operator in order to represent complex attitudinal characters. We also study different properties of the I-IFOWAWA operator and different particular cases.

We also analyze the applicability of the new approach and we see that it is possible to develop an astonishingly wide range of applications. For example, we can apply it in a lot of problems about statistics, economics and decision theory. In this paper, we focus on a decision-making problem concerning strategic selection of investments. The main advantage of the I-IFOWAWA in these problems is that it is possible to consider the subjective probability (or the degree of importance) and the attitudinal character of the decision maker at the same time.

This paper is organized as follows. In Section 2 we briefly review some basic concepts. Section 3 presents the I-IFOWAWA operator and analyzes different families of I-IFOWAWA operators in Section 4. In Section 5 we develop an application of the new approach. Section 6 summarizes the main conclusions of the paper.

2.   Preliminaries

In this Section, we briefly review some basic concepts about the intuitionistic fuzzy sets, the IOWA and the IOWAWA operator.

2.1.   Intuitionistic fuzzy sets

Intuitionistic fuzzy set (IFS) introduced by Atanassov [28] is an extension of the classical fuzzy set, which is a suitable way to deal with vagueness. It can be defined as follows.

Definition 1. Let a set $X = \{x_1, x_2, ..., x_n\}$ be fixed, an IFS A in X is given as following:

$$A = \{\langle x, \mu_A(x), v_A(x)\rangle | x \in X\}. \tag{1}$$

The numbers $\mu_A(x)$ and $v_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x to the set A, $0 \le \mu_A(x) + v_A(x) \le 1$, for all $x \in X$. The pair $(\mu_A(x), v_A(x))$ is called an intuitionistic fuzzy number (IFN) and each IFN can be simply denoted as $\alpha = (\mu_\alpha, v_\alpha)$, where $\mu_\alpha \in [0,1]$, $v_\alpha \in [0,1]$, $\mu_\alpha + v_\alpha \le 1$.

Additionally $S(\alpha) = \mu_\alpha - v_\alpha$ and $H(\alpha) = \mu_\alpha + v_\alpha$ are called the score and accuracy degree of α respectively.

For any three IFNs $\alpha = (\mu_\alpha, v_\alpha), \alpha_1 = (\mu_{\alpha_1}, v_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, v_{\alpha_2})$, the following operational laws are valid [10, 29].

(1) $\alpha_1 + \alpha_2 = (\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1} \cdot \mu_{\alpha_2}, v_{\alpha_1} \cdot v_{\alpha_2})$

(2) $\lambda\alpha = (1 - (1 - \mu_\alpha)^\lambda, v_\alpha{}^\lambda)$

To compare two IFNs α_1 and α_2, Xu and Yager [29] introduced a simple method as below:

- If $S(\alpha_1) < S(\alpha_2)$, then $\alpha_1 < \alpha_2$;

- If $S(\alpha_1) = S(\alpha_2)$, then

(1) If $H(\alpha_1) < H(\alpha_2)$, then $\alpha_1 < \alpha_2$;

(2) If $H(\alpha_1) = H(\alpha_2)$, then $\alpha_1 = \alpha_2$.

2.2. The Induced OWA Operator

The IOWA operator is an extension of the OWA operator. The main difference is that the reordering step is not carried out with the values of the argument a_i. In this case, the reordering step is developed with order-inducing variables that reflect a more complex reordering process. The IOWA operator also includes as particular cases maximum, minimum and average criteria. The IOWA operator can be defined as follows:

Definition 2. An IOWA operator of dimension n is a mapping *IOWA:* $R^n \times R^n \to R$ that has an associated weighting W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:

$$IOWA\left(\langle u_1, a_1 \rangle, ..., \langle u_n, a_n \rangle\right) = \sum_{j=1}^{n} w_j b_j .$$ (2)

where b_j is a_i value of the IOWA pair $\langle u_i, a_i \rangle$ having the jth largest u_i, u_i is the order inducing variable and a_i is the argument variable.

2.3. The intuitionistic fuzzy OWA operator

The intuitionistic fuzzy OWA (IFOWA) operator was introduced by Xu.[26] It is an extension of the OWA operator for uncertain situations where the available information can be assessed with IFNs. Let Ω be the set of all IFNs, it can be defined as follows:

Definition 3. Let $\alpha_i = (\mu_i, v_i)(i = 1, 2, ..., n)$ be a collection of IFNs, an IFOWA operator of dimension n is a mapping *IFOWA:* $\Omega^n \to \Omega$ that has an associated weighting W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:

$$IFOWA\left(\alpha_1, \alpha_2, ..., \alpha_n\right) = \sum_{j=1}^{n} w_j \beta_j = \left(1 - \prod_{j=1}^{n}\left(1 - \mu_j\right)^{w_j}, \prod_{j=1}^{n} v_j^{w_j}\right) .$$ (3)

where $\beta_j = (\mu_j, v_j)$ is the jth largest of the α_i and α_i is the argument variable represented in the form of IFN.

2.4.   The induced ordered weighted averaging-weighted average (IOWAWA) operator

The induced ordered weighted averaging–weighted average (IOWAWA) operator is a new model that unifies the IOWA operator and the WAs in the same formulation and considering a complex reordering process based on order-inducing variables. Therefore, both concepts can be seen as a particular case of a more general one. This approach seems to be complete, at least as an initial real unification between IOWA operators and WA. It can also be seen as a unification between decision-making problems under uncertainty (with IOWA operators) and under risk (with probabilities). It can be defined as follows.

Definition 4. An IOWAWA operator of dimension n is a mapping $IOWAWA$: $R^n \times R^n \to R$ that has an associated weighting W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:

$$IOWAWA\big(\langle u_1, a_1 \rangle, ..., \langle u_n, a_n \rangle\big) = \sum_{j=1}^{n} \hat{\upsilon}_j b_j . \qquad (4)$$

where b_j is a_i value of the IOWAWA pair $\langle u_i, a_i \rangle$ having the j th largest u_i, u_i is the order inducing variable and a_i is the argument variable, each argument a_i has an associated weight (WA) υ_j with $\sum_{j=1}^{n} \upsilon_j = 1$ and $\upsilon_j \in [0,1]$, $\hat{\upsilon}_j = \lambda w_j + (1-\lambda)\upsilon_j$ with $\lambda \in [0,1]$ and υ_j is the weight (WA) υ_i ordered according to b_j, that is, according to the j th largest u_i.

Note that it is also possible to formulate the IOWAWA operator separating the part that strictly affects the IOWA operator and the part that affects the WAs. This representation is useful to see both models in the same formulation but it does not seem to be as a unique equation that unifies both models.

Definition 5. An IOWAWA operator of dimension n is a mapping $IOWAWA$: $R^n \times R^n \to R$ that has an associated weighting W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, and a weighting vector V that affects the WA, with $\sum_{i=1}^{n} \upsilon_i = 1$ and $\upsilon_i \in [0,1]$, such that:

$$IOWAWA\left(\langle u_1, a_1 \rangle, ..., \langle u_n, a_n \rangle\right) = \lambda \sum_{j=1}^{n} w_j b_j + (1-\lambda) \sum_{i=1}^{n} \upsilon_i a_i. \qquad (5)$$

where b_j is a_i value of the IOWAWA pair $\langle u_i, a_i \rangle$ having the j th largest u_i, u_i is the order inducing variable and $\lambda \in [0,1]$.

3.   Induced intuitionistic fuzzy ordered weighted averaging-weighted average (I-IFOWAWA) operator

The induced intuitionistic fuzzy ordered weighted averaging-weighted average (I-IFOWAWA) operator is an extension of the IOWAWA operator that uses uncertain information in the aggregation represented in the form of IFNs. Note that the I-IFOWAWA can also be seen as an aggregation operator that uses the IOWA operator, the WA and IFNs in the same formulation. The reason for using this operator is that sometimes, the uncertain factors that affect our decisions are not clearly known and in order to assess the problem we need to use IFNs. This operator can be defined as follows.

Definition 6. An I-IFOWAWA operator of dimension n is a mapping *I-IFOWAWA*: $R^n \times \Omega^n \rightarrow \Omega$ that has an associated weighting W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ such that:

$$I\text{-}IFOWAWA\left(\langle u_1, \alpha_1 \rangle, ..., \langle u_n, \alpha_n \rangle\right) = \sum_{j=1}^{n} \hat{\upsilon}_j \beta_j. \qquad (6)$$

where β_j is α_i value of the I-IFOWAWA pair $\langle u_i, \alpha_i \rangle$ having the j th largest u_i, u_i is the order inducing variable and α_i is the argument variable, each argument α_i has an associated weight (WA) υ_j with $\sum_{j=1}^{n} \upsilon_j = 1$ and $\upsilon_j \in [0,1]$, $\hat{\upsilon}_j = \lambda w_j + (1-\lambda)\upsilon_j$ with $\lambda \in [0,1]$ and υ_j is the weight (WA) υ_i ordered according to β_j, that is, according to the j th largest u_i.

Note that it is also possible to formulate the I-IFOWAWA operator separating the part that strictly affects the IOWA operator and the WA.

Definition 7. An I-IFOWAWA operator of dimension n is a mapping *I-IFOWAWA:* $R^n \times \Omega^n \rightarrow \Omega$ that has an associated weighting W with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, and a weighting vector V that affects the WA, with $\sum_{i=1}^{n} \upsilon_i = 1$ and $\upsilon_i \in [0,1]$, such that:

$$I - IFOWAWA\left(\langle u_1, \alpha_1 \rangle, ..., \langle u_n, \alpha_n \rangle\right)$$
$$= \lambda \sum_{j=1}^{n} w_j \beta_j + (1-\lambda) \sum_{i=1}^{n} \upsilon_i \alpha_i . \qquad (7)$$

where β_j is α_i value of the I-IFOWAWA pair $\langle u_i, \alpha_i \rangle$ having the j th largest u_i, u_i is the order inducing variable and $\lambda \in [0,1]$.

Note that if $\gamma = 1$, we get the I-IFOWA operator and if $\gamma = 0$, the IFWA operator.

In the following, we are going to give a simple example on how to aggregate with the I-IFOWAWA operator. We consider the aggregation with both definitions.

Example 1. Assume the following arguments in an aggregation process: $((0.5,0.3),(0.4,0.5),(0.8,0.1),(0.6,0.3))$ with the following order-inducing variables $U = (4,7,1,9)$. Assume the following weighting vector W= (0.2, 0.2, 0.3, 0.3) and the following probabilistic weighting vector V = (0.3, 0.2, 0.4, 0.1). Note that the WA has a degree of importance of 70% while the weighting vector W of the IOWA a degree of 30%. If we want to aggregate this information by using the I-IFOWAWA operator, we will get the following result. The aggregation can be solved either with Eq. (6) or Eq. (7). With Eq. (6) we calculate the new weighting vector as:

$$\hat{\upsilon}_1 = 0.3 \times 0.2 + 0.7 \times 0.1 = 0.13$$
$$\hat{\upsilon}_2 = 0.3 \times 0.2 + 0.7 \times 0.2 = 0.2$$
$$\hat{\upsilon}_3 = 0.3 \times 0.3 + 0.7 \times 0.3 = 0.3$$
$$\hat{\upsilon}_4 = 0.3 \times 0.3 + 0.7 \times 0.4 = 0.37$$

And then, we calculate the aggregation process as follows:

$$I - IFOWAWA = 0.13 \times (0.6,0.3) + 0.2 \times (0.4,0.5) +$$

$$+0.3 \times (0.5,0.3) + 0.37 \times (0.8,0.1) = (0.64,0.22)$$

With Eq. (7), we aggregate as follows:

$$I - IFOWAWA = 0.3 \times (0.2 \times (0.6, 0.3) + 0.2 \times (0.4, 0.5) +$$
$$0.3 \times (0.5, 0.3) + 0.3 \times (0.8, 0.1)) + 0.7 \times (0.3 \times (0.5, 0.3) +$$
$$0.2 \times (0.4, 0.5) + 0.4 \times (0.8, 0.1) + 0.1 \times (0.6, 0.3)) = (0.64, 0.22)$$

Obviously, we get the same results with both methods.

From a generalized perspective of the reordering step, we can distinguish between the descending I-IFOWA (DI-IFOWA) operator and the ascending I-IFOWA (AI-IFOWA) operator by using $w_j = w*_{n-j+1}$, where w_j is the jth weight of the DI-IFOWA and $w*_{n-j+1}$ the jth weight of the AI-IFOWA operator.

Note that if the weighting vector is not normalized, i.e., $\hat{V} = \sum_{j=1}^{n} \hat{\upsilon}_j \neq 1$, then, the I-IFOWAWA operator can be expressed as:

$$I - IFOWAWA\left(\langle u_1, \alpha_1 \rangle, ..., \langle u_n, \alpha_n \rangle\right) = \frac{1}{\hat{V}} \sum_{j=1}^{n} \hat{\upsilon}_j \beta_j . \qquad (8)$$

Similar to the IOWAWA operator, the I-IFOWAWA operator is monotonic, bounded and idempotent. These properties can be proved with the following theorems.

Theorem 1 (Monotonic). Assume f is the I-IFOWAWA operator, if $\alpha_i \geq \alpha_i'$ for all i, then:

$$f\left(\langle u_1, \alpha_1 \rangle, ..., \langle u_n, \alpha_n \rangle\right) \geq f\left(\langle u_1, \alpha_1' \rangle, ..., \langle u_n, \alpha_n' \rangle\right). \qquad (9)$$

Theorem 2 (Idempotency). Assume f is the I-IFOWA operator, if $\alpha_i = \alpha$ for all i, then

$$f\left(\langle u_1, \alpha_1 \rangle, ..., \langle u_n, \alpha_n \rangle\right) = \alpha . \qquad (10)$$

Theorem 3 (Bounded). Assume f is the I-IFOWA operator, then

$$\min\{\alpha_i\} \leq f\left(\langle u_1, \alpha_1 \rangle, ..., \langle u_n, \alpha_n \rangle\right) \leq \max\{\alpha_i\}. \qquad (11)$$

Note that the proofs of these theorems are straightforward and thus omitted.

A further interesting issue is the problem of ties in the reordering process of the order inducing variables. In order to solve this problem, we recommend following the policy explained by Yager and Filev [17] about replacing the tied arguments by their average. Note that in this case, it would mean that we are replacing the tied arguments by their intuitionistic fuzzy average.

Another interesting issue to consider is the measures for characterizing the weighting vector $\hat{V}$ (V and W) of the I-IFOWA operator such as the attitudinal character, the entropy of dispersion, the divergence of W and the balance operator. As these features do not depend upon the intuitionistic fuzzy numbers arguments, the

formulation is the same as the IOWAWA operator. The entropy of dispersion is defined as follows:

$$H\left(\hat{V}\right) = -\left(\gamma \sum_{j=1}^{n} w_j \ln(w_j) + (1-\gamma) \sum_{i=1}^{n} \upsilon_i \ln(\upsilon_i)\right). \qquad \textbf{(12)}$$

Note that υ_i is the ith weight of the WA aggregation. As we can see, if $\gamma = 1$, we obtain the Yager entropy of dispersion and if $\gamma = 0$, we get the classical Shannon entropy. Note that strictly speaking, the Shannon entropy is extended by using the I-IFOWAWA operator as follows:

$$H\left(\hat{V}\right) = -\left(\gamma \sum_{j=1}^{n} w_j \log_2(w_j) + (1-\gamma) \sum_{i=1}^{n} \upsilon_i \log_2(\upsilon_i)\right). \qquad \textbf{(13)}$$

Thus, it is easy to see that we can extend all the analysis developed by Shannon and others in information theory by using this new approach. Note also that it is possible to consider other entropy measures in the I-IFOWAWA operator.

If we extend the analysis of the orness–andness measure to the I-IFOWAWA operator, we get the following expressions. For the degree of orness:

$$\alpha\left(\hat{V}\right) = \gamma \sum_{j=1}^{n} w_j^* \left(\frac{n-j}{n-1}\right) + (1-\gamma) \sum_{j=1}^{n} \upsilon_j^* \left(\frac{n-j}{n-1}\right). \qquad \textbf{(14)}$$

Note that w_j^* and υ_j^* are the w_j and υ_j weights of the I-IFOWAWA aggregation ordered according to the values of the arguments α_i. As we can see, if $\gamma = 1$, we get the usual orness measure of Yager[7] and if $\gamma = 0$, we obtain the orness measure of the weighted average. It is straightforward to calculate the andness measure by using the dual. That is, $Andnesse\left(\hat{V}\right) = 1 - \alpha\left(\hat{V}\right)$.

If we extend the divergence of W to the I-IFOWAWA operator, we get the following divergence of $\hat{V}$:

$$Div\left(\hat{V}\right) = \sum_{j=1}^{n} \hat{\upsilon}_j \left(\frac{n-j}{n-1} - \alpha\left(\hat{V}\right)\right)^2. \qquad \textbf{(15)}$$

or also as:

$$Div\left(\hat{V}\right) = \gamma \left(\sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2\right) + \qquad \textbf{(16)}$$

$$(1-\gamma)\left(\sum_{j=1}^{n} \upsilon_j \left(\frac{n-j}{n-1} - \alpha(W) \right)^2 \right).$$

Note that if $\gamma = 1$, we get the I-IFOWA divergence and if $\gamma = 0$, the IFWA divergence.

The balance operator can be defined as:

$$BAL(W) = \sum_{j=1}^{n} w_j \left(\frac{n+1-2j}{n-1} \right). \tag{17}$$

And the divergence of $\hat{V}$:

$$\begin{aligned} Bal(\hat{V}) &= \gamma g^{-1} \left(\sum_{j=1}^{n} g\left(\frac{n+1-2j}{n-1} \right) w_j \right) \\ &+ (1-\gamma) g^{-1} \left(\sum_{j=1}^{n} g\left(\frac{n+1-2j}{n-1} \right) \upsilon_j \right). \end{aligned} \tag{18}$$

Note that $\hat{\upsilon}_j = \lambda w_j + (1-\lambda)\upsilon_j$ is the jth weight of the I-IFOWAWA aggregation. And if $g(b) = b$, then, we get the usual balance operator applied to the I-IFOWAWA operator as follows:

$$Bal(\hat{V}) = \sum_{j=1}^{n} \hat{\upsilon}_j \left(\frac{n+1-2j}{n-1} \right). \tag{19}$$

Note that if $\gamma = 1$, we get the classic balance operator and if $\gamma = 0$, the IFWA divergence.

Analyzing the applicability of the I-IFOWAWA operator, we can see that it is applicable to similar situations already discussed in other types of induced aggregation operators where it is possible to use intuitionistic fuzzy numbers information. For example, we could use it in different decision-making problems, statistics, fuzzy set theory, soft computing, operational research, business administration, economics etc.

4.   Families of I-IFOWAWA Operators

In this Section we analyze different families of I-IFOWAWA operators. The main advantage is that we can consider a wide range of particular cases that can be used in the I-IFOWAWA operator leading to different results. Thus, we are able to provide a more complete representation of the aggregation process. Basically, we distinguish between the families found in the weighting vector $\hat{V}$ (V and W) and those found in the parameter γ.

Remark 1. If we analyze the parameter γ, we are going to consider the two main cases of the I-IFOWAWA operator. Basically:

- If $\gamma = 0$, we get the IFWA [10].
- If $\gamma = 1$, we get the I-IFOWA [33].
- If $\gamma = 1$ and the ordered position of u_i is the same as the ordered position of β_j such that β_j is the j th largest α_i, the IFOWA [10].

Remark 2. Another group of interesting families are the maximum-IFWA, the minimum-IFWA, the step-I-IFOWAWA operator and the usual average.

- The maximum-IFWA is found when $w_p = 1$, $w_j = 0$, for all $j \neq p$, and $u_p = \text{Max}\{\alpha_i\}$.
- The minimum-IFWA is found when $w_p = 1$, $w_j = 0$, for all $j \neq p$, and $u_p = \text{Min}\{\alpha_i\}$.
- More generally, the step- I-IFOWAWA is formed when $w_k = 1$ and $w_j = 0$ for all $j \neq k$.
- The intuitionistic fuzzy average is obtained when $w_j = 1/n$, and $\upsilon_j = 1/n$, for all α_i.

Remark 3. The arithmetic-IFWA is obtained when $w_j = 1/n$ for all j, and the weighted average is equal to the IFOWA when the ordered position of i is the same as the ordered position of j. The arithmetic-IFWA (A-IFWA) can be formulated as follows:

$$A - IFWA\left(\langle u_1, \alpha_1 \rangle, ..., \langle u_n, \alpha_n \rangle\right) = \frac{1}{n}\lambda\sum_{i=1}^{n}\alpha_i + (1-\lambda)\sum_{i=1}^{n}\upsilon_i\alpha_i . \tag{20}$$

Note that if $\upsilon_i = 1/n$, for all i, then, we get the unification between the intuitionistic fuzzy arithmetic mean (or simple intuitionistic fuzzy average) and the I-IFOWA operator, that is, the arithmetic-I-IFOWA (A-I-IFOWA). The A-I-IFOWA operator can be formulated as follows:

$$A - I - IFOWA\left(\langle u_1, \alpha_1 \rangle, ..., \langle u_n, \alpha_n \rangle\right) =$$
$$\lambda\sum_{j=1}^{n}w_j\beta_j + (1-\lambda)\frac{1}{n}\sum_{i=1}^{n}\alpha_i . \tag{21}$$

Remark 4. Following the OWA literature [4, 16, 20], we can develop many other families of I-IFOWAWA operators such as:

- The olympic-I-IFOWAWA operator ($w_1 = w_n = 0$ and for all others $w_j = 1/(n-2)$).

- The general olympic-I-IFOWAWA operator ($w_j = 0$ for $j = 1, 2, ..., k, n,$ $n-1, ..., n-k+1$; and for other $w_{j*} = 1/(n-2k)$ where $k < n/2$).

- The centered- I-IFOWAWA (if it is symmetric, strongly decaying from the center to the maximum and the minimum, and inclusive).

Remark 5. Using a similar methodology, we could develop numerous other families of I-IFOWAWA operators. For more information, refer to Ref. 16, 20-23, 27.

5.   Application in business decision-making

The I-IFOWAWA operator can be applied in a wide range of problems including statistics, economics and engineering.

In the following, we are going to develop a brief example where we will see the applicability of the new approach. We will focus on a decision-making problem about selection of strategies. Note that other business decision making applications could be developed such as financial decision making and human resource selection.

Assume a company that operates in North America and Europe is analyzing the general policy for the next year and they consider five possible strategies to follow.

- A_1 = Expand to the Asian market.

- A_2 = Expand to the South American market.

- A_3 = Expand to the African market.

- A_4 = Expand to the three continents.

- A_5 = Do not develop any expansion.

In order to evaluate these strategies, the experts consider that the key factor is the economic situation of the next year. Thus, depending on the situation, the expected benefits will be different. The experts have considered five possible situations for the next year:

- S_1 = Very bad.

- S_2 = Bad.

- S_3 = Regular.

- S_4 = Good.

- S_5 = Very good.

The group of experts of the company gives its opinion about the uncertain expected results that may occur in the future. The expected results depending on the situation S_i and the alternative A_i are shown in Tables 1. Note that the results are IFNs.

In this problem, the decision maker assumes the following degrees of importance (IFWA) of the characteristics: $V = (0.1, 0.2, 0.2, 0.2, 0.3)$. He assumes that the IFOWA weight is: $W = (0.1, 0.1, 0.2, 0.2, 0.4)$; with the following order-inducing variables: $U = (8, 4, 9, 3, 2)$. Note that IFWA has an importance of 70% and the IFOWA an importance of 30% ($\gamma = 0.3$) because he believes that the IFWA is more relevant in the problem.

Table 1. Available strategies

	S_1	S_2	S_3	S_4	S_5
A_1	(0.5,0.4)	(0.6,0.2)	(0.4,0.4)	(0.7,0.1)	(0.5,0.5)
A_2	(0.6,0.2)	(0.9,0.1)	[(0.7,0.2)	(0.3,0.5)	(0.4,0.5)
A_3	(0.4,0.4)	(0.6,0.3)	(0.8,0.1)	(0.5,0.4)	(0.6,0.3)
A_4	(0.7,0.2)	(0.4,0.6)	(0.5,0.3)	(0.3,0.6)	(0.7,0.1)
A_5	(0.8,0.2)	(0.5,0.3)	(0.6,0.3)	(0.6,0.4)	(0.4,0.5)

Due to the fact that the attitudinal character of the entrepreneurs is very complex because it involves a lot of complexities, the experts use order inducing variables to express it. Table 2 shows the results.

Table 2. Order inducing variables

	S_1	S_2	S_3	S_4	S_5
U	12	13	16	15	19

With this information, we can make an aggregation to make a decision. In this example, we will consider the Max-IFWA, the Min- IFWA, the IFWA, the IFOWA, the I-IFOWA, the A-IFWA, the A-I-IFOWA, the I-IFOWAWA. The results are shown in Table 3.

As we can see, depending on the aggregation operator used, the ordering of the strategies may be different. Therefore, the decision about which strategy select may be also different.

If we establish an ordering of the investments, a typical situation if we want to consider more than one alternative, we will get the following orders shown in Table 4. Note that the first alternative in each ordering is the optimal choice.

Table 3. Aggregated results

	Max-IFWA	Min-IFWA	IFWA	IFOWA	I-IFOWA	I-IFOWAWA
A_1	(0.7,0.1)	(0.4,0.4)	(0.55,0.28)	(0.50,0.34)	(0.56,0.29)	(0.55,0.28)
A_2	(0.9,0.1)	(0.3,0.5)	(0.64,0.28)	(0.54,0.32)	(0.61,0.30)	(0.63,0.28)
A_3	(0.8,0.1)	(0.4,0.4)	(0.62,0.26)	(0.54,0.32)	(0.59,0.29)	(0.61,0.27)
A_4	(0.7,0.1)	(0.3,0.6)	(0.55,0.27)	(0.46,0.39)	(0.57,0.24)	(0.55,0.26)
A_5	(0.8,0.2)	(0.4,0.5)	(0.56,0.36)	(0.54,0.37)	(0.54,0.37)	(0.55,0.36)

Table 4. Ordering of the Strategies

	Ordering
Max-IFWA	$A_2 \succ A_3 \succ A_5 \succ A_1 = A_4$
Min-IFWA	$A_1 = A_3 \succ A_5 \succ A_2 \succ A_4$
IFWA	$A_2 \succ A_3 \succ A_4 \succ A_1 \succ A_5$
IFOWA	$A_2 = A_3 \succ A_5 \succ A_1 \succ A_4$
I-IFOWA	$A_4 \succ A_2 \succ A_3 \succ A_1 \succ A_5$
I-IFOWAWA	$A_2 \succ A_3 \succ A_1 \succ A_4 \succ A_5$

As we can see, depending on the particular type of I-IFOWAWA operator used, the results may lead to different decisions. Note that the main advantage of using the I-IFOWAWA operator is that we can consider a wide range of scenarios in order to form a general picture. As each case may give different results, the decision maker will select the particular case that it is closest to his interests.

6.   Conclusions

We have introduced the I-IFOWAWA operator. It is an aggregation operator that uses the IOWA operator, the WA and uncertain information represented in the form of IFNs. It is very useful for uncertain situations where the decision maker can not assess the information with exact numbers or singletons but it is possible to assess it with

IFNs. By using the I-IFOWAWA, we are able to deal with complex situations that require a complex reordering process of the arguments before the aggregation step. Moreover, by using the I-IFOWAWA, we get a generalization that includes a wide range of intuitionistic fuzzy operators such as the IFWA, the IFOWA, the I-IFOWA, the A-IFWA and the A-I-IFOWA.

We have also developed an application of the new approach in a strategic decision-making problem. We have seen that the I-IFOWAWA is very useful because it represents very well the uncertain information by using IFNs. We have also seen that depending on the particular case of the I-IFOWAWA operator used the results may lead to different decisions.

In future research we expect to develop further extensions by adding new characteristics in the problem such as the use of probabilistic aggregations. We will also consider other decision making applications such as human resource management, investment selection and product management.

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