

# Fuzzy Claim Reserving In Non-Life Insurance

Jorge de Andrés-Sánchez

Social and Business Research Laboratory  
Rovira i Virgili University  
Avinguda de la Universitat 1, 43204, REUS, SPAIN  
jorge.deandres@urv.cat

**Abstract.** This paper develops several expressions to quantify claim provisions to account in financial statements of a non-life insurance company under the hypothesis of a fuzzy environment. Concretely, by applying the expected value of a fuzzy number and the more general concept of value of a fuzzy number to the ANOVA claim predicting model [2] we estimate claim reserves to account in insurer's balance sheet and income account.

**Key-words:** Fuzzy logic; Fuzzy numbers; Value of a Fuzzy Number, Expected Value of a Fuzzy Number; Insurance; Claims provisions

## 1. Introduction

As Dubois and Prade [15] point out Fuzzy Sets Theory (FST) and its extensions are applied, basically, in the following three circumstances: gradualness, epistemic uncertainty and bipolarity. So, although actuarial quantitative analysis is essentially based on statistical methods, academics and practitioners now tend to believe that FST is a useful complement to statistics in the cases that require a great deal of actuarial subjective judgement and problems for which the information available is scarce or vague. An extended survey can be found in [29]. So, the Encyclopaedia of Actuarial Science in [13] dedicated a chapter to FST. Any case, as it is pointed out in [27], fuzziness does not figure as central to any science and, of course, Actuarial Mathematics is not an exception.

One of the most interesting areas of FST for actuaries is Fuzzy Data Analysis (FDA). As Statistics, FST provides several techniques for searching and ordering the information contained in empirical data (e.g. for grouping elements, to find relations between variables, etc.). Within an actuarial context, Fuzzy Regression (FR) has been used intensively in several areas. In life insurance field, [1] and [22] use two different Fuzzy Regression (FR) methods to fit the temporal structure of interest rates whereas [21] develops a FR methodology to forecast mortality with a Lee-Carter model. In non-life insurance Berry-Stölze et al. in [4] use FR to evaluate solvency requirements for property-liability insurers.

In non-life claim reserving, the use of FR is motivated, basically, because of it is not advisable to use a wide data-base since data too far from the present can lead to unrealistic estimates. In this context, Andrés-Sánchez in [2] proposes a claim reserving method that mixes FR with the ANOVA reserving method [23], which has been used intensively in actuarial literature (see e.g. [8, 28]). Andrés-Sánchez's method estimates

future liabilities not only as a point values but also their variability with the use of Fuzzy Numbers (FNs) instead of random variables.

The fuzzy estimation of claiming costs finally needs to be transformed into a crisp equivalent in order, for example, to compute them in financial statements. So, [2] takes into account only non-discounted reserves (i.e. does not considers the profit of the assets that support liabilities) and uses the concept of the expected value [7]. This paper extends those results in two ways. Firstly, we also introduce in the analysis the expected profit of assets and so we obtain discounted reserves. We consider that the interest rate for finding the present value of future claims will be estimated subjectively by the actuary and so, it is a very natural to quantify that magnitude with a FN. Financial pricing with fuzzy parameters has been widely developed [5, 16, 20, 25]. Subsequently, fuzzy financial mathematics has been extended to life insurance pricing [3, 26] and non-life insurance pricing [1, 9, 12, 23]. Secondly, to transform fuzzy liabilities into crisp estimates we will use the concept of value of FNs [11] which generalises the expected value of a FN [7]. It will allow us weighting all the possible values of fuzzy estimates of liabilities taking into account their actual reliability.

The structure of the paper is as follows. In the next section we shall describe the aspects of fuzzy arithmetic and defuzzification that are used in this paper. Section 3 exposes claim reserving prediction exposed in [2] and develops several expressions for the value of reserves to be accounted. Those expressions will depend on the weighting function used for defuzzifying and if we discount future liabilities with an interest rate. Subsequently we develop a numerical application. Finally, we state the most important conclusions of the paper.

## 2. Fuzzy Numbers and related concepts

A Fuzzy Number (FN) is a fuzzy subset  $\tilde{a}$  defined over real numbers which is normal (i.e.  $\sup_{\forall x \in X} \mu_{\tilde{a}}(x) = 1$ ) and convex (i.e. its  $\alpha$ -cuts must be convex sets). For practical

purposes, Triangular Fuzzy Numbers (TFNs) are widely used FNs since they are easy to handle arithmetically and they can be interpreted intuitively. We shall symbolise a TFN  $\tilde{a}$  as  $\tilde{a} = (a, l_a, r_a)$  where  $a$  is the centre and  $l_a$  and  $r_a$  are the left and right spreads, respectively. Analytically, a TFN is characterised by its membership function  $\mu_{\tilde{a}}(x)$  and its  $\alpha$ -cuts,  $a_\alpha$ :

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x - a + l_a}{l_a} & a - l_a < x \leq a \\ \frac{a + r_a - x}{r_a} & a < x \leq a + r_a \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$a_\alpha = [\underline{a}(\alpha), \bar{a}(\alpha)] = [a - l_a(1 - \alpha), a + r_a(1 - \alpha)] \tag{2}$$

In many actuarial analyses, it is often necessary to evaluate functions (e.g. the net present value of an annuity), which we shall name  $y=f(x_1, x_2, \dots, x_n)$ . Then, if  $x_1, x_2, \dots, x_n$  are not crisp numbers but the FNs  $\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n$ ,  $f(\cdot)$  induces the FN  $\tilde{b} = f(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n)$ . Buckley and Qu in [6] demonstrate that if the function  $f(\cdot)$  that induces  $\tilde{b}$  is increasing with respect to the first  $m$  variables, where  $m \leq n$ , and decreasing with respect to the last  $n-m$  variables, then  $b_\alpha$  is:

$$b_\alpha = [b(\alpha), \bar{b}(\alpha)] = \left[ f\left(\underline{a}_1(\alpha), \dots, \underline{a}_m(\alpha), \overline{a}_{m+1}(\alpha), \dots, \overline{a}_n(\alpha)\right), f\left(\overline{a}_1(\alpha), \dots, \overline{a}_m(\alpha), \underline{a}_{m+1}(\alpha), \dots, \underline{a}_n(\alpha)\right) \right] \quad (3)$$

It is very usual in real insurance situations to estimate magnitudes as approximate quantities, for example, by means of a sentence like “the claim provisions must be around 2000 monetary units”. Clearly, FNs can be used to represent these magnitudes. However, these magnitudes also often need to be quantified with crisp values. For example, in our context, this will occur when the definitive amount of claim provisions needs to be specified in financial statements. This paper proposes using the concept of value of a FN by Delgado et al. in [11], but adapted in order to introduce decision maker’s risk aversion in the sense of Hurwitz. For a FN  $\tilde{a}$ , it will be symbolised as  $V[\tilde{a}, \beta]$ . If we use a decision-maker risk aversion  $\beta$ , where  $0 \leq \beta \leq 1$  we obtain:

$$V[\tilde{a}, \beta] = (1 - \beta) \int_0^1 w(\alpha) \underline{a}(\alpha) d\alpha + \beta \int_0^1 w(\alpha) \overline{a}(\alpha) d\alpha \quad (4a)$$

where  $w(\alpha)$  is the weighting function. In this paper, following [11], we will consider  $w(\alpha) = 2\alpha$ . Likewise, as it is done in [2], we will also consider  $w(\alpha) = 1$  (there is no weighting). Notice that under this hypothesis, (3) is now the expected value of a FN (Campos and González, 1989). In this case, the value will be symbolised as  $EV[\tilde{a}, \beta]$ .

Notice that the value of a FN is an additive measure, and so:

$$V\left[\sum_{i=1}^n \tilde{a}_i, \beta\right] = \sum_{i=1}^n V[\tilde{a}_i, \beta] \quad (4b)$$

### 3. Fuzzy claim reserving

#### 3.1. Fitting future claiming with fuzzy ANOVA

Let us symbolise as  $s_{i,j}$  the claim cost of the insurance contracts originated in the  $i$ th period ( $i=0,1,\dots,n$ ) within the  $j$ th claiming period ( $j=0,1,\dots,n$ ). Given that the past claiming costs of the  $i$ th occurrence period are  $s_{i,j}$ ,  $j=0,1,\dots,n-i$ , calculating claim reserves implies forecasting and adding the claim costs of future development periods:  $s_{i,j}$ ,  $i=1,2,\dots,n$ ;  $j=n-i+1, n-i+2,\dots, n$ .

Andrés-Sánchez in [2] extends ANOVA claim reserving method [23], to the use of Fuzzy Regression. ANOVA chain ladder, as classical chain ladder, supposes that  $s_{i,j}$  can be represented by the product  $C_i p_j$  where  $C_i$  is the total claiming cost in the  $i$ th origin period, whereas  $p_j$  is the proportion of this cost paid in the  $j$ th development period. Therefore, the parameters  $C_i, i=0,1,\dots,n$  and  $p_j, j=0,1,\dots,n$  can be obtained by using linear regression since  $\ln s_{i,j} = \ln C_i + \ln p_j$ . To introduce uncertainty [23] supposes that:

$$s_{i,j} = C_i p_j \varepsilon_{i,j} \tag{5}$$

where  $\varepsilon_{i,j}$  is a random variable whose mean is 1. The theoretical model (5) can be transformed into the following linear regression equation:

$$\ln s_{i,j} = a + b_i + c_j + \varepsilon_{i,j}, i=1,2,\dots,n; j=1,2,\dots,n \tag{6}$$

where here,  $\varepsilon_{i,j}, i=0,1,\dots,n; j=0,1,\dots,n-i$  are identical and uncorrelated distributed normal random variables with mean 0 and variance  $\sigma^2$ . Notice that in this model the incremental cost of claims  $s_{i,j}$  is a log-normal random variable, because (6) lets us deduce:

$$s_{i,j} = e^{a+b_i+c_j+\varepsilon_{i,j}} \tag{7}$$

Here,  $e^a$  can be interpreted as the incremental claim cost in the initial origin and development periods ( $i=0, j=0$ ). Thus,  $b_i$  can be understood as the logarithmical growth rate of total claiming during the origin period  $i, (i=1,2,\dots,n)$  with respect to the initial origin period. Analogously, for a given origin period,  $c_j$  is the logarithmical growth rate of the incremental cost of claims during the development period  $j (j=1,2,\dots,n)$  with respect to the cost of claims declared during the development period  $j=0$ .

On the other hand [2] considers that the uncertainty about incremental claims is due to fuzziness and it is not exogenous to total claiming cost,  $C_i$ , and proportions  $p_j, j=1,2,\dots,n$ . So, we will obtain an estimate of the incremental claim cost  $s_{i,j}$  by means of the fuzzy number  $\tilde{s}_{i,j}$ :

$$\tilde{s}_{i,j} = \tilde{C}_i \tilde{p}_j \tag{8a}$$

then, the linear regression model to fit is analogous to (6):

$$\ln \tilde{s}_{i,j} = \tilde{a} + \tilde{b}_i + \tilde{c}_j \tag{8b}$$

Andrés-Sánchez in [2] supposes that  $\tilde{a}, \tilde{b}_i$  and  $\tilde{c}_j$  are TFNs and therefore  $\ln \tilde{s}_{i,j}$  is also a TFN. If we symbolise  $\tilde{a} = (a, l_a, r_a), \tilde{b}_i = (b_i, l_{b_i}, r_{b_i}), i=1,2,\dots,n$  and  $\tilde{c}_j = (c_j, l_{c_j}, r_{c_j}), j=1,2,\dots,n$ , we obtain from (6) and (8b):

$$\begin{aligned} \ln \tilde{s}_{i,j} &= (\ln s_{i,j}, l_{\ln s_{i,j}}, r_{\ln s_{i,j}}) = (a, l_a, r_a) + (b_i, l_{b_i}, r_{b_i}) + (c_j, l_{c_j}, r_{c_j}) \\ &= (a + b_i + c_j, l_a + l_{b_i} + l_{c_j}, r_a + r_{b_i} + r_{c_j}) \end{aligned} \tag{8c}$$

Notice that in the fuzzy regression model (8b)  $\tilde{a}$  is the independent term whereas  $\tilde{b}_i, \tilde{c}_j$  are the coefficients of dichotomous explanatory variables, in such a way that

their observations are equal to one when the period in which they are located is equal to the parameter period, and zero in the other case. Therefore, although in the FR model that we use, the spreads of the estimate response are increasing with respect to the magnitude of the explanatory variables, this is not a problem since we are handling dummy independent variables.

After fitting the model (8a)-(8c) that is done with the FR model described in [19], we can now predict the claiming cost of all origin periods in the development periods in which they are unknown. Given that the logarithm of incremental claim costs  $\ln \tilde{s}_{i,j}$ ,  $i=1,2,\dots,n; j \geq n-i+1$  are the TFN in (8b) and that the incremental claim cost  $\tilde{s}_{i,j}$  is:

$$\tilde{s}_{i,j} = e^{\ln \tilde{s}_{i,j}} \tag{9a}$$

Then, the  $\alpha$ -cuts of  $\tilde{s}_{i,j}$ ,  $s_{i,j_\alpha}$  are obtained from (2) and (3):

$$s_{i,j_\alpha} = \left[ \underline{s}_{i,j}(\alpha), \overline{s}_{i,j}(\alpha) \right] = \left[ e^{a+b_i+c_j - (l_a+l_{b_i}+l_{c_j})(1-\alpha)}, e^{a+b_i+c_j + (r_a+r_{b_i}+r_{c_j})(1-\alpha)} \right] \tag{9b}$$

### 3.2. Fitting accounting value for claim reserves

Firstly, we fit a crisp quantification for non-discounted claim reserves, i.e., without taking into account the return of the assets that support these reserves. In this case, we must obtain the value of  $\tilde{s}_{i,j}$ ,  $V[\tilde{s}_{i,j}; \beta]$  by applying (3) to (9b):

$$V[\tilde{s}_{i,j}; \beta] = (1-\beta) \int_0^1 w(\alpha) e^{a+b_i+c_j - (l_a+l_{b_i}+l_{c_j})(1-\alpha)} d\alpha + \beta \int_0^1 w(\alpha) e^{a+b_i+c_j + (r_a+r_{b_i}+r_{c_j})(1-\alpha)} d\alpha \tag{10a}$$

In the case where  $w(\alpha)=2\alpha$ , (10a) must be integrated by parts and so:

$$V[\tilde{s}_{i,j}; \beta] = 2 \left[ (1-\beta) e^A \frac{(B-1)e^B + 1}{B^2} + \beta e^D \frac{1 - (C+1)e^{-C}}{C^2} \right] \tag{10b}$$

where:

$$A = a + b_i + c_j - (l_a + l_{b_i} + l_{c_j}); B = l_a + l_{b_i} + l_{c_j}; C = r_a + r_{b_i} + r_{c_j}; D = a + b_i + c_j + (r_a + r_{b_i} + r_{c_j})$$

Likewise if  $w(\alpha)=1$  we obtain  $EV[\tilde{s}_{i,j}; \beta]$  in such a way that:

$$EV[\tilde{s}_{i,j}; \beta] = (1-\beta) \frac{e^A}{B} [e^B - 1] + \beta \frac{e^D}{C} [1 - e^{-C}] \tag{11}$$

where the parameters  $A$ ,  $B$ ,  $C$  and  $D$  are equal to the case (10b). Of course, in both cases, (10b) and (10c), if  $l_a = l_{b_i} = l_{c_j} = 0$ , the first summand is simply  $(1-\beta)e^{a+b_i+c_j}$

and when  $r_a = r_{b_i} = r_{c_j} = 0$ , the second summand must be  $\beta e^{a+b_i+c_j}$ .

The non-discounted provision corresponding to the  $i$ th origin period is obtained by doing:

$$P\tilde{R}O_i = \sum_{j=n-i+1}^n \tilde{s}_{i,j} \tag{12}$$

therefore, the provision for all the occurrence periods is:

$$P\tilde{R}O = \sum_{i=1}^n P\tilde{R}O_i = \sum_{i=1}^n \sum_{j=n-i+1}^n \tilde{s}_{i,j} \tag{13}$$

From the  $\alpha$ -cuts of  $\tilde{s}_{i,j}$  in (9b) we can derive the exact value for the  $\alpha$ -cuts of  $P\tilde{R}O_i$  and  $P\tilde{R}O$  by applying (3):

$$PRO_{i,\alpha} = \left[ \underline{PRO}_i(\alpha), \overline{PRO}_i(\alpha) \right] = \left[ \sum_{j=n-i+1}^n e^{a+b_i+c_j - (l_a+l_{b_i}+l_{c_j})(1-\alpha)}, \sum_{j=n-i+1}^n e^{a+b_i+c_j + (r_a+r_{b_i}+r_{c_j})(1-\alpha)} \right] \tag{14}$$

$$PRO_\alpha = \left[ \underline{PRO}(\alpha), \overline{PRO}(\alpha) \right] = \left[ \sum_{i=1}^n \sum_{j=n-i+1}^n e^{a+b_i+c_j - (l_a+l_{b_i}+l_{c_j})(1-\alpha)}, \sum_{i=1}^n \sum_{j=n-i+1}^n e^{a+b_i+c_j + (r_a+r_{b_i}+r_{c_j})(1-\alpha)} \right] \tag{15}$$

To account for the claim provision in the financial statements, we must transform  $P\tilde{R}O_i$  and  $P\tilde{R}O$  into the crisp numbers  $PRO_i^*$  and  $PRO^*$  respectively. To do so, we will use the concept of value of a FN which we have described in (3) and (4). In our problem, to fix  $\beta$ , we must bear in mind that actuarial decisions must be prudent, that is,  $\beta > 0.5$ . Given that the expected value of FN is an additive measure, we can obtain  $PRO_i^*$  and  $PRO^*$  by aggregating  $V[\tilde{s}_{i,j}; \beta]$ ,  $i=1,2,\dots,n$ ;  $j=n-i+1,\dots,n$ , that we have obtained in (10a)-(10c). Thus:

$$PRO_i^* = V[P\tilde{R}O_i; \beta] = \sum_{j=n-i+1}^n V[\tilde{s}_{i,j}; \beta] \tag{16}$$

Therefore, the crisp provision for all the occurrence years is:

$$PRO^* = V[P\tilde{R}O; \beta] = \sum_{i=1}^n \sum_{j=n-i+1}^n V[\tilde{s}_{i,j}; \beta] \tag{17}$$

However, these developments do not take into account that the assets that support future liabilities produce financial returns and so, non-discounted provisions overrate the real value of reserves. To avoid this problem, the regulation of many countries allows discounting future liabilities to quantify claim reserves. We consider that the interest rate for finding the present value of future claims will be estimated subjectively by the actuary and so, it is a very natural to quantify that magnitude with a FN. This is a relatively common hypothesis in non-life insurance, [1, 9, 12, 23], that extend financial pricing with fuzzy parameters [5, 16, 20, 25] to non-life insurance pricing.

This paper supposes a constant force of interest throughout the temporal horizon estimated with the TFN  $\tilde{\rho} = (\rho, l_\rho, r_\rho)$ , that the periodicity of claiming is annual and

that its distribution is uniform within the year. As is pointed out in [18], this last hypothesis is applied in practice by supposing that the cost of claims is paid in the middle of the year. Therefore, the discounted value of the incremental claiming of the  $i$ th origin year during the  $j$ th development period will be symbolised as  $d\tilde{s}_{i,j}$ ,  $i=1,2,\dots,n; j \geq n-i+1$  and it is obtained as:

$$d\tilde{s}_{i,j} = \tilde{s}_{i,j} e^{\tilde{\rho}[n+1/2-(i+j)]} = e^{\tilde{a}+\tilde{b}_i+\tilde{c}_j+\tilde{\rho}[n+1/2-(i+j)]} \tag{14a}$$

Then, the  $\alpha$ -cuts of  $d\tilde{s}_{i,j}$  are obtained by using (2) and (3):

$$ds_{i,j,\alpha} = \left[ \underline{ds}_{i,j}(\alpha), \overline{ds}_{i,j}(\alpha) \right] = \left[ e^{a+b_i+c_j+\rho[n+1/2-(i+j)]-\{l_a+l_{b_i}+l_{c_j}-r_\rho[n+1/2-(i+j)]\}(1-\alpha)}, e^{a+b_i+c_j+\rho[n+1/2-(i+j)]+\{r_a+r_{b_i}+r_{c_j}-l_\rho[n+1/2-(i+j)]\}(1-\alpha)} \right] \tag{18}$$

So, the value of the discounted incremental cost of claims  $V[d\tilde{s}_{i,j}; \beta]$  is obtained by applying (3) to (14b):

$$V[d\tilde{s}_{i,j}; \beta] = (1-\beta) \int_0^1 w(\alpha) e^{a+b_i+c_j+\rho[n+1/2-(i+j)]-\{l_a+l_{b_i}+l_{c_j}-r_\rho[n+1/2-(i+j)]\}(1-\alpha)} d\alpha + \beta \int_0^1 w(\alpha) e^{a+b_i+c_j+\rho[n+1/2-(i+j)]+\{r_a+r_{b_i}+r_{c_j}-l_\rho[n+1/2-(i+j)]\}(1-\alpha)} d\alpha \tag{19}$$

In the case where  $w(\alpha)=2\alpha$ , (15a) is also (10b) but in this case the parameters  $A, B, C$  and  $D$  are:

$$\begin{aligned} A &= a + b_i + c_j + \rho[n+1/2-(i+j)] - \{l_a + l_{b_i} + l_{c_j} - r_\rho[n+1/2-(i+j)]\} \\ B &= l_a + l_{b_i} + l_{c_j} - r_\rho[n+1/2-(i+j)] \\ C &= r_a + r_{b_i} + r_{c_j} - l_\rho[n+1/2-(i+j)] \\ D &= a + b_i + c_j + \rho[n+1/2-(i+j)] - \{r_a + r_{b_i} + r_{c_j} - l_\rho[n+1/2-(i+j)]\} \end{aligned} \tag{15b}$$

Likewise if  $w(\alpha)=1$  we obtain  $EV[\tilde{s}_{i,j}; \beta]$  and also the result is (10c) but the values of  $A, B, C$  and  $D$  are in (15b).

The discounted reserve to account for the  $i$ th origin period is then:

$$dP\tilde{R}O_i = \sum_{j=n-i+1}^n d\tilde{s}_{i,j} \tag{20}$$

and therefore, the provision for all the occurrence years is:

$$dP\tilde{R}O = \sum_{i=1}^n P\tilde{R}O_i = \sum_{i=1}^n \sum_{j=n-i+1}^n d\tilde{s}_{i,j} \tag{21}$$

From the  $\alpha$ -cuts of  $d\tilde{s}_{i,j}$  in (16b) we derive the value for the  $\alpha$ -cuts of  $dP\tilde{R}O_i$  and  $dP\tilde{R}O$  with (3):

$$dPRO_{i\alpha} = [dPRO_i(\alpha), \overline{dPRO_i}(\alpha)] = \left[ \sum_{j=n-i+1}^n e^{a+b_i+c_j+\rho[n+1/2-(i+j)]-\{a+l_{b_i}+l_{c_j}-r_\rho[n+1/2-(i+j)]\}}(1-\alpha), \sum_{j=n-i+1}^n e^{a+b_i+c_j+\rho[n+1/2-(i+j)]+\{r_a+r_{b_i}+r_{c_j}-l_\rho[n+1/2-(i+j)]\}}(1-\alpha) \right] \tag{22}$$

$$dPRO_\alpha = [dPRO(\alpha), \overline{dPRO}(\alpha)] = \left[ \sum_{i=1}^n \sum_{j=n-i+1}^n e^{a+b_i+c_j+\rho[n+1/2-(i+j)]-\{l_a+l_{b_i}+l_{c_j}-r_\rho[n+1/2-(i+j)]\}}(1-\alpha), \sum_{i=1}^n \sum_{j=n-i+1}^n e^{a+b_i+c_j+\rho[n+1/2-(i+j)]+\{r_a+r_{b_i}+r_{c_j}-l_\rho[n+1/2-(i+j)]\}}(1-\alpha) \right] \tag{23}$$

Finally, to fit the claim provision to account in financial statements we must transform the fuzzy value of the discounted reserves into the crisp values. Analogously to non-discounted reserves, we obtain:

$$dPRO_i^* = V[dP\tilde{R}O_i; \beta] = \sum_{j=n-i+1}^n EV[d\tilde{s}_{i,j}; \beta] \tag{24}$$

Therefore, the provision for all the occurrence years is:

$$dPRO^* = V[dP\tilde{R}O; \beta] = \sum_{i=1}^n \sum_{j=n-i+1}^n EV[d\tilde{s}_{i,j}; \beta] \tag{25}$$

#### 4. Numerical application

Our numerical example is developed over the run-off triangle in Table 2 [8, p. D5.4] that was also used in [2]. From Table 1, we can immediately obtain the log-incremental payments in Table 2.

**Table 1.** Run off triangle. The amounts are the cost of the claims from the  $i$ th origin year paid at the  $j$ th development year ( $s_{i,j}, i=0,1,2,3, j=0,1,\dots,3-i$ )

		Development year			
		$i \setminus j$	0	1	2
Origin year	0	11073	6427	1839	766
	1	14799	9357	2344	
	2	15636	10523		
	3	16913			

**Table 2.** Run off triangle in which the quantities are  $\ln s_{i,j}, i=0,1,2,3, j=0,1,\dots,3-i$

		Development year			
		$i \setminus j$	0	1	2
Origin year	0	9.312	8.768	7.517	6.641
	1	9.602	9.144	7.760	
	2	9.657	9.261		
	3	9.736			

Now, we develop our numerical with certain detail.

a) First we need to fit the parameters  $\tilde{a} = (a, l_a, r_a), \tilde{b}_i = (b_i, l_{b_i}, r_{b_i}), i=1,2,3$  and  $\tilde{c}_j = (c_j, l_{c_j}, r_{c_j}), j=1,2,3$ . Thus, following [2] we implement two steps.

a.1) We obtain the estimates of the centres of  $\tilde{a}, \tilde{b}_i, i=1,2,3$  and  $\tilde{c}_j, j=1,2,3$ , with the algorithm used in [23]. See the results in Table 3.

**Table 3.** Least Squares estimates for the centres of  $\tilde{a}, \tilde{b}_i, i=1,2,3$  and  $\tilde{c}_j, j=1,2,3$

Coefficient	$a^*$	$b_1^*$	$b_2^*$	$b_3^*$	$c_1^*$	$c_2^*$	$c_3^*$
Value	9.288	0.303	0.04	0.447	-0.466	-1.801	-2.647
Standard deviation	0.040	0.043	0.050	0.066	0.043	0.050	0.066

a.2) We fit the spreads of  $\tilde{a}, \tilde{b}_i, i=1,2,3$  and  $\tilde{c}_j, j=1,2,3$ , being the result:

$$\tilde{a} = (9.288, 0.024, 0.000)$$

$$\tilde{b}_1 = (0.303, 0.000, 0.000)$$

$$\tilde{b}_2 = (0.404, 0.011, 0.016)$$

$$\tilde{b}_3 = (0.447, 0.000, 0.000)$$

$$\tilde{c}_1 = (-0.466, 0.030, 0.019)$$

$$\tilde{c}_2 = (-1.801, 0.006, 0.030)$$

$$\tilde{c}_3 = (-2.647, 0.000, 0.000)$$

b) Now, we must evaluate (9a) to calculate the future cost of claims  $\tilde{s}_{i,j}$ ,  $i=1,2,3$ ;  $j=3-i+1, \dots, 3$ . Table 4 indicates the values of  $\alpha$ -cuts  $s_{i,j\alpha}$  for  $\alpha=0, 0.5, 1$ . The value  $s_{i,j_1}$  gives us a prediction of the most feasible point estimate of the future incremental claim whereas the 0-cut quantifies its estimated range. Subsequently, by applying (12a) and (12b) to the results of Table 4 we calculate the  $\alpha$ -cuts of provisions (see Table 5).

**Table 4.** Values of  $s_{i,j\alpha}$   $i=1,2,3, j=3-i+1, \dots, 3$  for the  $\alpha$ -levels  $\alpha=0, 0.5, 1$ .

	$s_{1,3\alpha}$	$s_{2,2\alpha}$	$s_{2,3\alpha}$
$\alpha=1$	[1036.86, 1036.86]	[2672.95, 2672.95]	[1147.35, 1147.35]
$\alpha=0.5$	[1012.38, 1036.86]	[2564.96, 2799.47]	[1107.81, 1166.06]
$\alpha=0$	[988.48, 1036.86]	[2461.33, 2931.96]	[1069.64, 1185.08]
	$s_{3,1\alpha}$	$s_{3,2\alpha}$	$s_{3,3\alpha}$
$\alpha=1$	[10611.44, 10611.44]	[2791.62, 2791.62]	[1198.29, 1198.29]
$\alpha=0.5$	[10054.03, 10813.80]	[2708.94, 2876.83]	[1170.00, 1198.29]
$\alpha=0$	[9525.91, 11020.01]	[2628.71, 2964.63]	[1142.37, 1198.29]

**Table 5.** Values of the reserves for the  $\alpha$ -levels  $\alpha=0, 0.5, 1$ .

	$PROV_{1\alpha}$	$PROV_{2\alpha}$	$PROV_{3\alpha}$	$PROV_{\alpha}$
$\alpha=1$	[1036.86, 1036.86]	[3820.30, 3820.30]	[14601.35, 14601.35]	[19458.51, 19458.51]
$\alpha=0.5$	[1012.38, 1036.86]	[3672.77, 3965.53]	[13932.97, 14888.91]	[18618.12, 19891.30]
$\alpha=0$	[988.48, 1036.86]	[3530.97, 4117.05]	[13296.99, 15182.94]	[17816.43, 20336.84]

To account non-discounted provisions in financial statements it is necessary to defuzzify the FNs that estimate future claiming. By using (10c), (13a) and (13b) and taking a risk aversion coefficient  $\beta=1$ , we obtain:

$$V[\tilde{s}_{1,3};1] = 1036.86; V[\tilde{s}_{2,2};1] = 2759.29; V[\tilde{s}_{2,3};1] = 1159.93;$$

$$V[\tilde{s}_{3,1};1] = 10747.63; V[\tilde{s}_{3,2};1] = 2849.29; V[\tilde{s}_{3,3};1] = 1198.29$$

$$PRO_1^* = V[P\tilde{R}O_1;1] = 1036.86$$

$$PRO_2^* = V[P\tilde{R}O_2;1] = 3919.21$$

$$PRO_3^* = V[P\tilde{R}O_3;1] = 14795.21$$

$$PRO^* = V[P\tilde{R}O_i;1] = 19751.28$$

On the other hand, by using a risk aversion coefficient  $\beta=1$  but the expected value of claims (10c), we find:

$$EV[\tilde{s}_{1,3};1]=1036.86; EV[\tilde{s}_{2,2};1]=2800.46; EV[\tilde{s}_{2,3};1]=1166.11;$$

$$EV[\tilde{s}_{3,1};1]=10814.44; EV[\tilde{s}_{3,2};1]=2877.26; EV[\tilde{s}_{3,3};1]=1198.29$$

$$PRO_1^* = EV[P\tilde{R}O_1;1]=1036.86$$

$$PRO_2^* = EV[P\tilde{R}O_2;1]=3966.58$$

$$PRO_3^* = EV[P\tilde{R}O_3;1]=14889.99$$

$$PRO^* = EV[P\tilde{R}O_i;1]=19893.42$$

c) To obtain discounted reserves, we will suppose that the actuary predicts for the coming years that the returns of the insurer’s investments will be approximately 3%, and that deviations over that value may be about 0.5%. That rate can be quantified by the TFN  $\tilde{p} = (0.03, 0.005, 0.005)$ . Subsequently, by applying (14b) we calculate the  $\alpha$ -cuts of the discounted value of incremental claims (see Table 6). Likewise, the discounted value of reserves in Table 7 is calculated with (17a) and (17b).

**Table 6.** Values of  $ds_{i,j_\alpha}$   $i=1,2,3, j=2,3$  for the  $\alpha$ -levels  $\alpha=0, 0.5, 1$ .

	$ds_{1,3_\alpha}$	$ds_{2,2_\alpha}$	$ds_{2,3_\alpha}$
$\alpha=1$	[1021.42, 1021.42]	[2633.16, 2633.16]	[1096.86, 1096.86]
$\alpha=0.5$	[996.06, 1022.70]	[2523.62, 2761.24]	[1055.10, 1118.94]
$\alpha=0$	[971.33, 1023.98]	[2418.63, 2895.54]	[1014.93, 1141.46]
	$ds_{3,1_\alpha}$	$ds_{3,2_\alpha}$	$ds_{3,3_\alpha}$
$\alpha=1$	[10453.46, 10453.46]	[2668.78, 2668.78]	[1111.70, 1111.70]
$\alpha=0.5$	[9891.98, 10666.13]	[2580.05, 2760.57]	[1078.69, 1118.67]
$\alpha=0$	[9360.65, 10883.12]	[2494.26, 2855.52]	[1046.66, 1125.69]

**Table 7.** Estimated values of the discounted reserves for the  $\alpha$ -levels  $\alpha=0, 0.5, 1$ .

	$dPROV_{1\alpha}$	$dPROV_{2\alpha}$	$dPROV_{3\alpha}$	$dPROV_\alpha$
$\alpha=1$	[1021.42, 1021.42]	[3730.02, 3730.02]	[14233.94, 14233.94]	[18985.39, 18985.39]
$\alpha=0.5$	[996.06, 1022.70]	[3578.72, 3880.18]	[13550.72, 14545.37]	[18125.49, 19448.25]
$\alpha=0$	[971.33, 1023.98]	[3433.56, 4037.01]	[12901.58, 14864.33]	[17306.47, 19925.31]

To obtain a crisp value for the discounted value of incremental claims and discounted provisions, we calculate  $d\tilde{s}_{i,j}$ ,  $i=1,2,3; j \geq 3-i+1$  and apply (10b) and (15b). Subsequently we apply (18a) and (18b), considering again that  $\beta= 1$ :

$$V[d\tilde{s}_{1,3};1]=1022.27; V[d\tilde{s}_{2,2};1]=2720.62; V[d\tilde{s}_{2,3};1]=1111.73;$$

$$V[d\tilde{s}_{3,1};1]=10596.68; V[d\tilde{s}_{3,2};1]=2731.03; V[d\tilde{s}_{3,3};1]=1116.36$$

$$PRO_1^* = V[dP\tilde{R}O_1;1] = 1022.27;$$

$$PRO_2^* = V[dP\tilde{R}O_2;1] = 3832.35$$

$$PRO_3^* = V[dP\tilde{R}O_3;1] = 14444.07$$

$$PRO^* = V[dP\tilde{R}O_i;1] = 19298.69$$

From (10c) and (15b) we obtain  $EV[d\tilde{s}_{i,j};\beta]$ ,  $i=1,2,3$ ;  $j \geq 3-i+1$ . If we establish  $\beta=1$ , those values are:

$$EV[d\tilde{s}_{1,3};1] = 1022.70; EV[d\tilde{s}_{2,2};1] = 2762.27; EV[d\tilde{s}_{2,3};1] = 1119.02;$$

$$EV[d\tilde{s}_{3,1};1] = 10666.85; EV[d\tilde{s}_{3,2};1] = 2761.10; EV[d\tilde{s}_{3,3};1] = 1118.68$$

And so, we obtain the following crisp values for provisions:

$$PRO_1^* = EV[dP\tilde{R}O_1;1] = 1022.70$$

$$PRO_2^* = EV[dP\tilde{R}O_2;1] = 3881.29$$

$$PRO_3^* = EV[dP\tilde{R}O_3;1] = 14546.63$$

$$PRO^* = EV[dP\tilde{R}O_i;1] = 19450.62$$

## 5. Conclusions

Few data must be used to adjust claim reserves. In this context we think that Fuzzy Set Theory is a suitable alternative to the usual statistical methods. Specifically, to quantify claim provisions we have used the method developed by Andrés-Sánchez in [12].

This paper deals with the fact that fuzzy estimation of claim reserves needs to be transformed into a crisp equivalent in order, for example, to compute them in balance sheet and income account. In this paper, we propose using the concept of value of a FN [11] because this defuzzification method makes it possible to introduce the actuarial risk aversion easily and intuitively and, likewise, to graduate the weights of the values embedded in fuzzy quantification. In a first approach we have fitted non-discounted reserves, whose value overrates real present value of liabilities. So, subsequently we have introduced in the analysis the return of the assets that cover future claims. By using financial mathematics with fuzzy parameters we calculate fuzzy discounted reserves and propose several expressions that allow their crisp quantification.

Any case, we think that other alternative FDA instruments like FR method [32] or fuzzy transforms [14] may give a viable quantitative basis to calculate claim reserves within the framework of Fuzzy Sets Theory. Likewise, in our opinion non-classical statistical methods like grey models [24] or gray correlation methods may also offer interesting solutions approaches to quantify claim reserves.

**Acknowledgements.** We thank the helpful comments of anonymous reviewers on previous versions of this paper.

## References

1. Andrés, J. de, Terceño, A.: Applications of Fuzzy Regression in Actuarial Analysis. *Journal of Risk and Insurance*, Vol. 70, No. 4, 665-699. (2003)
2. Andrés-Sánchez, J. de: Claim reserving with fuzzy regression and the two ways of ANOVA. *Applied Soft Computing*, Vol. 12, No. 8, 2435-2441. (2012)
3. Andrés-Sánchez, J. de, González-Vila, L. : Using fuzzy random variables in life annuity pricing. *Fuzzy Sets and Systems*, Vol. 188, No. 1, 27-44. (2012)
4. Berry-Stölzle, T. R., Koissi, M.-C., Shapiro, A. F.: Detecting fuzzy relationships in regression models: The case of insurer solvency surveillance in Germany. *Insurance: Mathematics and Economics*, Vol. 46, No. 3, 554-567. (2009)
5. Buckley, J.J.: The fuzzy mathematics of finance. *Fuzzy Sets and Systems* , Vol. 21, No. 1, 57-73. (1987)
6. Buckley, J.J., Qu, Y. : On using  $\alpha$ -cuts to evaluate fuzzy equations. *Fuzzy Sets and Systems*, Vol. 38, No. 3, 309-312. (1990)
7. Campos, L.M., González, A.: A subjective approach for ranking fuzzy numbers. *Fuzzy Sets and Systems*, Vol. 29, No. 2, 145-153. (1989)
8. Christofides, S.: Regression models based on log-incremental payments. *Claims reserving manual*, Volume 2, Institute of Actuaries, London. (1989)
9. Cummins, J.D., Derrig, R.A.: Fuzzy financial pricing of property-liability insurance. *North American Actuarial Journal*, Vol. 1, No.4, 21-44. (1997)
10. Da, L., Dongxiao, N., Yuanyuan, L., Guanjuan, C.: Combined models for day-ahead electricity price forecasting based on improved gray correlation methodology. *Kybernetes*, Vol. 38, No. 3/4, 354 – 361. (2009)
11. Delgado, M.; Vila, M.A.; Woxman, W.: On a canonical representation of a fuzzy number. *Fuzzy Sets and Systems*, Vol. 93, No. 1, 125-135. (1998)
12. Derrig, R.A., Ostaszewski, K.: Managing the tax liability of a property liability insurance company. *Journal of Risk and Insurance*, Vol. 64, No. 4, 695-711. (1997)
13. Derrig, R.A., Ostaszewski, K.: *Fuzzy Sets*. Encyclopedia of Actuarial Science, Volume 2, John Wiley & Sons, Chichester, 745-750. (2004)
14. Di Martino, F., Loia, V., Sessa S.: Fuzzy transforms method in prediction data analysis. *Fuzzy Sets and Systems*, Vol. 180, No. 1, 146-156. (2011)
15. Dubois, D., Prade, H.: Gradualness, uncertainty and bipolarity: Making sense of fuzzy sets. *Fuzzy Sets and Systems*, Vol. 192, No. 1, 3-24. (2012)
16. Gil-Aluja, J.: *Investment on uncertainty*, Kluwer Academic Publishers, Dordrecht. (1998)
17. Gil-Lafuente, A.M.: *Fuzzy logic in financial analysis*, Springer-Verlag, Berlin. (2005)
18. Institute Of Actuaries: *Claims reserving manual*, Volume 1, Institute of Actuaries, London. (1989)
19. Ishibuchi, H., Nii, M.: Fuzzy regression using asymmetric fuzzy coefficients and fuzzified neural networks. *Fuzzy Sets and Systems*, Vol. 119, No. 2, 273-290. (2001)
20. Kaufmann, A.; Gupta, M.M.: *Fuzzy mathematical models in engineering and management science*, North-Holland, Amsterdam (Netherlands) (1988)
21. Koissi, M.C. Shapiro, A.F.: Fuzzy formulation of the Lee–Carter model for mortality forecasting. *Insurance: Mathematics and Economics*, Vol. 39, No. 3, 289-307. (2006)
22. Koissi, M.C. Shapiro, A.F.: Fuzzy regression and the term structure of interest rates- A least squares approach. *Actuarial Research Clearing House Newsletters*, Vol. 2009, No. 1. (2009)
23. Kremer, E.: IBNR claims and the two way model of ANOVA. *Scandinavian Actuarial Journal*, Vol. 1, No. 1, 47-55. (1982)

23. Lai, L.-H.: Underwriting Underwriting Systematic Risk and Profit Margin in Fuzzy CAPM and ICAPM Models: The Case of Aviation Coverage. *Contemporary Management Research*, Vol. 5, No. 4, 387-396. (2009),
24. Li, Q.-X., Liu, S.-F.: The grey input-occupancy-output analysis. *Kybernetes*, Vol. 38, No. 3/4, 306 – 313. (2009)
25. Li Calzi, M.: Towards a general setting for the fuzzy mathematics of finance. *Fuzzy Sets and Systems*, Vol. 35, No. 3, 265-280. (1990)
26. Ostaszewski, K.: An investigation into possible applications of fuzzy sets methods in actuarial science. *Society of Actuaries, Schaumburg (USA)*. (1993)
27. Negoita, C.V.: Fuzzy systems and cybernetics. *Kybernetes*, Vol. 38, No. 1/2, 58-60. (2009)
28. Renshaw, A.E.: Chain-ladder and interactive modelling (claims reserving and GLIM). *Journal of the Institute of Actuaries*, Vol. 116, 559-587. (1989)
29. Shapiro, A.F.: Fuzzy logic in insurance. *Insurance: Mathematics and Economics*, Vol. 35, No. 2, 399-424. (2004)
30. Tanaka, H.: Fuzzy data analysis by possibilistic linear models. *Fuzzy Sets and Systems*, Vol. 24, No. 3, 363-375. (1987)
31. Tanaka, H., Ishibuchi H.: A possibilistic regression analysis based on linear programming, In: Kacprzyk, J.; Fedrizzi, M. (eds.): *Fuzzy Regression Analysis*, Physica-Verlag, Heidelberg, 47-60. (1992)
32. Xu, S.-Q., Luo, Q.-Y. Xu, G.-H. Lei, Z.: Asymmetrical interval regression using extended  $\varepsilon$ -SVM with robust algorithm. *Fuzzy Sets and Systems*, Vol. 160, No. 7, 988-1002. (2009)

**Jorge de Andrés-Sánchez** is an Associate Professor and a member of the Social and Business Research Laboratory of the Rovira i Virgili University, Spain. He is MSc degree in Actuarial Sciences from University of Barcelona and PhD in Business Administration from Rovira i Virgili University. He has published more than 100 papers in journals, books and conference proceedings including journals such as *Fuzzy Sets and Systems*, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *Applied Soft Computing*, *Insurance: Mathematics and Economics*. He is currently interested in Fuzzy Mathematics applications in Finance and Insurance.

*Received: December 25, 2012; Accepted: February 22, 2014*