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An Uncertain Optimal Control Model with n Jumps and Application

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Abstract. Optimal control theory is an important branch of modern control theory which has been widely applied in various sciences. Uncertain optimal control is a theory dealing with optimal control problems which are based a new uncertainty theory and differs from the stochastic optimal control based on probability theory and fuzzy optimal control based on fuzzy set theory or credibility theory. As the further work of the uncertain optimal control with jump in the one-dimensional case and multidimensional linear-quadratic (LQ) uncertain optimal control problem with jump which has a quadratic objective function for a linear uncertain control system with jump, a general uncertain optimal control problem with n jumps in the multi-dimensional cases is considered in this paper. The principle of optimality is presented and the equation of optimality is obtained about multidimensional uncertain optimal control with n jumps. Finally, as an application, an optimal control problem in R&D (Research and Development) fiscal subsidy policy is discussed and the optimal control decisions are obtained.

Keywords: optimal control, uncertainty, jump, multidimensional, R&D fiscal subsidy policy.

1. Introduction

Optimal control theory is an important branch of modern control theory. It is to seek the optimal control decision among the admissible control strategies for maximizing or minimizing some objective functions, which relate to a dynamic process driven by a differential equation. With the more use of methods and results on mathematics and computer science, optimal control theory has made considerable advances, and has been used widely in real-life problems such as production engineering, national defence, programming, finance, and economic management.

There has been an enormously rich theory on deterministic optimal control problem. The Pontryagin's maximum principle, Bellman's dynamic programming and Kalman's optimal linear feedback regulator design theory are the main tools studying deterministic optimal control problems. When a control system is an uncertain dynamic system, we are facing an uncertain optimal control problem.

Randomness and fuzziness are two traditional uncertainties. By applying probability theory and the Zadeh's fuzzy set theory or the credibility theory [16], the stochastic optimal control problem and the fuzzy optimal control problem have been developed by many researchers. However, the complexity of the world makes the events we face uncertain in various forms. A lot of surveys show that some imprecise quantities, such as information and knowledge represented by human language, behave neither like randomness nor like fuzziness. In order to model these imprecise quantities, an uncertainty theory was founded by Liu [17] in 2007 and refined in 2011 [18]. Nowadays, uncertainty theory has become a new branch of axiomatic mathematics and is well developed on both theory and applications.

Based on uncertain canonical process in uncertainty theory, Zhu [33] first introduced and dealt with an uncertain optimal control problem without jump by using dynamic programming in 2010. It is reasonable model for uncertain systems with the continuous uncertain canonical process without jump. Nevertheless, in real world, some external extreme events or noises have a great influence on uncertain dynamic systems. In [5–7], Deng and Zhu studied uncertain optimal control problems with jump, including a general uncertain optimal control problem with one jump in one-dimensional case, a linear quadratic (LQ) uncertain optimal control problem with jump which has a quadratic objective function for a linear uncertain control system with one jump, and a multidimensional linear quadratic (LQ) uncertain optimal control model with n jumps. In this paper, we will extend the special multidimensional uncertain LQ model to a general one.

The subsequent sections of the paper are organized as follows. In the next section related work will be reviewed. In Section 3, some preliminaries will be given. In Section 4, a multidimensional uncertain control model with n jumps will be put forward. Then, the equation of optimality will be obtained for the proposed model. As an application of the equation, an optimal control problem in R&D (Research and Development) fiscal subsidy policy will be discussed. In Section 5, some concluding remarks will be made.

2. Related Work

Merton [21] studied stochastic optimal control for finance in the late of 1960s. In recent decades, the study of stochastic optimal control greatly attracted the attention of many researchers, and has been made considerable advances, especially for finance. For example, Fleming and Rishel [8], Harrison [10] and Karatzas [13] studied optimal control problems of Brownian motion or stochastic differential equations and applications in finance and engineering. Dixit and Pindyck [4] discussed the use of dynamic programming in optimization over Ito's process. Zhou and Li [29] introduced the stochastic LQ control as a general framework to analyze the continuous-time mean-variance portfolio optimization and hedging problem. Jensen [11] studied stochastic optimal stopping problem in risk theory. Cairns [3] and Boulier [1] established the optimal control models of stochastic pension fund. Ou-Yang [22] and Sung [25] designed optimal contracts of dynamic delegated portfolio management by applying stochastic optimal control approach under Merton's continuous-time stochastic finance framework.

Komolov et al [14] presented the concept of optimal fuzzy control in 1979 based on the fuzzy set theory introduced by scientist on cybernetics Zadeh in 1965. Following that, the optimal fuzzy control problem has been studied in some work, see e.g., [9, 12, 23, 26] and references therein. In recent years, based on the credibility theory found by Liu [16] in 2004, which gives a self-dual measure for fuzzy events, a fuzzy optimal control problem with fuzzy Liu process was studied in some literature [20, 24, 27, 30–32].

Zhu [33] put forward an uncertain optimal control problem without jump based on the uncertainty theory in 2010. The equation of optimality was obtained and applied to solve a portfolio selection problem. The results showed the effectiveness of the proposed method. Then Kang and Zhu [15] and Xu and Zhu [28] discussed uncertain bang-bang optimal control problems for multi-stage and continuous time uncertain systems without jump in 2012, respectively. A general uncertain optimal control problem with one jump in one dimensional case was presented and dealt with by Deng and Zhu [5] in 2012. The principle of optimality was presented and the equation of optimality was obtained. Furthermore, in one-dimensional case, a special model, i.e., linear quadratic (LQ) uncertain optimal control model with one jump, was discussed by Deng and Zhu [6]. A necessary and sufficient condition for the existence of optimal control was derived. Deng [7] studied a linear quadratic (LQ) uncertain optimal control model with n jumps in multidimensional case. However, for many multidimensional uncertain optimal control problems, uncertain dynamic system with jumps may be nonlinear and the objective function may be not quadratic. Therefore, it is a meaningful work to extend the special multidimensional uncertain LQ model to general one. In this paper, we will focus on a general multidimensional uncertain optimal control problem with n jumps.

3. Preliminary

The concepts about uncertain measure, uncertain variable, uncertain process, *et al* may be referred to Liu [17], [18] and references therein.

Definition 1. *(Liu [19]) An uncertain process C_t is said to be a canonical process if (i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous; (ii) C_t has stationary and independent increments; (iii) every increment $C_{s+t} - C_s$ is a normally distributed uncertain variable with expected value 0 and variance t^2, whose uncertainty distribution is*

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in \Re. \tag{1}$$

Theorem 1. *(Zhu [33]) Let ξ be a normally distributed uncertain variable with expected value 0 and variance σ^2, whose uncertainty distribution is*

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re. \tag{2}$$

Then for any real number a, we have $\frac{\sigma^2}{2} \le E[a\xi + \xi^2] \le \sigma^2$.

Theorem 2. *(Liu [18]) Let f be a convex function on $[a, b]$, and ξ an uncertain variable that takes values in $[a, b]$ and has expected value e. Then*

$$E[f(\xi)] \le \frac{b-e}{b-a} f(a) + \frac{e-a}{b-a} f(b). \tag{3}$$

In order to model the discontinuous jump part of an uncertain system, an uncertain V-jump process, which is associated with an uncertain Z-jump variable $Z(r_1, r_2, t)$ defined by a jump uncertainty distribution was introduced by Deng and Zhu [5]. For the sake of simplicity, only one jump point is considered there. Now we generalize them to the following case of n jump points.

Definition 2. *An uncertain variable $Z(r_{i1}, r_{i2}, n, t)$ is said to be an uncertain Z-n jumps variable with parameters r_{i1} and r_{i2} ($0 < r_{i1} < r_{i2} < r_{(i+1)1} < r_{(i+1)2} < r_{(n+1)1} = 1$, $i = 1, 2, \cdots, n-1$) for $t > 0$ if it has a jump uncertainty distribution*

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 0, \\ \dfrac{(n+1)r_{11}}{t} x, & \text{if } 0 \le x < \dfrac{t}{n+1}, \\ r_{i2} + \dfrac{(n+1)(r_{(i+1)1} - r_{i2})}{t}\left(x - \dfrac{it}{n+1}\right), & \\ \qquad \text{if } \dfrac{it}{n+1} \le x < \dfrac{(i+1)t}{n+1},\ i = 1, \cdots, n, & \\ 1, & \text{if } x \ge t. \end{cases} \tag{4}$$

Theorem 3. *Assume ξ_1 and ξ_2 are independent uncertain Z-n jumps variables $Z(r_{i1}, r_{i2}, n, t_1)$ and $Z(r_{i1}, r_{i2}, n, t_2)$, respectively. Then the sum $\xi_1 + \xi_2$ is also an uncertain Z-n jumps variable $Z(r_{i1}, r_{i2}, n, t_1 + t_2)$, i.e.,*

$$Z(r_{i1}, r_{i2}, n, t_1) + Z(r_{i1}, r_{i2}, n, t_2) = Z(r_{i1}, r_{i2}, n, t_1 + t_2). \tag{5}$$

The product of an uncertain Z-n jumps variable $Z(r_{i1}, r_{i2}, n, t)$ and a scalar number $\bar{k} > 0$ is also an uncertain Z-n jumps variable $Z(r_{i1}, r_{i2}, n, \bar{k}t)$, i.e.,

$$\bar{k} \cdot Z(r_{i1}, r_{i2}, n, t) = Z(r_{i1}, r_{i2}, n, \bar{k}t) \tag{6}$$

Definition 3. *An uncertain process V_t is said to be an uncertain V-n jumps process with parameters r_{i1} and r_{i2} ($0 < r_{i1} < r_{i2} < r_{(i+1)1} < r_{(i+1)2} < r_{(n+1)1} = 1$, $i = 1, 2, \cdots, n-1$) for $t > 0$ if (i) $V_0 = 0$; (ii) V_t has stationary and independent increments; (iii) every increment $V_{s+t} - V_s$ is an uncertain Z-n jumps variable $Z(r_{i1}, r_{i2}, n, t)$.*

Let V_t be an uncertain V-n jumps process, and $\Delta V_t = V_{t+\Delta t} - V_t$. Then by using definition of expected value of uncertain variable, we get

$$\begin{aligned}\tilde{k} = E[\Delta V_t] &= \int_0^{+\infty} (1 - \Phi(x))dx = \int_0^{\frac{\Delta t}{n+1}} \left(1 - \frac{(n+1)r_{11}}{\Delta t}x\right)dx \\ &+ \sum_{i=1}^{n} \int_{\frac{i\Delta t}{n+1}}^{\frac{(i+1)\Delta t}{n+1}} \left(1 - r_{i2} - \frac{(n+1)(r_{(i+1)1} - r_{i2})}{\Delta t}\left(x - \frac{i\Delta t}{n+1}\right)\right)dx \\ &= \left(1 - \frac{1}{2(n+1)}\left(\sum_{i=1}^{n}(r_{i1} + r_{i2}) + 1\right)\right)\Delta t.\end{aligned}$$

Theorem 4. *(Existence Theorem) There is an uncertain V-n jumps process.*

The proof of the theorem 3 and 4 are parallel to the corresponding results in Deng and Zhu [5].

Theorem 5. *Let V_t be an uncertain V-n jumps process and $\zeta = \Delta V_t$. Then for any real number a,*

$$E[a\zeta + \zeta^2] = a\tilde{k}\Delta t + o(\Delta t).$$

Proof: To begin with we have

$$E[a\zeta + \zeta^2] \geq E[a\zeta] = aE[\zeta] = a\tilde{k}\Delta t. \tag{7}$$

On the other hand, we can get

$$E[a\zeta + \zeta^2] \leq \frac{E[\zeta]}{\Delta t}(a\Delta t + (\Delta t)^2) = a\tilde{k}\Delta t + o(\Delta t) \tag{8}$$

by Theorem 2. Combining inequality (7) and inequality (8), we can obtain

$$E[a\zeta + \zeta^2] = a\tilde{k}\Delta t + o(\Delta t).$$

Definition 4. *(Deng and Zhu [5]) Suppose that C_t is an uncertain canonical process, V_t is an uncertain V-n-jumps process, and g_1, g_2 and g_3 are some given functions. Then*

$$dX_t = g_1(X_t, t)\,dt + g_2(X_t, t)\,dC_t + g_3(X_t, t)\,dV_t \tag{9}$$

is called an uncertain differential equation with jump. A solution is an uncertain process X_t that satisfies (9) identically in t.

4. Multidimensional Uncertain Optimal Control Model with n Jumps and Application

4.1. Multidimensional Uncertain Optimal Control Model with n Jumps

An uncertain optimal control model with jump under one-dimensional case was proposed and the equation of optimality for solving the model was obtained by

Deng and Zhu [5]. In this subsection, we present the following general multidimensional uncertain optimal control model with n jumps:

$$\begin{cases} J(t,\mathbf{x}) \equiv \sup\limits_{\mathbf{u}_s\in\mathbf{U}} E\left[\int_t^T L\left(s,\mathbf{X}_s,\mathbf{u}_s\right)ds + F\left(T,\mathbf{X}_T\right)\right], \\ \text{subject to} \\ \quad d\mathbf{X_s} = \mathbf{A}\left(s,\mathbf{X}_s,\mathbf{u}_s\right)ds + \mathbf{B}\left(s,\mathbf{X}_s,\mathbf{u}_s\right)d\mathbf{C}_s + \mathbf{H}\left(s,\mathbf{X}_s,\mathbf{u}_s\right)d\mathbf{V}_s, \\ \quad \mathbf{X}_t = \mathbf{x}, \end{cases}$$

where $\mathbf{X}_s = (X_{s1}, X_{s2}, \cdots, X_{sn})^\tau$ is a n-dimensional state vector with the initial condition $\mathbf{X}_t = \mathbf{x} = (x_1, x_2, \cdots, x_n)^\tau$ at time t, $\mathbf{u}_s$ the decision vector of dimension r (represents the function $\mathbf{u}(s,\mathbf{X}_s)$ of time s and state $\mathbf{X}_s$) in a domain $\mathbf{U}$, $L : [0,T) \times \Re^n \times \Re^r \to \Re$ the objective function, and $F : [0,T) \times \Re^n \to \Re$ the terminal reward function. In addition, $\mathbf{A} : [0,T) \times \Re^n \times \Re^r \to \Re^n$ is a column-vector function, $\mathbf{B} : [0,T) \times \Re^n \times \Re^r \to \Re^n \times \Re^k$ and $\mathbf{H} : [0,T) \times \Re^n \times \Re^r \to \Re^n \times \Re^k$ are two matrix functions, and $\mathbf{C}_s = (C_{s1}, C_{s2}, \cdots, C_{sk})^\tau$, $\mathbf{V}_s = (V_{s1}, V_{s2}, \cdots, V_{sk})^\tau$, where $C_{s1}, C_{s2}, \cdots, C_{sk}$ are independent uncertain canonical processes, $V_{s1}, V_{s2}, \cdots, V_{sk}$ are independent uncertain V-n jumps processes, and C_{si} and V_{sj} for any $i,j = 1,2,\cdots,k$ $(i \neq j)$ are independent. Note that $\mathbf{y}^\tau$ represents the transpose vector of the vector $\mathbf{y}$, and the final time $T > 0$ is fixed or free.

Theorem 6 (Principle of optimality). *For any* $(t,\mathbf{x}) \in [0,T) \times \Re^n$*, and* $\Delta t > 0$ *with* $t + \Delta t < T$*, we have*

$$J(t,\mathbf{x}) = \sup_{\mathbf{u}_t\in\mathbf{U}} E\left[L\left(t,\mathbf{x},\mathbf{u}_t\right)\Delta t + J\left(t+\Delta t, \mathbf{x}+\Delta\mathbf{X}_t\right) + o(\Delta t)\right], \tag{10}$$

where $\mathbf{x} + \Delta\mathbf{X}_t = \mathbf{X}_{t+\Delta t}$.

The proof of the theorem is parallel to the corresponding result in Deng and Zhu [5].

Theorem 7 (Equation of optimality). *Let* $J(t,\mathbf{x})$ *be twice differentiable on* $[0,T] \times \Re^n$*. Then we have*

$$\begin{aligned} -J_t(t,\mathbf{x}) = \sup_{\mathbf{u}_t\in\mathbf{U}} \Big\{ & L\left(t,\mathbf{x},\mathbf{u}_t\right) + \nabla_{\mathbf{x}} J(t,\mathbf{x})^\tau \mathbf{A}\left(t,\mathbf{x},\mathbf{u}_t\right) \\ & + \tilde{k}\nabla_{\mathbf{x}} J(t,\mathbf{x})^\tau \mathbf{H}\left(t,\mathbf{x},\mathbf{u}_t\right)\mathbf{1} \Big\} \end{aligned} \tag{11}$$

where $\tilde{k} = 1 - \frac{1}{2(n+1)}\left(\sum_{i=1}^n (r_{i1} + r_{i2}) + 1\right)$*,* $J_t(t,\mathbf{x})$ *is the partial derivative of the function* $J(t,\mathbf{x})$ *in* t*,* $\nabla_{\mathbf{x}} J(t,\mathbf{x})$ *is the gradient of* $J(t,\mathbf{x})$ *in* $\mathbf{x}$ *and* $\mathbf{1}$ *is a* k*-dimension column-vector with the all terms being* 1*.*

The proof of theorem 7 is showed in the Appendix.

4.2. Optimal Control Problem in R&D Fiscal Subsidy Policy

The technology improvement is increasingly playing an important role in modern economic growth and economic development. As the source of the technology improvement, R&D (Research and Development) exert a direct or indirect effect on economic growth. The R&D investment related to new technology or new knowledge is very important and necessary. However, only market power in such field makes not enough incentive because for the risk contained in the process of R&D. So the government's R&D fiscal subsidy policy is one of the effective means to guide the market corresponding input.

The government departments usually cannot directly take part in R&D activities with fiscal subsidy policy. The relationship between government and R&D departments is a principle-agent relationship in which the officer is the principle and the other(representative consumer) is the agent. The problem of moral hazard may arise because the principle is unable to directly monitor R&D subsidy program implementation, and agent can change the use of the grant of R&D investment from principle in real world.

In this subsection, we employ the result obtained in the above subsection to solve the principle-agent problem in R&D fiscal subsidy policy with more than one uncertain factor. We present the dynamic optimization model of the principle and the agents with considering the possible moral hazard under the framework of uncertain environment, and seek their optimal paths.

We consider the principle-agent problem with a principle(government) and n agents(different industries' representative consumers).

Agents' Optimization Problem Let $\mathbf{K}$ denote the n-dimensional agents' capital stock column-vector at time t with the initial condition $\mathbf{K}_0 = \mathbf{k}_0$ (state vector), $\mathbf{c}$ and $\mathbf{w}$ denote the n-dimensional consumption level vector selected by n agents at time t and the fraction vector of the capital stock allocated by n agents in R&D activities (control vector), respectively. $\mathbf{C}_t$ and $\mathbf{V}_t$ denote, respectively, independent canonical process vector and uncertain V-n jumps process vector. $\mathbf{r}_y$, $\mathbf{r}_c$, $\mathbf{\Theta}_1$, $\mathbf{\Theta}_2$, $\mathbf{s}$ and $\mathbf{a}$ are all diagonal matrixes with the diagonal terms being n representative consumers' capital tax rate, the consumption tax rate collected by the government, the constant diffusion coefficient of volatility, the constant jump coefficient of volatility, the input/output ratio in daily production activities and the average rate of return in R&D activities, respectively. $\mathbf{I}$ is the unit matrix. In addition, $\mathbf{W}$ and $\tilde{\mathbf{K}}$ are also the diagonal matrixes with the diagonal terms vector being $\mathbf{w}$ and $\mathbf{K}$, respectively.

Thus n representative consumers' dynamic budget constraint equation may be described by a multidimensional uncertain differential equation with n jumps as follows

$$\begin{aligned} d\mathbf{K} = & \left[(\mathbf{I}-\mathbf{r}_y)(\mathbf{I}-\mathbf{W})\mathbf{s}\mathbf{K} + (\mathbf{I}-\mathbf{r}_y)\mathbf{W}\mathbf{a}\mathbf{K} - (\mathbf{I}+\mathbf{r}_c)\mathbf{c}\right]dt \\ & + (\mathbf{I}-\mathbf{r}_y)\mathbf{W}\mathbf{\Theta}_1\tilde{\mathbf{K}}d\mathbf{C}_t + (\mathbf{I}-\mathbf{r}_y)\mathbf{W}\mathbf{\Theta}_2\tilde{\mathbf{K}}d\mathbf{V}_t. \end{aligned} \tag{12}$$

Assume that the agents are interested in maximizing their discount value of expected total utilities over an infinite time horizon. Then n representative consumer's optimization problem may be described as follows

$$\max_{\mathbf{c},\mathbf{w}} E\left[\int_0^\infty e^{-\rho t} U\left(\mathbf{c},\mathbf{G}\right) dt\right], \tag{13}$$

where ρ and $U(\cdot)$ denote the consumer's subjective discount rate and instantaneous utility function at time t, respectively. $\mathbf{G}$ denotes the government's unproductive public input vector.

Further assume that agents' instantaneous utility function is the specific form below [2]

$$U\left(\mathbf{c},\mathbf{G}\right) = \frac{1}{b}\left[\left(\mathbf{W}\mathbf{c}\right)^b\right]^\tau \mathbf{G}^{1-b}, \tag{14}$$

where coefficient b satisfies $0 < b < 1$.

Unless in particular stated otherwise, we assume $\mathbf{A}^s = (a_{ij}{}^s)_{m\times n}$ for any $\mathbf{A} = (a_{ij})_{m\times n}$. $(\mathbf{AB})^s = \mathbf{A}^s\mathbf{B}^s$. Then the agents' optimal control model is provided by

$$\begin{cases} J(0,\mathbf{k}_0) \equiv \max\limits_{\mathbf{c},\mathbf{w}} E\left[\int_0^\infty \frac{1}{b} e^{-\rho t}\left[\left(\mathbf{W}\mathbf{c}\right)^b\right]^\tau \mathbf{G}^{1-b}\right], \\ \text{subject to} \\ \quad d\mathbf{K} = \left[\left(\mathbf{I}-\mathbf{r}_y\right)\left(\mathbf{I}-\mathbf{W}\right)\mathbf{s}\mathbf{K} + \left(\mathbf{I}-\mathbf{r}_y\right)\mathbf{W}\mathbf{a}\mathbf{K} - \left(\mathbf{I}+\mathbf{r}_c\right)\mathbf{c}\right] dt \\ \qquad\quad + \left(\mathbf{I}-\mathbf{r}_y\right)\mathbf{W}\boldsymbol{\Theta}_1\tilde{\mathbf{K}} d\mathbf{C}_t + \left(\mathbf{I}-\mathbf{r}_y\right)\mathbf{W}\boldsymbol{\Theta}_2\tilde{\mathbf{K}} d\mathbf{V}_t, \quad \mathbf{K}_0 = \mathbf{k}_0. \end{cases} \tag{15}$$

By the equation of optimality (11), we have

$$\begin{aligned} -J_t(t,\mathbf{k}) = \max_{\mathbf{c},\mathbf{w}} \Big\{ & \frac{1}{b} e^{-\rho t}\left[\left(\mathbf{W}\mathbf{c}\right)^b\right]^\tau \mathbf{G}^{1-b} + \nabla_{\mathbf{k}} J(t,\mathbf{k})^\tau \left[\left(\mathbf{I}-\mathbf{r}_y\right)\left(\mathbf{I}-\mathbf{W}\right)\mathbf{s}\mathbf{k}\right. \\ & \left. + \left(\mathbf{I}-\mathbf{r}_y\right)\mathbf{W}\mathbf{a}\mathbf{k} - \left(\mathbf{I}+\mathbf{r}_c\right)\mathbf{c} + \tilde{k}\left(\mathbf{I}-\mathbf{r}_y\right)\mathbf{W}\boldsymbol{\Theta}_2\mathbf{k}\right]\Big\} = \max_{\mathbf{c},\mathbf{w}} L(\mathbf{c},\mathbf{w}), \end{aligned} \tag{16}$$

where $L(\mathbf{c},\mathbf{w})$ represents the term in the braces. The optimal $(\mathbf{c},\mathbf{w})$ satisfies

$$\frac{\partial L(\mathbf{c},\mathbf{w})}{\partial \mathbf{c}} = e^{-\rho t}\left[\left(\mathbf{W}\mathbf{c}\right)^{b-1}\right]^\tau \mathbf{W}\mathbf{G}^{1-b} - \nabla_{\mathbf{k}} J(t,\mathbf{k})^\tau \left(\mathbf{I}+\mathbf{r}_c\right) = 0, \tag{17}$$

$$\begin{aligned} \frac{\partial L(\mathbf{c},\mathbf{w})}{\partial \mathbf{w}} &= e^{-\rho t}\left[\left(\mathbf{W}\mathbf{c}\right)^{b-1}\right]^\tau \mathbf{C}\mathbf{G}^{1-b} \\ &+ \nabla_{\mathbf{k}} J(t,\mathbf{k})^\tau \left(\mathbf{I}-\mathbf{r}_y\right)\left(\mathbf{a}+\tilde{k}\boldsymbol{\Theta}_2-\mathbf{s}\right)\mathbf{k} = 0. \end{aligned} \tag{18}$$

Solving the equation (17) and (18), respectively, we get

$$\left(\mathbf{W}\mathbf{c}\right)^b = e^{\rho t}\mathbf{C}\left(\tilde{\mathbf{G}}^{1-b}\right)^{-1}\left(\mathbf{I}+\mathbf{r}_c\right)\nabla_{\mathbf{k}} J(t,\mathbf{k}), \tag{19}$$

$$(\mathbf{W}\mathbf{c})^b = e^{\rho t}\mathbf{W}\left(\tilde{\mathbf{G}}^{1-b}\right)^{-1}\tilde{\mathbf{k}}\left(\mathbf{I}-\mathbf{r}_y\right)\left(\mathbf{s}-\mathbf{a}-\tilde{k}\mathbf{\Theta}_2\right)\nabla_{\mathbf{k}}J(t,\mathbf{k}), \tag{20}$$

where $\mathbf{C}, \tilde{\mathbf{G}}$ and $\tilde{\mathbf{k}}$ are the diagonal matrixes with the diagonal terms vector being $\mathbf{c}$, $\mathbf{G}$ and $\mathbf{k}$, respectively.

By the equation (19) and (20), we have

$$\mathbf{C}\left(\mathbf{I}+\mathbf{r}_c\right) = \mathbf{W}\tilde{\mathbf{k}}\left(\mathbf{I}-\mathbf{r}_y\right)\left(\mathbf{s}-\mathbf{a}-\tilde{k}\mathbf{\Theta}_2\right).$$

Hence

$$\mathbf{c}^* = \tilde{\mathbf{k}}\left(\mathbf{I}-\mathbf{r}_y\right)\left(\mathbf{s}-\mathbf{a}-\tilde{k}\mathbf{\Theta}_2\right)\left(\mathbf{I}+\mathbf{r}_c\right)^{-1}\mathbf{w}^*. \tag{21}$$

We conjecture that $J(t,\mathbf{k}) = e^{-\rho t}\mathbf{A}\tilde{\mathbf{k}}^b\mathbf{G}^{1-b}$, where $\mathbf{A}$ is a row-vector. Then

$$J_t(t,\mathbf{k}) = -\rho e^{-\rho t}\mathbf{A}\tilde{\mathbf{k}}^b\mathbf{G}^{1-b}, \tag{22}$$

$$\nabla_{\mathbf{k}}J(t,\mathbf{k}) = be^{-\rho t}\mathbf{A}\mathbf{G}^{1-b}\mathbf{k}^{b-1}. \tag{23}$$

Substituting Eq. (21), (22) and (23) into Eq. (16) yields

$$\begin{aligned}\rho\mathbf{A}\tilde{\mathbf{k}}^b\mathbf{G}^{1-b} &= \left(\mathbf{k}^b\right)^\tau\mathbf{A}\left(\mathbf{I}-\mathbf{r}_y\right)\left(\mathbf{s}-\mathbf{a}-\tilde{k}\mathbf{\Theta}_2\right)\mathbf{w}^*\mathbf{G}^{1-b}\\ &+b\left(\mathbf{k}^{b-1}\right)^\tau\mathbf{A}\mathbf{G}^{1-b}\left[\left(\mathbf{I}-\mathbf{r}_y\right)\mathbf{s}-2\left(\mathbf{I}-\mathbf{r}_y\right)\left(\mathbf{s}-\mathbf{a}-\tilde{k}\mathbf{\Theta}_2\right)\mathbf{W}^*\right]\mathbf{k}.\end{aligned} \tag{24}$$

Solving Eq. (24), we may get

$$\mathbf{w}^* = \frac{1}{1-2b}\left(\mathbf{s}-\mathbf{a}-\tilde{k}\mathbf{\Theta}_2\right)^{-1}\left[\rho\left(\mathbf{I}-\mathbf{r}_y\right)^{-1}-b\mathbf{s}\right]\mathbf{1}. \tag{25}$$

Substituting Eq. (25) into Eq. (21), we have

$$\mathbf{c}^* = \frac{1}{1-2b}\left(\mathbf{I}+\mathbf{r}_c\right)^{-1}\left[\rho\mathbf{I}-b\mathbf{s}\left(\mathbf{I}-\mathbf{r}_y\right)\right]\mathbf{k}. \tag{26}$$

Thus agents' optimal fraction of capital stock allocated in R&D and optimal consumption path are obtained.

Substituting Eq. (23) into Eq. (19) yields

$$(\mathbf{W}\mathbf{c})^b = b\mathbf{A}\left(\mathbf{I}+\mathbf{r}_c\right)\mathbf{c}\mathbf{k}^{b-1}. \tag{27}$$

Substituting Eq. (25) and (26) into Eq. (27), we can derive

$$\begin{aligned}\mathbf{A} = &\frac{1}{b(1-2b)^{2b-1}}\mathbf{1}^\tau\left(\mathbf{s}-\mathbf{a}-\tilde{k}\mathbf{\Theta}_2\right)^{-b}\left(\mathbf{I}-\mathbf{r}_y\right)^{b-1}\\ &\left[\rho\left(\mathbf{I}-\mathbf{r}_y\right)^{-1}-b\mathbf{s}\right]^{2b-1}\left(\mathbf{I}+\mathbf{r}_c\right)^{-b}.\end{aligned} \tag{28}$$

Then agent' optimal value may be obtained.

Principle's Optimization Problem Next, we consider the optimization problem of principle (government).

The principle's dynamic budget constraint equation may be described by a multidimensional uncertain differential equation with n jumps as follows

$$d\mathbf{K} = \left[(\mathbf{I} - \mathbf{W})\,\mathbf{sK} + \mathbf{aWK} - \mathbf{c} - \mathbf{G}\right] dt + \mathbf{\Theta}_1 \mathbf{W}\tilde{\mathbf{K}} d\mathbf{C}_t + \mathbf{\Theta}_2 \mathbf{W}\tilde{\mathbf{K}} d\mathbf{V}_t. \quad (29)$$

Let instantaneous utility function of principle be also the form by (14). And assume that the principle is interested in maximizing the expected total utilities over an infinite time horizon. Then the principle's optimization problem may be described as follows

$$\begin{cases} J(0, \mathbf{K}_0) \equiv \max\limits_{\mathbf{G}} E\left[\int_0^\infty \frac{1}{b} e^{-\rho t} \left[(\mathbf{Wc})^b\right]^\tau \mathbf{G}^{1-b}\right], \\ \text{subject to} \\ \quad d\mathbf{K} = \left[(\mathbf{I} - \mathbf{W})\,\mathbf{sK} + \mathbf{aWK} - \mathbf{c} - \mathbf{G}\right] dt \\ \qquad\quad + \mathbf{\Theta}_1 \mathbf{W}\tilde{\mathbf{K}} d\mathbf{C}_t + \mathbf{\Theta}_2 \mathbf{W}\tilde{\mathbf{K}} d\mathbf{V}_t, \quad \mathbf{K}_0 = \mathbf{k}_0. \end{cases} \quad (30)$$

By the equation of optimality (11), we have

$$\begin{aligned} -J_t(t, \mathbf{k}) = \max_{\mathbf{G}} \Big\{ & \frac{1}{b} e^{-\rho t} \left[(\mathbf{Wc})^b\right]^\tau \mathbf{G}^{1-b} + \nabla_{\mathbf{k}} J(t, \mathbf{k})^\tau \left[(\mathbf{I} - \mathbf{W})\,\mathbf{sk}\right. \\ & \left. + \mathbf{aWk} - \mathbf{c} - \mathbf{G} + \tilde{k}\mathbf{\Theta}_2 \mathbf{Wk}\right] \Big\} = \max_{\mathbf{G}} L(\mathbf{G}), \end{aligned} \quad (31)$$

where $L(\mathbf{G})$ represents the term in the braces. The optimal $\mathbf{G}$ satisfies

$$\frac{\partial L(\mathbf{G})}{\partial \mathbf{G}} = \frac{1-b}{b} e^{-\rho t} \left[(\mathbf{Wc})^b\right]^\tau \mathbf{G}^{-b} - \nabla_{\mathbf{k}} J(t, \mathbf{k})^\tau \mathbf{1} = 0,$$

or

$$\left[(\mathbf{Wc})^b\right]^\tau \mathbf{G}^{-b} = \frac{b e^{\rho t}}{1-b} \nabla_{\mathbf{k}} J(t, \mathbf{k})^\tau \mathbf{1}.$$

So

$$\mathbf{G}^{-b} = \frac{b e^{\rho t}}{1-b} \mathbf{W}^{-b} \tilde{\mathbf{c}}^{-b} \nabla_{\mathbf{k}} J(t, \mathbf{k}). \quad (32)$$

We conjecture that $J(t, \mathbf{k}) = e^{-\rho t} \mathbf{B}\tilde{\mathbf{c}}^b \mathbf{k}^{1-b}$, where $\mathbf{B}$ is a row-vector. Then

$$J_t(t, \mathbf{k}) = -\rho e^{-\rho t} \mathbf{B}\tilde{\mathbf{c}}^b \mathbf{k}^{1-b}, \quad (33)$$

$$\nabla_{\mathbf{k}} J(t, \mathbf{k}) = (1-b) e^{-\rho t} \mathbf{B}\mathbf{c}^b \mathbf{k}^{-b}. \quad (34)$$

Substituting Eq. (34) into Eq. (32), we have

$$\mathbf{G} = b^{-\frac{1}{b}} \mathbf{B}^{-\frac{1}{b}} \mathbf{wk}. \quad (35)$$

Substituting Eq. (33), (34) and (35) into Eq. (31) yields

$$\rho\mathbf{B}\tilde{\mathbf{c}}^b\mathbf{k}^{1-b} = b^{-\frac{1}{b}}\mathbf{B}^{1-\frac{1}{b}}\mathbf{c}^b\left(\mathbf{k}^{1-b}\right)^{\tau}\mathbf{w} + (1-b)\mathbf{B}\mathbf{c}^b\left(\mathbf{k}^{-b}\right)^{\tau}\Big[(\mathbf{I}-\mathbf{W})\,\mathbf{sk} + \mathbf{aWk} - \mathbf{c} - b^{-\frac{1}{b}}\mathbf{B}^{-\frac{1}{b}}\mathbf{wk} + \tilde{k}\mathbf{\Theta}_2\mathbf{Wk}\Big]. \quad (36)$$

Solving Eq. (36), we derive

$$\mathbf{B}^{-\frac{1}{b}} = b^{\frac{1}{b}-1}\left\{\rho\mathbf{1}^{\tau}\mathbf{W}^{-1} - (1-b)\mathbf{1}^{\tau}\left[\left(\mathbf{s}-\mathbf{C}\tilde{\mathbf{k}}^{-1}\right)\mathbf{W}^{-1} - \left(\mathbf{s}-\mathbf{a}-\tilde{k}\mathbf{\Theta}_2\right)\right]\right\} \quad (37)$$

where $\mathbf{W}$ and $\mathbf{C}$ satisfies (25) and (26), respectively.

Substituting Eq. (37) and (25) into Eq. (35), we can get government' optimal R&D fiscal subsidy $\mathbf{G}^*$. At the same time, government' optimal value can be obtained.

5. Conclusion

Based on the one-dimensional uncertain optimal control with one jump, in this paper, a multidimensional uncertain optimal control problem with n jumps is studied. The equation of optimality is presented to solve the problem. Finally, an optimal control model in R&D fiscal subsidy policy is solved by the equation of optimality to show the usefulness of the equation. The difference between our work and the previous works is that in the previous work, only one-dimensional model with one jump or a special multidimensional LQ model with n jumps was considered, and in this paper, a general multidimensional uncertain optimal control model with n jumps is established. In our previous work and this work, we adapted expected value operator to measure the objective rewards in uncertain optimal control problems with jump. We may consider to do so by employing optimistic value or pessimistic value operator in future work.

Appendix: Proof of Theorem 7

Proof: By Taylor formula, we get

$$J(t+\Delta t, \mathbf{x}+\Delta\mathbf{X}_t) = J(t,\mathbf{x}) + J_t(t,\mathbf{x})\Delta t + \nabla_{\mathbf{x}}J(t,\mathbf{x})^{\tau}\Delta\mathbf{X}_t + \frac{1}{2}J_{tt}(t,\mathbf{x})\Delta t^2 + \frac{1}{2}\Delta\mathbf{X}_t{}^{\tau}\nabla_{\mathbf{xx}}J(t,\mathbf{x})\Delta\mathbf{X_t} + \nabla_{\mathbf{x}}J_t(t,\mathbf{x})^{\tau}\Delta\mathbf{X}_t\Delta t + o(\Delta t) \quad (38)$$

where $\nabla_{\mathbf{xx}}J(t,\mathbf{x})$ is the Hessian matrix of $J(t,\mathbf{x})$. Since

$$\Delta\mathbf{X_t} = \mathbf{A}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta t + \mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{C}_t + \mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{V}_t,$$

the expansion of (38) may be rewritten as

$$J(t+\Delta t, \mathbf{x}+\Delta\mathbf{X}_t)$$

$$
\begin{aligned}
&= J(t,\mathbf{x}) + J_t(t,\mathbf{x})\Delta t + \nabla_{\mathbf{x}} J(t,\mathbf{x})^{\tau}\mathbf{A}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta t \\
&\quad +\nabla_{\mathbf{x}} J(t,\mathbf{x})^{\tau}\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{C}_t + \nabla_{\mathbf{x}} J(t,\mathbf{x})^{\tau}\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{V}_t \\
&\quad +\frac{1}{2}\mathbf{A}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{A}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta t^2 \\
&\quad +\mathbf{A}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{C}_t\Delta t \\
&\quad +\mathbf{A}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{V}_t\Delta t \\
&\quad +\frac{1}{2}\big(\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{C}_t\big)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{C}_t \\
&\quad +(\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{C}_t)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{V}_t \\
&\quad +\frac{1}{2}\big(\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{V}_t\big)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{V}_t \\
&\quad +\nabla_{\mathbf{x}} J_t(t,\mathbf{x})^{\tau}\mathbf{A}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta t^2 + \nabla_{\mathbf{x}} J_t(t,\mathbf{x})^{\tau}\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{C}_t\Delta t \\
&\quad +\nabla_{\mathbf{x}} J_t(t,\mathbf{x})^{\tau}\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{V}_t\Delta t + o(\Delta t) \\
&= J(t,\mathbf{x}) + J_t(t,\mathbf{x})\Delta t + \nabla_{\mathbf{x}} J(t,\mathbf{x})^{\tau}\mathbf{A}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta t \\
&\quad +\big\{\nabla_{\mathbf{x}} J(t,\mathbf{x})^{\tau}\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right) + \nabla_{\mathbf{x}} J_t(t,\mathbf{x})^{\tau}\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta t \\
&\quad +\mathbf{A}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta t\big\}\Delta\mathbf{C}_t \\
&\quad +\big\{\nabla_{\mathbf{x}} J(t,\mathbf{x})^{\tau}\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right) + \nabla_{\mathbf{x}} J_t(t,\mathbf{x})^{\tau}\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta t \\
&\quad +\mathbf{A}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta t\big\}\Delta\mathbf{V}_t \\
&\quad +\frac{1}{2}\big(\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{C}_t\big)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{C}_t \\
&\quad +(\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{C}_t)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{V}_t \\
&\quad +\frac{1}{2}\big(\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{V}_t\big)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta\mathbf{V}_t + o(\Delta t).
\end{aligned}
\tag{39}
$$

Denote

$$
\begin{aligned}
\mathbf{m} &= \nabla_{\mathbf{x}} J(t,\mathbf{x})^{\tau}\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right) + \nabla_{\mathbf{x}} J_t(t,\mathbf{x})^{\tau}\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta t \\
&\quad +\mathbf{A}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta t, \\
\mathbf{n} &= \nabla_{\mathbf{x}} J(t,\mathbf{x})^{\tau}\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right) + \nabla_{\mathbf{x}} J_t(t,\mathbf{x})^{\tau}\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta t \\
&\quad +\mathbf{A}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta t, \\
\mathbf{P} &= \frac{1}{2}\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right), \\
\mathbf{Q} &= \mathbf{B}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right), \\
\mathbf{R} &= \frac{1}{2}\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)^{\tau}\nabla_{\mathbf{xx}} J(t,\mathbf{x})\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right).
\end{aligned}
$$

Therefore the equation (39) may be simply expressed as

$$
\begin{aligned}
&J(t+\Delta t,\mathbf{x}+\Delta\mathbf{X}_t) \\
&= J(t,\mathbf{x}) + J_t(t,\mathbf{x})\Delta t + \nabla_{\mathbf{x}} J(t,\mathbf{x})^{\tau}\mathbf{A}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta t + \mathbf{m}\Delta\mathbf{C}_t \\
&\quad +\mathbf{n}\Delta\mathbf{V}_t + \Delta\mathbf{C}_t{}^{\tau}\mathbf{P}\Delta\mathbf{C}_t + \Delta\mathbf{C}_t{}^{\tau}\mathbf{Q}\Delta\mathbf{V}_t + \Delta\mathbf{V}_t{}^{\tau}\mathbf{R}\Delta\mathbf{V}_t + o(\Delta t).
\end{aligned}
\tag{40}
$$

Substituting equation (40) into equation (10) yields

$$0 = \sup_{\mathbf{u}_t \in \mathbf{U}} \{L(\mathbf{x}, \mathbf{u}_t, t)\Delta t + J_t(t, \mathbf{x})\Delta t + \nabla_{\mathbf{x}} J(t, \mathbf{x})^\tau \mathbf{A}(t, \mathbf{X}_t, \mathbf{u}_t)\Delta t$$
$$+E[\mathbf{m}\Delta\mathbf{C}_t + \mathbf{n}\Delta\mathbf{V}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{P}\Delta\mathbf{C}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{Q}\Delta\mathbf{V}_t + \Delta\mathbf{V}_t{}^\tau\mathbf{R}\Delta\mathbf{V}_t]$$
$$+o(\Delta t)\}. \tag{41}$$

Let $\mathbf{m} = (m_i)_{1\times k}, \mathbf{n} = (n_i)_{1\times k}, \mathbf{P} = (p_{ij})_{k\times k}, \mathbf{Q} = (q_{ij})_{k\times k}$ and $\mathbf{R} = (r_{ij})_{k\times k}$. Then we have

$$\mathbf{m}\Delta\mathbf{C}_t + \mathbf{n}\Delta\mathbf{V}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{P}\Delta\mathbf{C}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{Q}\Delta\mathbf{V}_t + \Delta\mathbf{V}_t{}^\tau\mathbf{R}\Delta\mathbf{V}_t$$
$$= \sum_{i=1}^{k}(m_i\Delta C_{ti} + n_i\Delta V_{ti}) + \sum_{i=1}^{k}\sum_{j=1}^{k}(p_{ij}\Delta C_{ti}\Delta C_{tj} + q_{ij}\Delta C_{ti}\Delta V_{tj}$$
$$+r_{ij}\Delta V_{ti}\Delta V_{tj}).$$

Since $|p_{ij}\Delta C_{ti}\Delta C_{tj}| \le \frac{1}{2}|p_{ij}|(\Delta C_{ti}{}^2 + \Delta C_{tj}{}^2),\ |q_{ij}\Delta C_{ti}\Delta V_{tj}| \le \frac{1}{2}|q_{ij}|(\Delta C_{ti}{}^2 + \Delta V_{tj}{}^2),\ |r_{ij}\Delta V_{ti}\Delta V_{tj}| \le \frac{1}{2}|r_{ij}|(\Delta V_{ti}{}^2 + \Delta V_{tj}{}^2)$, we have

$$\sum_{i=1}^{k}\left[m_i\Delta C_{ti} + n_i\Delta V_{ti} - \sum_{j=1}^{k}\left\{\left(|p_{ij}| + \frac{|q_{ij}|}{2}\right)\Delta C_{ti}{}^2 + \left(|r_{ij}| + \frac{|q_{ij}|}{2}\right)\Delta V_{ti}{}^2\right\}\right]$$
$$\le \mathbf{m}\Delta\mathbf{C}_t + \mathbf{n}\Delta\mathbf{V}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{P}\Delta\mathbf{C}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{Q}\Delta\mathbf{V}_t + \Delta\mathbf{V}_t{}^\tau\mathbf{R}\Delta\mathbf{V}_t$$
$$\le \sum_{i=1}^{k}\left[m_i\Delta C_{ti} + n_i\Delta V_{ti} + \sum_{j=1}^{k}\left\{\left(|p_{ij}| + \frac{|q_{ij}|}{2}\right)\Delta C_{ti}{}^2\right.\right.$$
$$\left.\left.+\left(|r_{ij}| + \frac{|q_{ij}|}{2}\right)\Delta V_{ti}{}^2\right\}\right].$$

It follows from the independence of C_{ti} and $V_{tj}(i, j = 1, 2, \cdots, k)$ that

$$\sum_{i=1}^{k}E\left[m_i\Delta C_{ti} + n_i\Delta V_{ti} - \sum_{j=1}^{k}\left\{\left(|p_{ij}| + \frac{|q_{ij}|}{2}\right)\Delta C_{ti}{}^2 + \left(|r_{ij}| + \frac{|q_{ij}|}{2}\right)\Delta V_{ti}{}^2\right\}\right]$$
$$\le E\big[\mathbf{m}\Delta\mathbf{C}_t + \mathbf{n}\Delta\mathbf{V}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{P}\Delta\mathbf{C}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{Q}\Delta\mathbf{V}_t + \Delta\mathbf{V}_t{}^\tau\mathbf{R}\Delta\mathbf{V}_t\big]$$
$$\le \sum_{i=1}^{k}E\left[m_i\Delta C_{ti} + n_i\Delta V_{ti} + \sum_{j=1}^{k}\left\{\left(|p_{ij}| + \frac{|q_{ij}|}{2}\right)\Delta C_{ti}{}^2\right.\right.$$
$$\left.\left.+\left(|r_{ij}| + \frac{|q_{ij}|}{2}\right)\Delta V_{ti}{}^2\right\}\right].$$

Theorem 1 implies that

$$E\left[m_i\Delta C_{ti} - \sum_{j=1}^{k}\left(|p_{ij}| + \frac{|q_{ij}|}{2}\right)\Delta C_{ti}{}^2\right] = o(\Delta t),$$

$$E\left[m_i\Delta C_{ti}+\sum_{j=1}^{k}\left(|p_{ij}|+\frac{|q_{ij}|}{2}\right)\Delta {C_{ti}}^2\right]=o(\Delta t).$$

Theorem 5 implies that

$$E\left[n_i\Delta V_{ti}-\sum_{j=1}^{k}\left(|r_{ij}|+\frac{|q_{ij}|}{2}\right)\Delta {V_{ti}}^2\right]=n_i\tilde{k}\Delta t+o(\Delta t),$$
$$E\left[n_i\Delta V_{ti}+\sum_{j=1}^{k}\left(|r_{ij}|+\frac{|q_{ij}|}{2}\right)\Delta {V_{ti}}^2\right]=n_i\tilde{k}\Delta t+o(\Delta t).$$

Therefore

$$E\big[\mathbf{m}\Delta\mathbf{C}_t+\mathbf{n}\Delta\mathbf{V}_t+\Delta{\mathbf{C}_t}^\tau\mathbf{P}\Delta\mathbf{C}_t+\Delta{\mathbf{C}_t}^\tau\mathbf{Q}\Delta\mathbf{V}_t+\Delta{\mathbf{V}_t}^\tau\mathbf{R}\Delta\mathbf{V}_t\big]$$
$$=\sum_{i=1}^{k}n_i\tilde{k}\Delta t+o(\Delta t).$$

Obviously, we have

$$\sum_{i=1}^{k}n_i\tilde{k}\Delta t=\mathbf{n}\mathbf{1}\tilde{k}\Delta t=\ \nabla_{\mathbf{x}}J(t,\mathbf{x})^\tau\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\mathbf{1}\tilde{k}\Delta t+o(\Delta t).$$

Hence

$$E\big[\mathbf{m}\Delta\mathbf{C}_t+\mathbf{n}\Delta\mathbf{V}_t+\Delta{\mathbf{C}_t}^\tau\mathbf{P}\Delta\mathbf{C}_t+\Delta{\mathbf{C}_t}^\tau\mathbf{Q}\Delta\mathbf{V}_t+\Delta{\mathbf{V}_t}^\tau\mathbf{R}\Delta\mathbf{V}_t\big]$$
$$=\nabla_{\mathbf{x}}J(t,\mathbf{x})^\tau\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\mathbf{1}\tilde{k}\Delta t+o(\Delta t). \tag{42}$$

Substituting equation (42) into equation (41) yields

$$0=\sup_{\mathbf{u}_t\in\mathbf{U}}\Big\{L(\mathbf{x},\mathbf{u}_t,t)\Delta t+J_t(t,\mathbf{x})\Delta t+\nabla_{\mathbf{x}}J(t,\mathbf{x})^\tau\mathbf{A}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\Delta t$$
$$+\nabla_{\mathbf{x}}J(t,\mathbf{x})^\tau\mathbf{H}\left(t,\mathbf{X}_t,\mathbf{u}_t\right)\mathbf{1}\tilde{k}\Delta t+o(\Delta t)\Big\}. \tag{43}$$

Dividing the equation (43) by Δt, and letting $\Delta t\to 0$, we can obtain the result (11). The theorem is proved.

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