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# A Method for Decision Making with the OWA Operator

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**Abstract.** A new method for decision making that uses the ordered weighted averaging (OWA) operator in the aggregation of the information is presented. It is used a concept that it is known in the literature as the index of maximum and minimum level (IMAM). This index is based on distance measures and other techniques that are useful for decision making. By using the OWA operator in the IMAM, we form a new aggregation operator that we call the ordered weighted averaging index of maximum and minimum level (OWAIMAM) operator. The main advantage is that it provides a parameterized family of aggregation operators between the minimum and the maximum and a wide range of special cases. Then, the decision maker may take decisions according to his degree of optimism and considering ideals in the decision process. A further extension of this approach is presented by using hybrid averages and Choquet integrals. We also develop an application of the new approach in a multi-person decision-making problem regarding the selection of strategies.

**Keywords:** decision making, OWA operator, aggregation operator, index of maximum and minimum level, selection of strategies.

## 1. Introduction

The index of maximum and minimum (IMAM) level [1] is a very useful technique that provides similar results with the Hamming distance with some differences that makes it more complete. It includes the Hamming distance and the adequacy coefficient [2-6] in the same formulation. Since its appearance, it has been used in a wide range of applications such as fuzzy set theory, business decisions and multicriteria decision making [7-8]. Often, we prefer to use the normalized IMAM (NIMAM) because we want an average result of all the individual comparisons. This type of index is also known as the weighted IMAM (WIMAM) when we prefer to give different degrees of importance to the individual comparisons instead of giving them the same importance.

Sometimes, when calculating the NIMAM, it would be interesting to consider the attitudinal character of the decision maker. A very useful tool for aggregating the information considering the attitudinal character of the decision maker is the ordered weighted averaging (OWA) operator [9]. The OWA operator is an aggregation operator that includes the maximum, the minimum and the average criteria, as special cases. It has been used in a wide range of applications [10-21].

The aim of this paper is to present a new type of IMAM operator that uses the OWA operator in the aggregation process. We call this new aggregation operator, the ordered weighted averaging index of maximum and minimum level (OWAIMAM) operator. The fundamental characteristic of this index is that it normalizes the IMAM with the OWA operator. Therefore, it is possible to develop a more general IMAM that includes the maximum, the minimum and the NIMAM, as special cases. The main advantage of the OWAIMAM is the possibility of over or under estimate the results of an aggregation in order to take a decision according to a certain degree of optimism. Then, in a decision making problem, the decision maker will be able to take decisions according to his degree of optimism. Some of its main properties and different families of OWAIMAM operators are studied.

A further extension of this approach is presented by using the hybrid average [22-27]. The main advantage of this approach is that it uses the weighted average and the OWA operator in the same formulation. Thus, it is possible to consider the subjective probability and the attitudinal character of the decision maker. We call it the hybrid averaging IMAM (HAIMAM) operator. Moreover, we generalize this approach by using Choquet integrals [28-32] obtaining the Choquet integral IMAM aggregation (CIIMAMA). Thus, a more robust and general formulation of the IMAM operator is obtained.

We also develop an application of this new method in a business multi-person decision-making problem. This decision-making model can be summarized in one aggregation operator called the multi-person OWAIMAM (MP-OWAIMAM) operator. We apply it in the selection of strategies because this problem can be considered as a general one that includes a wide range of business situations. Note that other applications could be developed such as in human resource management, supplier selection and product management. For further information on other decision-making methods, refer, e.g., to [33-42].

This paper is organized as follows. In Section 2 some basic concepts such as the OWA operator and the IMAM are described. Section 3 presents the OWAIMAM operator and Section 4 analyzes some of its families. Section 5 presents an extension by using the hybrid average and Section 6 a generalization by using Choquet integrals. In Section 7 a multi-person decision-making model is presented and in Section 8 an application of the new approach in the selection of strategies. Finally, Section 9 summarizes the main findings of the paper.

## 2. Preliminaries

In this Section, we briefly review some basic concepts to be used throughout the paper such as the IMAM and the OWA operator.

### 2.1. The Index of Maximum and Minimum Level

The NIMAM [1] is an index used for calculating the differences between two elements, two sets, etc. In decision making, it is very useful for comparing alternatives in different business decision making problems such as financial management, human resource management, product management, etc. In fuzzy set theory, it can be useful, for example, for the calculation of distances between fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, etc. It is a very useful technique that provides similar results than the Hamming distance but with some differences that makes it more complete. Basically, it can be defined as a measure that includes the Hamming distance and the adequacy coefficient [2-6] in the same formulation. For two sets $P$ and $P_j$, it can be defined as follows.

**Definition 1.** A NIMAM of dimension $n$ is a mapping $K$: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ such that:

$$K(P, P_j) = \frac{1}{u+v}\left[\sum_u \left|\mu_i(u) - \mu_i^{(j)}(u)\right| + \sum_v \left(0 \vee (\mu_i(v) - \mu_i^{(j)}(v))\right)\right], \quad (1)$$

where $\mu_i$ and $\mu_i^{(j)}$ are the $i$th arguments of the sets $P$ and $P_j$ respectively, $u$ and $v$ are the number of elements used with the Hamming distance and with the dual adequacy coefficient, respectively, and $u + v = n$.

Sometimes, when normalizing the IMAM it is better to give different weights to each individual element. Then, the index is known as the WIMAM. It can be defined as follows.

**Definition 2.** A WIMAM of dimension $n$ is a mapping $K$: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector $W$ of dimension $n$ with the following properties:

$$\sum_{i=1}^{n} Z_i = 1 \text{ with } \sum_u Z_i(u) + \sum_v Z_i(v) = 1$$

$$Z_i \in [0, 1]$$

and such that:

$$K(P, P_j) = \quad (2)$$

$$\sum_{u} Z_i(u) \times \left| \mu_i(u) - \mu_i^{(j)}(u) \right| + \sum_{v} Z_i(v) \times \left[ 0 \vee (\mu_i(v) - \mu_i^{(j)}(v)) \right],$$

where $\mu_i$ and $\mu_i^{(j)}$ are the $i$th arguments of the sets $P$ and $P_j$ respectively, $u$ and $v$ are the number of elements used with the Hamming distance and with the dual adequacy coefficient, respectively, and $u + v = n$.

Note that if $u = n$, the WIMAM operator becomes the usual weighted Hamming distance (WHD) that can be defined as follows.

**Definition 3.** A weighted Hamming distance of dimension $n$ is a mapping $WHD$: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector $W$ of dimension $n$ with $W = \sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$WHD\ (A, B) = \left( \sum_{i=1}^{n} w_i \mid a_i - b_i \mid \right), \qquad (3)$$

where $a_i$ and $b_i$ are the $i$th arguments of the sets $A$ and $B$ respectively.

Moreover, the WIMAM operator accomplishes similar properties than the distance measures [29] although it does not always accomplish commutativity, from the perspective of a distance measure, because it uses norms in the aggregation process. In this case we have that a WIMAM aggregation fulfils:

Non-negativity: $K\ (A_1, A_2) \geq 0$.
Reflexivity: $K\ (A_1, A_1) = 0$.
Triangle inequality: $K\ (A_1, A_2) + K\ (A_2, A_3) \geq K\ (A_1, A_3)$.

### 2.2. The OWA Operator

The OWA operator [9] provides a parameterized family of aggregation operators which have been used in many applications. It can be defined as follows.

**Definition 4.** An OWA operator of dimension $n$ is a mapping OWA: $R^n \rightarrow R$ that has an associated weighting vector $W$ of dimension $n$ having the properties:

$w_j \in [0, 1]$
$\sum_{j=1}^{n} w_j = 1$

and such that:

$$OWA\ (a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j\ , \qquad (4)$$

where $b_j$ is the $j$th largest of the $a_i$.

From a generalized perspective of the reordering step it is possible to distinguish between the descending OWA (DOWA) operator and the ascending OWA (AOWA) operator. Note that the weights of these two operators are related by $w_j = w^*_{n-j+1}$, where $w_j$ is the $j$th weight of the DOWA and $w^*_{n-j+1}$ the $j$th weight of the AOWA operator. For further properties and applications on the OWA operator, refer, e.g., to [10,21,43-45].

## 3. The OWAIMAM Operator

In this Section, the use of the OWA operator in the IMAM operator is introduced. We call it the ordered weighted averaging index of maximum and minimum level (OWAIMAM). It can be defined as follows.

**Definition 5.** An OWAIMAM operator of dimension $n$, is a mapping $OWAIMAM$: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector $W$, with $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$OWAIMAM\ (\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = \sum_{j=1}^{n} w_j K_j\ , \qquad (5)$$

where $K_j$ represents the $j$th largest of all the $|x_i - y_i|$ and the $[0 \vee (x_i - y_i)]$.

In the following, a simple numerical example concerning the aggregation process with the OWAIMAM operator is presented.

**Example 1.** Assume the following arguments in an aggregation process: $X$ = (0.3, 0.4, 0.8, 0.6), $Y$ = (0.5, 0.7, 0.3, 0.7). Assume the following weighting vector $W$ = (0.1, 0.2, 0.3, 0.4). If we calculate the similarity between $X$ and $Y$ using the OWAIMAM operator, we get the following. Assume that the first two arguments have to be treated with the Hamming distance and the other two with the dual adequacy coefficient.

$OWAIMAM\ (X, Y) = 0.1 \times [0 \vee (0.8 - 0.3)] + 0.2 \times |0.4 - 0.7| + 0.3 \times |0.3 - 0.5| + 0.4 \times [0 \vee (0.6 - 0.7)] = 0.17$.

Note that from a generalized perspective of the reordering step it is possible to distinguish between descending and ascending orders. The weights of these operators are related by $w_j = w^*_{n-j+1}$, where $w_j$ is the $j$th weight of the descending OWAIMAM (DOWAIMAM) and $w^*_{n-j+1}$ the $j$th weight of the ascending OWAIMAM (AOWAIMAM) operator.

If $K$ is a vector corresponding to the ordered arguments $K_j$, we shall call this the ordered argument vector, and $W^T$ is the transpose of the weighting vector, then the OWAIMAM can be expressed as:

$$OWAIMAM\ (\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = W^T K \ . \tag{6}$$

Note that if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, then, the OWAIMAM operator can be expressed as:

$$OWAIMAM\ (\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = \frac{1}{W} \sum_{j=1}^{n} w_j K_j \ . \tag{7}$$

Analogously to the OWAIMAM operator, we can suggest a removal index that it is a dual of the OWAIMAM because $Q$ ($\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle$) = 1 – $K$ ($\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle$). We refer to it as the ordered weighted averaging dual index of maximum and minimum level (OWADIMAM). Note that it can be seen as a dissimilarity measure. It is defined as follows.

**Definition 6.** An OWADIMAM operator of dimension $n$, is a mapping $OWADIMAM$: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector $W$, with $w_j \in [0, 1]$ and the sum of the weights is equal to 1, then:

$$OWADIMAM\ (\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = \sum_{j=1}^{n} w_j Q_j \ , \tag{8}$$

where $Q_j$ represents the $j$th largest of all the $[1 - |x_i - y_i|]$ and the $[1 \wedge (1 - x_i + y_i)]$; with $k = 1,2,\ldots,m$.

The final result will be a number between [0, 1]. Note that in this case the recommendation is to select the lowest value as the best result.

In this case, we can also distinguish between the descending OWADIMAM (DOWADIMAM) and the ascending OWADIMAM (AOWADIMAM) operator.

Note also that the OWAIMAM operator follows the usual methodology of the aggregation operators. Thus, it is commutative, monotonic, bounded, idempotent, nonnegative and reflexive. As we can see, it accomplishes the usual properties excepting commutativity from the perspective of a distance measure because of the use of norms in the aggregation. These properties can be proved with the following theorems.

**Theorem 1** (Monotonicity). Assume $f$ is the OWAIMAM operator, if $|x_i - y_i| \geq |u_i - v_i|$, for all $i$, then:

$$f\ (\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) \geq f\ (\langle u_1, v_1\rangle, \langle u_2, v_2\rangle, \ldots, \langle u_n, v_n\rangle). \tag{9}$$

Proof. Let

$$f(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = \sum_{j=1}^{n} w_j K_j \,, \tag{10}$$

$$f(\langle u_1, v_1\rangle, \langle u_2, v_2\rangle, \ldots, \langle u_n, v_n\rangle) = \sum_{j=1}^{n} w_j Q_j \,. \tag{11}$$

Since $|x_i - y_i| \geq |u_i - v_i|$, for all $i$, it follows that, $K_j \geq Q_j$, and then

$$f(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) \geq f(\langle u_1, v_1\rangle, \langle u_2, v_2\rangle, \ldots, \langle u_n, v_n\rangle). \quad \blacksquare$$

**Theorem 2** (Commutativity). Assume $f$ is the OWAIMAM operator, then:

$$f(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = f(\langle u_1, v_1\rangle, \langle u_2, v_2\rangle, \ldots, \langle u_n, v_n\rangle). \tag{12}$$

where $(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle)$ is any permutation of the arguments $(\langle u_1, v_1\rangle, \langle u_2, v_2\rangle, \ldots, \langle u_n, v_n\rangle)$.

Proof. Let

$$f(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = \sum_{j=1}^{n} w_j K_j \,, \tag{13}$$

$$f(\langle u_1, v_1\rangle, \langle u_2, v_2\rangle, \ldots, \langle u_n, v_n\rangle) = \sum_{j=1}^{n} w_j Q_j \,. \tag{14}$$

Since $(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle)$ is a permutation of $(\langle u_1, v_1\rangle, \langle u_2, v_2\rangle, \ldots, \langle u_n, v_n\rangle)$, we have $K_j = Q_j$, for all $j$, and then

$$f(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = f(\langle u_1, v_1\rangle, \langle u_2, v_2\rangle, \ldots, \langle u_n, v_n\rangle). \quad \blacksquare$$

**Theorem 3** (Idempotency). Assume $f$ is the OWAIMAM operator, if $|x_i - y_i| = |x - y|$, for all $i$, then:

$$f(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = |x - y|. \tag{15}$$

**Proof.** Since $|x_i - y_i| = |x - y|$, for all $i$,

$$f(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) = \sum_{j=1}^{n} w_j K_j = \sum_{j=1}^{n} w_j \,|\, x - y \,|\, = \,|\, x - y \,|\, \sum_{j=1}^{n} w_j \,. \tag{16}$$

Since $\sum_{j=1}^{n} w_j = 1$,

$$f(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = |x - y|. \quad \blacksquare$$

**Theorem 4** (Bounded). Assume $f$ is the OWAIMAM operator, then:

$$\text{Min}\{|x_i - y_i|\} \leq f(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) \leq \text{Max}\{|x_i - y_i|\}. \quad (17)$$

**Proof.** Let $\max\{|x_i - y_i|\} = b$, and $\min\{|x_i - y_i|\} = a$, then

$$f(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) = \sum_{j=1}^{n} w_j K_j \leq \sum_{j=1}^{n} w_j b = b\sum_{j=1}^{n} w_j, \quad (18)$$

$$f(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) = \sum_{j=1}^{n} w_j K_j \geq \sum_{j=1}^{n} w_j a = a\sum_{j=1}^{n} w_j. \quad (19)$$

Since $\sum_{j=1}^{n} w_j = 1$,

$$f(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) \leq b. \quad (20)$$

$$f(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) \geq a. \quad (21)$$

Therefore,

$$\text{Min}\{|x_i - y_i|\} \leq f(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) \leq \text{Max}\{|x_i - y_i|\}. \quad \blacksquare$$

**Theorem 5** (Nonnegativity). Assume $f$ is the IOWAD operator, then:

$$f(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) \geq 0. \quad (22)$$

Proof. Let

$$f(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) = \sum_{j=1}^{n} w_j b_j, \quad (23)$$

Since $|x_i - y_i| \geq 0$, for all $i$, we obtain

$$f(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) \geq 0. \quad \blacksquare$$

**Theorem 6** (Reflexivity). Assume $f$ is the IOWAD operator, then:

$$f(\langle x_1, x_1\rangle, \ldots, \langle x_n, x_n\rangle) = 0. \quad (24)$$

Proof. Let

$$f(\langle x_1, x_1\rangle, \ldots, \langle x_n, x_n\rangle) = \sum_{j=1}^{n} w_j b_j, \quad (25)$$

Since $x_i = x_i$, $|x_i - x_i| = 0$, for all $i$, therefore,

$$f(\langle x_1, x_1\rangle, \ldots, \langle x_n, x_n\rangle) = 0. \qquad \blacksquare$$

A further interesting feature to consider in the OWAIMAM operator is the unification point with distance measures. The unification point between the IMAM and the Hamming distance appears when $x_i \geq y_i$ for all $i$. In the OWAIMAM operator, we find a similar situation with the difference that now the unification is with the ordered weighted averaging distance (OWAD) operator [6]. Then, we get the following.

**Theorem 7.** Assume $OWAD(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle)$ is the OWAD operator and $OWADIMAM(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle)$ the OWADIMAM operator. If $x_i \geq y_i$ for all $i$, then:

$$OWAD(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = OWADIMAM(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle). \tag{26}$$

Proof. Let

$$OWAD(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = \sum_{j=1}^{n} w_j b_j, \tag{27}$$

$$OWADIMAM(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = \sum_{j=1}^{n} w_j Q_j. \tag{28}$$

Since $x_i \geq y_i$ for all $i$, $[0 \vee (x_i - y_i)] = (x_i - y_i)$ for all $i$, then

$$OWADIMAM(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle) = \sum_{j=1}^{n} w_j (x_i - y_i) = OWAD(\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle). \qquad \blacksquare$$

As we can see, the unification with the Hamming distance is found with the dual IMAM. Note that it is possible to distinguish between different types of unifications depending on the problem analyzed such as partial and total unification point. The partial unification point appears if at least one of the alternatives but not all of them is in a situation of unification point and the total unification point appears if all the alternatives accomplish the conditions of the unification point. Note that it is straightforward to prove these unifications by looking to [5,46] and following Theorem 7.

Another interesting issue to analyze is the different measures used to characterize the weighting vector of the OWAIMAM operator. Based on the measures developed for the OWA operator in [9,19], they can be defined as follows. For the attitudinal character, we get the following:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right). \quad (29)$$

It can be shown that $\alpha \in [0, 1]$. Note that for the optimistic criteria $\alpha(W) = 1$, for the pessimistic criteria $\alpha(W) = 0$, and for the average criteria $\alpha(W) = 0.5$.

The dispersion is a measure that provides the type of information being used. It can be defined as follows.

$$H(W) = -\sum_{j=1}^{n} w_j \ln(w_j). \quad (30)$$

For example, if $w_j = 1$ for some $j$, then $H(W) = 0$, and the least amount of information is used. If $w_j = 1/n$ for all $j$, then, the amount of information used is maximum. The divergence can be defined as follows.

$$Div(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2. \quad (31)$$

Note that the divergence can also be formulated with an ascending order in a similar way as it has been shown in the attitudinal character.

## 4. Families of OWAIMAM Operators

The OWAIMAM operator provides a parameterized family of aggregation operators. Therefore, it includes a wide range of special cases. In Table 1, some of these families of OWAIMAM operators are presented.

For more information on these and other families of OWAIMAM operators that are based on the OWA methodology, refer, e.g., to [9,15,17,47-49].

In the following, the main features of the families presented in Table 1 are commented.

**Table 1.** Families of OWAIMAM operators.

| *Basic families* | *Weighting vector W* |
|---|---|
| • The OWA operator<br>• The Hamming distance<br>• The OWAD operator<br>• The adequacy coefficient<br>• The OWAAC operator<br>• Etc. | • Maximum and Minimum<br>• NIMAM and WIMAM<br>• Olympic-OWAIMAM<br>• Window-OWAIMAM<br>• S-OWAIMAM<br>• Centered-OWAIMAM<br>• BUM function – OWAIMAM<br>• Nonmonotonic-OWAIMAM<br>• Etc. |

*Remark 1.* If the second set is empty, the OWAIMAM operator becomes the OWA operator. If all the individual similarities use the Hamming distance, we get the OWAD operator [6] and if all of them use the adequacy coefficient, we obtain the ordered weighted averaging adequacy coefficient (OWAAC) operator [5].

*Remark 2.* The maximum is obtained if $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$ and the minimum if $w_n = 1$ and $w_j = 0$, for all $j \neq n$. More generally, if $w_k = 1$ and $w_j = 0$, for all $j \neq k$, we get, $OWAIMAM$ $(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = b_k$, where $b_k$ is the $k$th largest argument $a_i$. The NIMAM is formed when $w_j = 1/n$, and $\omega_i = 1/n$, for all $a_i$. The WIMAM is obtained when $w_j = 1/n$, for all $a_i$. Note that the construction of the WIMAM from the OWAIMAM is artificial in the sense that it considers the importance of the attributes while the OWAIMAM focuses on the degree of optimism of the aggregation.

*Remark 3.* The olympic-OWAIMAM is found when $w_1 = w_n = 0$, and for all others $w_{j*} = 1/(n - 2)$. Note that it is possible to present a general form of the olympic-OWAIMAM considering that $w_j = 0$ for $j = 1, 2, \ldots, k, n, n - 1, \ldots, n - k + 1$; and for all others $w_{j*} = 1/(n - 2k)$, where $k < n/2$. Note that if $k = 1$, then, this general form becomes the usual olympic-OWAIMAM. If $k = (n - 1)/2$, then, it becomes the median-OWAIMAM aggregation. That is, if $n$ is odd we assign $w_{(n+1)/2} = 1$ and $w_{j*} = 0$ for all others. If $n$ is even we assign for example, $w_{n/2} = w_{(n/2)+1} = 0.5$ and $w_{j*} = 0$ for all others.

*Remark 4.* Additionally, it is also possible to form the contrary case of the general olympic-OWAIMAM operator. This case is obtained when $w_j = (1/2k)$ for $j = 1, 2, \ldots, k, n, n - 1, \ldots, n - k + 1$; and $w_j = 0$, for all others, where $k < n/2$. Note that if $k = 1$, then, the contrary case of the median-OWAIMAM is obtained.

*Remark 5.* Following the ideas of Yager [49], the window-OWAIMAM operator can be formed. It is obtained when $w_{j*} = 1/m$ for $k \leq j^* \leq k + m - 1$ and $w_{j*} = 0$ for $j^* > k + m$ and $j^* < k$. Note that $k$ and $m$ must be positive integers such that $k + m - 1 \leq n$. Also note that if $m = k = 1$, the window-OWAIMAM becomes the maximum and if $m = 1$, $k = n$, it becomes the minimum. And if $m = n$ and $k = 1$, it is obtained the NIMAM.

*Remark 6.* A further interesting family is the S-OWAIMAM operator [49]. It can be classified in three classes: the “orlike”, the “andlike” and the generalized S-OWAIMAM operator. The generalized S-OWAIMAM operator is obtained when $w_1 = (1/n)(1 - (\alpha + \beta)) + \alpha$, $w_n = (1/n)(1 - (\alpha + \beta)) + \beta$, and $w_j = (1/n)(1 - (\alpha + \beta))$ for $j = 2$ to $n - 1$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Note that if $\alpha = 0$, the generalized S-OWAIMAM operator becomes the “andlike” S-OWAIMAM operator and if $\beta = 0$, it becomes the “orlike” S-OWAIMAM operator. Also note that if $\alpha + \beta = 1$, we get the Hurwicz IMAM criteria.

*Remark 7.* Another family of aggregation operator that could be used is the centered-OWAIMAM operator. An OWAIMAM operator is defined as a centered aggregation operator if it is symmetric, inclusive and strongly decaying. It is symmetric if $w_j = w_{j+n-1}$. It is inclusive if $w_j > 0$. It is strongly decaying when $i < j \leq (n + 1)/2$ then $w_i < w_j$ and when $i > j \geq (n + 1)/2$ then $w_i < w_j$. Note that it is possible to consider a softening of the third condition by using $w_i \leq w_j$ instead of $w_i < w_j$ and it is possible to remove the second condition.

*Remark 8.* Another interesting method for determining the OWAIMAM weights is the functional method. It can be described as follows. Let $f$ be a function $f$: [0, 1] $\rightarrow$ [0, 1] such that $f(0) = f(1)$ and $f(x) \geq f(y)$ for $x > y$. We call this function a basic unit interval monotonic function (BUM). Using this BUM function we form the OWAIMAM weights $w_j$ for $j$ = 1 to $n$ as

$$w_j = f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right). \quad (32)$$

It is easy to see that using this method, the $w_j$ satisfy that the sum of the weights is 1 and $w_j \in [0,1]$.

*Remark 9.* Another interesting family is the nonmonotonic-OWAIMAM operator based on [18]. It is possible to form it when at least one of the weights $w_j$ is lower than 0 and $\sum_{j=1}^{n} w_j = 1$. Note that a key aspect of this operator is that it does not always accomplish the monotonicity property. Then, this property could not be included in this special type of the OWAIMAM operator.

## 5. Using the Hybrid Average in the IMAM Operator

A further generalization of the OWAIMAM operator can be introduced by using the HA operator [26]. Thus, we can use in the IMAM operator, the weighted average and the OWA operator, considering both the attitudinal character of the decision maker and its subjective probability (or degree of importance). This new approach is called the hybrid averaging IMAM (HAIMAM) operator. Before defining the HAIMAM operator, let us briefly recall the definition of the HA operator.

**Definition 7.** A HA operator of dimension $n$ is a mapping $HA$: $R^n \rightarrow R$ that has an associated weighting vector $W$ of dimension $n$ with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$HA\ (a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j\ , \quad (33)$$

where $b_j$ is the $j$th largest of the $\hat{a}_i$ ($\hat{a}_i = n\omega_i a_i$, $i$ = 1,2,…,$n$), $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of the $a_i$, with $\omega_i \in [0, 1]$ and the sum of the weights is 1.

With this introduction, the HAIMAM operator can be introduced as follows. Note that the main advantage of this approach is that the WIMAM and the OWAIMAM operators can be used in the same formulation.

**Definition 8.** A HAIMAM operator of dimension $n$ is a mapping $HAIMAM$: $[0, 1]^n \times [0, 1]^n \rightarrow [0, 1]$ that has an associated weighting vector $W$ of dimension $n$ with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$HAIMAM\ (a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j K_j\ , \quad (34)$$

where $K_j$ represents the $j$th largest of all the $|x_i - y_i|^* = n\omega_i\,|x_i - y_i|$ and the $[0 \vee (x_i - y_i)]^* = n\omega_i\ [0 \vee (x_i - y_i)]$, with $i$ = 1, 2, …,$n$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weighting vector of the $a_i$, with $\omega_i \in [0, 1]$ and the sum of the weights is 1.

As we can see, if $w_j = 1/n$, for all $j$, we obtain the WIMAM operator and if $\omega_i = 1/n$, for all $i$, the OWAIMAM operator. If $w_j = 1/n$ and $\omega_i = 1/n$, for all $i$ and $j$, the NIMAM operator is obtained.

The HAIMAM operator accomplishes similar properties than the OWAIMAM operator. However, it is not idempotent nor commutative. Moreover, we can also study a wide range of families of HAIMAM operators following the methodology explained in Section 4.

## 6. Choquet Integrals in the IMAM Operator

By using Choquet integrals [28-32], another generalization of the IMAM operator can be developed. It is called the Choquet integral IMAM aggregation (CIIMAMA). Before introducing this new result, let us define the concept of fuzzy measure and the discrete Choquet integral. The fuzzy measure was introduced by Sugeno [50] and it can be defined as follows.

**Definition 9.** Let $X$ be a universal set $X = \{x_1, x_2, \ldots, x_n\}$ and $P(X)$ the power set of $X$. A fuzzy measure on $X$ is a set function on $m$: $P(X) \rightarrow [0, 1]$, that satisfies the following conditions:

$m(\varnothing) = 0$, $m(X) = 1$ (boundary conditions) and
If $A, B \in P(X)$ and $A \subseteq B$, then $m(A) \leq m(B)$ (monotonicity).

The discrete Choquet integral [28] can be defined in the following way.

**Definition 10.** Let $f$ be a positive real-valued function $f$: $X \rightarrow R^{+}$ and $m$ be a fuzzy measure on $X$. The (discrete) Choquet integral of $f$ with respect to $m$ is:

$$C_m\,(f_1, f_2, \ldots, f_n) = \sum_{i=1}^{n} f_{(i)}\left[m(A_{(i)}) - m(A_{(i-1)})\right], \tag{35}$$

where $(\cdot)$ indicates a permutation on domain and range $X$ such that $f_{(1)} \geq f_{(2)} \geq \ldots \geq f_{(n)}$, i.e. $f_{(i)}$ is the $i$th largest value in the finite set $\{f_1, f_2, \ldots, f_n\}$, $A_{(i)} = \{x_{(1)},\ldots, x_{(i)}\}$ $i \geq 1$ and $A_{(0)} = \varnothing$ being $\{x_{(1)},\ldots, x_{(i)}\}$ in the domain of $f$ such that $f(x_i) = f_i$.

With this information, we can present the CIIMAMA operator as an aggregation operator that uses the Choquet integral and the IMAM operator in the same formulation. It can be defined as follows.

**Definition 11.** Let $m$ be a fuzzy measure on $X$. A Choquet integral index of maximum and minimum level aggregation (CIIMAMA) operator of dimension $n$ is a function $CIIMAMA$: $R^n \times R^n \rightarrow R$, such that:

$$CIIMAMA\,(\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle) = \sum_{j=1}^{n} b_j\left[m(A_{(i)}) - m(A_{(i-1)})\right], \tag{36}$$

where $b_j$ is the $j$th largest of all the $|x_i - y_i|$ and the $[0 \vee (x_i - y_i)]$, the $x_i$ and the $y_i$ are the argument variables of the sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$, $A_{(i)} = \{x_{(1)}, \ldots, x_{(i)}\}$, $i \geq 1$ and $A_{(0)} = \varnothing$.

This approach can be generalized by using generalized and quasi-arithmetic means [15,17]. For example, by using quasi-arithmetic means, we get the quasi-arithmetic Choquet integral index of maximum and minimum level aggregation (Quasi-CIIMAMA) operator. It can be defined as follows.

**Definition 12.** Let $m$ be a fuzzy measure on $X$. A Quasi-CIIMAMA operator of dimension $n$ is a function $QICDIA$: $R^n \times R^n \times R^n \rightarrow R$, such that:

$$QICDIA\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = g^{-1}\left(\sum_{j=1}^{n} g(b_j)\left[m(A_{(i)}) - m(A_{(i-1)})\right]\right), \tag{37}$$

where $g$ is a strictly continuous monotonic function, $b_j$ is the $j$th largest of all the $|x_i - y_i|$ and the $[0 \vee (x_i - y_i)]$, the $x_i$ and the $y_i$ are the argument variables of the sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$, $A_{(i)} = \{x_{(1)}, \ldots, x_{(i)}\}$, $i \geq 1$ and $A_{(0)} = \varnothing$.

## 7. Multi-Person Decision Making with the OWAIMAM Operator

The OWAIMAM operator can be applied in a wide range of fields. In this paper, we consider a decision-making application in the selection of strategies by using a multi-person analysis. Note that in the literature we find a wide range of methods for decision making [43,45,51-53]. The process to follow can be summarized as follows.

*Step* 1: Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of finite alternatives, $C = \{C_1, C_2, \ldots, C_n\}$ a set of finite characteristics (or attributes), forming the matrix $(x_{hi})_{m\times n}$. Let $E = \{E_1, E_2, \ldots, E_p\}$ be a finite set of decision makers. Let $V = (v_1, v_2, \ldots, v_p)$ be the weighting vector of the decision makers such that $\sum_{k=1}^{p} v_k = 1$ and $v_k \in [0, 1]$. Each decision maker provides his own payoff matrix $(x_{hi}^{(k)})_{m\times n}$.

*Step* 2: Calculate the ideal values of each characteristic in order to form the ideal strategy shown in Table 2. Note that the ideal strategy is an unreal strategy where we imagine an optimal situation where we are able to reach all our objectives, perfectly.

**Table 2.** Ideal strategy.

| | $C_1$ | $C_2$ | … | $C_i$ | … | $C_n$ |
|---|---|---|---|---|---|---|
| I = | $y_1$ | $y_2$ | … | $y_i$ | … | $y_n$ |

where $I$ is the ideal strategy expressed by a fuzzy subset, $C_i$ is the $i$th characteristic to consider and $y_i \in [0, 1]$, $i = 1, 2, \ldots, n$, is a number between 0 and 1 for the $i$th characteristic. Each decision maker provides his own ideal strategy $y_i^{(k)}$.

*Step* 3: Calculate the weighting vector $W$ to be used in the OWAIMAM aggregation for each alternative $h$ and characteristic $i$. Note that $W = (w_1, w_2, \ldots, w_n)$ such that $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$.

*Step* 4: Comparison between the ideal strategy and the different characteristics considered using the OWAIMAM operator for each expert (person).

*Step* 5: Use the weighted average (WA) to aggregate the information of the decision makers $E$ by using the weighting vector $V$. The result is the collective payoff matrix $(x_{hi}, y_{hi})_{m\times n}$. Thus, $(x_{hi}, y_{hi}) = \sum_{k=1}^{p} v_k (x_{hi}^k, y_{hi}^k)$. Note that $(x_{hi}, y_{hi})$ represents either $|x_{hi} - y_{hi}|$ or $[0 \vee (x_{hi} - y_{hi})]$, for the comparison of each tuple of arguments.

*Step* 6: Calculate the aggregated results by using the OWAIMAM operator explained in Eq. (4). Consider different types of OWAIMAM operators by using different expressions in the weighting vector as it has been explained in Section 4.

*Step* 7: Select the alternative/s that provides the best result/s. Moreover, establish a ranking of the alternatives from the most to the less preferred alternative in order to be able to consider more than one selection.

Note that this decision-making process can be summarized using the following aggregation operator that it is called the multi-person – OWAIMAM (MP-OWAIMAM) operator.

**Definition 13.** A MP-OWAIMAM operator is an aggregation operator that has a weighting vector $V$ of dimension $p$ with $\sum_{k=1}^{p} v_k = 1$ and $v_k \in [0, 1]$ and a weighting vector $W$ of dimension $n$ with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$f(\langle(x_1^1, \ldots, x_1^p), (y_1^1, \ldots, y_1^p)\rangle, \ldots, \langle(x_n^1, \ldots, x_n^p), (y_n^1, \ldots, y_n^p)\rangle) = \sum_{j=1}^{n} w_j b_j , \tag{38}$$

where $b_j$ is the $j$th largest of all the similarities $(x_i, y_i)$ (either $|x_i - y_i|$ or $[0 \vee (x_i - y_i)]$), $(x_i, y_i) = \sum_{k=1}^{p} v_k (x_i^k, y_i^k)$, $(x_i, y_i)$ is the argument variable provided by each person (or expert) represented in the form of individual similarities.

The MP-OWAIMAM operator has similar properties than those explained in Section 3 such as the measures for characterizing the weighting vector $W$, the distinction between descending and ascending orders, and so on.

The MP-OWAIMAM operator includes a wide range of particular cases following the methodology explained in Section 4. Thus, we can find as special cases:

The multi-person – normalized Hamming distance (MP-NHD) operator.
The multi-person – weighted Hamming distance (MP-WHD) operator.
The multi-person – OWAD (MP-OWAD) operator.
The multi-person – normalized adequacy coefficient (MP-NAC) operator.
The multi-person – weighted adequacy coefficient (MP-WAC) operator.
The multi-person – OWAAC (MP-OWAAC) operator.
The multi-person – NIMAM operator.
The multi-person – OWA (MP-OWA) operator.

Additionally, it is possible to consider more complex formulations by using other types of aggregation operators in the aggregation of the experts opinion because in Definition 12, we assume that the experts opinions are aggregated by using the WA operator. Moreover, note that it is possible to develop a similar model by using Choquet integrals obtaining the multi-person – CIIMAMA (MP-CIIMAMA) operator and by using hybrid averages, obtaining the multi-person – HAIMAM (MP-HAIMAM) operator.

## 8. Numerical Example

In the following, we are going to present an illustrative example where we will see the applicability of the new approach. We consider a decision making problem regarding the selection of strategies. Different types of OWAIMAM operators such as the NIMAM, the WIMAM, the OWAIMAM, the AOWAIMAM and the olympic-OWAIMAM are used. The dual results are also considered.

Assume an enterprise that operates in Europe and in North America is considering an expansion for the next year and they consider 5 strategies to follow.

$A_1$: Expand to Asian market.

$A_2$: Expand to the South American market.

$A_3$: Expand to the African market.

$A_4$: Expand to the Oceanian market.

$A_5$: Expand to the 4 continents.

$A_6$: Do not develop any expansion.

In order to evaluate these strategies, the company has brought together a group of experts. They consider different characteristics about the strategies that can be summarized in the following ones: $C_1$ = Risk of the strategy; $C_2$ = Difficulty; $C_3$ = Benefits in the short term; $C_4$ = Benefits in the mid term; $C_5$ = Benefits in the long term; $C_6$ = Other characteristics.

**Table 3.** Payoff matrix – Expert 1.

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ |
|---|---|---|---|---|---|---|
| $A_1$ | 0.7 | 0.8 | 0.9 | 0.9 | 0.3 | 0.6 |
| $A_2$ | 0.9 | 0.7 | 0.7 | 0.5 | 0.5 | 1 |
| $A_3$ | 1 | 0.5 | 0.7 | 0.8 | 0.6 | 0.7 |
| $A_4$ | 0.7 | 0.5 | 0.6 | 0.7 | 0.8 | 0.8 |
| $A_5$ | 0.9 | 0.7 | 0.2 | 0.7 | 0.8 | 0.8 |
| $A_6$ | 0.6 | 0.8 | 0.7 | 0.8 | 0.7 | 0.7 |

**Table 4.** Payoff matrix – Expert 2.

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ |
|---|---|---|---|---|---|---|
| $A_1$ | 0.6 | 0.7 | 0.8 | 0.9 | 0.7 | 0.6 |
| $A_2$ | 0.8 | 0.6 | 0.7 | 0.6 | 0.5 | 1 |
| $A_3$ | 0.9 | 0.7 | 0.6 | 0.8 | 0.6 | 0.7 |
| $A_4$ | 0.6 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
| $A_5$ | 0.6 | 0.7 | 0.5 | 0.8 | 0.8 | 0.7 |
| $A_6$ | 0.7 | 0.8 | 0.7 | 0.9 | 0.5 | 0.7 |

**Table 5.** Payoff matrix – Expert 3.

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ |
|---|---|---|---|---|---|---|
| $A_1$ | 0.4 | 0.7 | 0.8 | 0.8 | 0.7 | 0.8 |
| $A_2$ | 0.8 | 0.7 | 0.7 | 0.6 | 0.5 | 1 |
| $A_3$ | 0.8 | 0.7 | 0.6 | 0.8 | 0.5 | 0.8 |
| $A_4$ | 0.6 | 0.5 | 0.6 | 0.8 | 0.6 | 0.9 |
| $A_5$ | 0.5 | 0.7 | 0.4 | 0.8 | 0.7 | 0.8 |
| $A_6$ | 0.7 | 0.8 | 0.6 | 0.9 | 0.6 | 0.6 |

The group of experts of the company is constituted by three persons that give its own opinion regarding the results obtained with each strategy. The results are shown in Tables 3, 4 and 5. Note that the results are valuations (numbers) between 0 and 1.

According to the objectives of the decision-maker, each expert establishes his own ideal strategy. The results are shown in Table 6.

**Table 6.** Collective results.

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ |
|---|---|---|---|---|---|---|
| $A_1$ | 0.28 | 0.07 | 0.07 | 0.11 | 0.38 | 0.26 |
| $A_2$ | 0 | 0.13 | 0.2 | 0.4 | 0.46 | 0 |
| $A_3$ | 0.03 | 0.16 | 0.27 | 0.17 | 0.4 | 0.2 |
| $A_4$ | 0.2 | 0.3 | 0.3 | 0.23 | 0.24 | 0.07 |
| $A_5$ | 0.18 | 0.1 | 0.53 | 0.2 | 0.2 | 0.17 |
| $A_6$ | 0.16 | 0 | 0.24 | 0.1 | 0.36 | 0.28 |

With this information, we can aggregate it in order to make a decision. First, the information of the three experts is aggregated in order to obtain a collective matrix represented in the form of individual similarities between the available alternatives and the ideal ones. We use the WA to obtain this matrix and assuming that $V$ = (0.2, 0.4, 0.4). The results are shown in Table 7.

**Table 7.** Ideal strategy.

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ |
|---|---|---|---|---|---|---|
| $E_1$ | 0.9 | 0.8 | 0.9 | 0.9 | 1 | 0.9 |
| $E_2$ | 0.8 | 0.8 | 0.9 | 1 | 1 | 0.9 |
| $E_3$ | 0.8 | 0.8 | 0.9 | 1 | 0.9 | 1 |

The group of experts considers the following weighting vector for all the cases: $W$ = (0.1, 0.1, 0.1, 0.2, 0.2, 0.3). In this example, we assume that the group of experts considers the three first characteristics with the Hamming distance and the other three with the adequacy coefficient. The usefulness of the IMAM is that we can use the Hamming distance or the adequacy coefficient depending on the particular interests of the decision maker in the analysis.

With this information, we can aggregate the expected results in order to obtain a representative result for each alternative. First, we consider the NIMAM, the WIMAM, the OWAIMAM, the AOWAIMAM and the olympic-OWAIMAM. Note that in the olympic-OWAIMAM, we consider $w_1 = w_6 = 0$, and for all others $w_j = 1/(n - 2)$. Then, in this case, we have: $w_2 = w_3 = w_4 = w_5 = 0.25$. The results are shown in Table 8.

**Table 8.** Aggregated results 1.

| | NIMAM | WIMAM | OWAIMAM | AOWAIMAM | Olympic |
|---|---|---|---|---|---|
| $A_1$ | 0.195 | 0.218 | 0.149 | 0.247 | 0.18 |
| $A_2$ | 0.198 | 0.205 | 0.132 | 0.271 | 0.1825 |
| $A_3$ | 0.205 | 0.22 | 0.162 | 0.25 | 0.2 |
| $A_4$ | 0.223 | 0.195 | 0.191 | 0.248 | 0.2425 |
| $A_5$ | 0.23 | 0.212 | 0.193 | 0.284 | 0.1875 |
| $A_6$ | 0.19 | 0.216 | 0.14 | 0.238 | 0.195 |

Now, the results obtained by using the OWADIMAM operator are considered. Obviously, the results obtained are the dual of the previous ones. The results are shown in Table 9.

**Table 9.** Aggregated results 2.

| | NDIMAM | WDIMAM | OWADIMAM | AOWADIMAM | Olympic |
|---|---|---|---|---|---|
| $A_1$ | 0.805 | 0.782 | 0.851 | 0.753 | 0.82 |
| $A_2$ | 0.802 | 0.795 | 0.868 | 0.729 | 0.8175 |
| $A_3$ | 0.795 | 0.78 | 0.838 | 0.75 | 0.8 |
| $A_4$ | 0.777 | 0.805 | 0.809 | 0.752 | 0.7575 |
| $A_5$ | 0.77 | 0.788 | 0.807 | 0.716 | 0.7125 |
| $A_6$ | 0.81 | 0.784 | 0.86 | 0.762 | 0.805 |

As we can see, depending on the aggregation operator used the results are different. $A_6$ is optimal choice with the NIMAM and the AOWAIMAM operator, $A_1$ with the olympic-OWAIMAM, $A_4$ with the WIMAM and $A_2$ with the OWAIMAM operator. Obviously, the same results are found with the dual indexes.

Another interesting issue is to establish an ordering of the alternatives. Note that this is useful when we want to consider more than one alternative. The results are shown in Table 10.

**Table 10.** Ordering of the strategies.

| | Ordering | | Ordering |
|---|---|---|---|
| NIMAM | $A_6 \succ A_1 \succ A_2 \succ A_3 \succ A_4 \succ A_5$ | NDIMAM | $A_6 \succ A_1 \succ A_2 \succ A_3 \succ A_4 \succ A_5$ |
| WIMAM | $A_4 \succ A_2 \succ A_5 \succ A_6 \succ A_1 \succ A_3$ | WDIMAM | $A_4 \succ A_2 \succ A_5 \succ A_6 \succ A_1 \succ A_3$ |
| OWAIMAM | $A_2 \succ A_6 \succ A_1 \succ A_3 \succ A_4 \succ A_5$ | OWADIMAM | $A_2 \succ A_6 \succ A_1 \succ A_3 \succ A_4 \succ A_5$ |
| AOWAIMAM | $A_6 \succ A_1 \succ A_4 \succ A_3 \succ A_2 \succ A_5$ | AOWADIMAM | $A_6 \succ A_1 \succ A_4 \succ A_3 \succ A_2 \succ A_5$ |
| Olympic | $A_1 \succ A_2 \succ A_5 \succ A_6 \succ A_3 \succ A_4$ | Olympic | $A_1 \succ A_2 \succ A_5 \succ A_6 \succ A_3 \succ A_4$ |

As we can see, depending on the aggregation operator used, the ordering of the strategies is different. Thus, these results may lead to different decisions.

## 9. Conclusions

We have analyzed the use of the OWA operator in the index of maximum and minimum level. As a result, we have obtained a new aggregation operator: the OWAIMAM operator. This operator is very useful because it provides a parameterized family of aggregation operators in the IMAM operator that includes the maximum, the minimum and the average. The main advantage of the OWAIMAM is that we can manipulate the neutrality of the aggregation so the decision maker can be more or less optimistic according to his interests. We have studied some of its main properties.

We have further extended the OWAIMAM operator by using the HA operator, obtaining the HAIMAM operator. We have seen that this operator is more general because it includes the weighted average and the OWA operator in the same formulation.

We have also studied another extension by using the Choquet integral. We have called it the CIIMAMA operator. Moreover, we have presented a further generalization by using quasi-arithmetic means, the Quasi-CIIMAMA operator.

We have applied the new approach in a multi-person decision-making problem about selection of strategies. We have seen that sometimes, depending on the particular type of OWAIMAM operator used, the results are different. Thus, the decisions of the decision maker may be also different. Moreover, we have developed the MP-OWAIMAM operator as a more general aggregation operator for the multi-person decision-making process.

In future research, we expect to develop further extensions of the OWAIMAM operator by adding new characteristics in the problem such as the use of order inducing variables and applying it to other decision making problems such as product management and investment selection.

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## References

1. Gil-Lafuente, J.: The index of maximum and minimum level in the selection of players in sport management (In Spanish). Proceedings 10th International Conference of the European Academy of Management and Business Economics (AEDEM), Reggio Calabria, Italy, pp.439-443. (2001)
2. Gil-Aluja, J.: The interactive management of human resources in uncertainty. Kluwer Academic Publishers, Dordrecht, Netherlands. (1998)
3. Gil-Lafuente, A. M.: Fuzzy logic in financial analysis. Springer, Berlin, Germany. (2005)

4. Kaufmann, A.; Gil-Aluja, J.: Introduction to the theory of fuzzy subsets in business management (In Spanish). Milladoiro, Santiago de Compostela, Spain. (1986)
5. Merigó, J. M.; Gil-Lafuente, A. M.: The generalized adequacy coefficient and its application in strategic decision making. Fuzzy Economic Review, Vol. 13, No. 2, 17-36. (2008)
6. Merigó, J. M.; Gil-Lafuente, A. M.: New decision-making techniques and their application in the selection of financial products. Information Sciences, Vol. 180, No. 11, 2085-2094. (2010)
7. Gil-Lafuente, A. M.; Merigó, J. M.: Computational Intelligence in Business and Economics. World Scientific, Singapore. (2010)
8. Gil-Lafuente, J.: Algorithms for excellence: Keys for being successful in sport management (In Spanish). Milladoiro, Vigo, Spain. (2002)
9. Yager, R. R.: On ordered weighted averaging aggregation operators in multi-criteria decision making. IEEE Transactions on Systems, Man and Cybernetics B, Vol. 18, No. 1, 183-190. (1988)
10. Beliakov, G., Pradera, A., Calvo, T.: Aggregation Functions: A Guide for Practitioners. Springer-Verlag, Berlin, Germany. (2007)
11. Canós, L., Liern, V.: Soft Computing-Based Aggregation Methods for Human Resource Management. European Journal of Operational Research, Vol. 189, No. 3, 669-681. (2008)
12. Kacprzyk, J.; Zadrozny, S.: Towards a generalized and unified characterization of individual and collective choice functions under fuzzy and nonfuzzy preferences and majority via ordered weighted average operators. International Journal of Intelligent Systems, Vol. 24, No. 1, 4-26. (2009)
13. Merigó, J. M.; Casanovas, M.: The fuzzy generalized OWA operator and its application in strategic decision making. Cybernetics & Systems, Vol. 41, No. 5, 359-370. (2010)
14. Merigó, J. M.; Casanovas, M.: Induced and heavy aggregation operators with distance measures. Journal of Systems Engineering and Electronics, Vol. 21, No. 3, 431-439. (2010)
15. Merigó, J. M.; Casanovas, M.: The uncertain induced quasi-arithmetic OWA operator. International journal of Intelligent Systems, Vol. 26, No. 1, 1-24. (2011)
16. Merigó, J. M.; Casanovas, M.: Induced and uncertain heavy OWA operators. Computers & Industrial Engineering, Vol. 60, No. 1, 106-116. (2011)
17. Merigó, J. M.; Gil-Lafuente, A. M.: The induced generalized OWA operator. Information Sciences, Vol. 179, No. 6, 729-741. (2009)
18. Yager, R. R.: Nonmonotonic OWA operators. Soft Computing, Vol. 3, No. 3, 187-196. (1999)
19. Yager, R. R.: Heavy OWA operators. Fuzzy Optimization and Decision Making, Vol. 1, No. 4, 379-397. (2002)
20. Yager, R. R.: Norms induced from OWA operators. IEEE Transactions on Fuzzy Systems, Vol. 18, No. 1, 57-66. (2010)
21. Yager, R. R.; Kacprzyk, J.: The ordered weighted averaging operators: Theory and Applications. Kluwer Academic Publishers, Norwell, MA, USA. (1997)
22. Merigó, J. M.; Casanovas, M.: Fuzzy generalized hybrid aggregation operators and its application in decision making. International Journal of Fuzzy Systems, Vol. 12, No. 1, 15-24. (2010)
23. Merigó, J. M.; Casanovas, M.; Martínez, L.: Linguistic aggregation operators for linguistic decision making based on the Dempster-Shafer theory of evidence.

International Journal of Uncertainty, Fuzziness Knowledge-Based Systems, Vol. 18, No. 3, 287-304. (2010)
24. Wei, G. W.: Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making. Applied Soft Computing, Vol. 10, No. 1, 423-431. (2010)
25. Wei, G. W.; Zhao, X.; Lin, R.: Some induced aggregating operators with fuzzy number intuitionistic fuzzy information and their applications to group decision making. International Journal of Computational Intelligence Systems, Vol. 3, No. 1, 84-95. (2010)
26. Xu, Z. S.; Da, Q. L.: An overview of operators for aggregating information. International Journal of Intelligent Systems, Vol. 18, No. 9, 953-969. (2003)
27. Zhao, H.; Xu, Z. S.; Ni, M.; Liu, S.: Generalized aggregation operators for intuitionistic fuzzy sets. International Journal of Intelligent Systems, Vol. 25, No. 1, 1-30. (2010)
28. Choquet, G.: Theory of Capacities. Annals Institute Fourier, Vol. 5, 131-295. (1953)
29. Merigó, J. M.; Casanovas, M.: Decision making with distance measures and induced aggregation operators. Computers & Industrial Engineering, Vol. 60, No. 1, 66-76. (2011)
30. Mesiar, R.: Choquet-like integrals. Journal of Mathematical Analysis and Applications, Vol. 194, No. 2, 477-488. (1995)
31. Tan, C.; Chen, X.: Induced Choquet ordered averaging operator and its application to group decision making. International journal of Intelligent Systems, Vol. 25, No. 1, 59-82. (2010)
32. Xu, Z. S.: Choquet integrals of weighted intuitionistic fuzzy information. Information Sciences, Vol. 180, No. 5, 726-736. (2010)
33. Figueira, J.; Greco, S.; Ehrgott, M.: Multiple criteria decision analysis: State of the art surveys. Springer, Boston, USA. (2005)
34. Gil-Aluja, J.; Gil-Lafuente, A. M.; Klimova, A.: M-attributes algorithm for the selection of a company to be affected by a public offering. International Journal of Uncertainty, Fuzziness Knowledge-Based Systems, Vol. 17, No. 3, 333-343. (2009)
35. Merigó, J. M.: Fuzzy decision making with immediate probabilities. Computers & Industrial Engineering, Vol. 58, No. 4, 651-657. (2010)
36. Merigó, J. M.; Casanovas, M.: Induced aggregation operators in decision making with the Dempster-Shafer belief structure. International Journal of Intelligent Systems, Vol. 24, No. 8, 934-954. (2009)
37. Xu, Y. J.; Da, Q. L.; Liu, X.: Some properties of linguistic preference relation and its ranking in group decision making. Journal of Systems Engineering and Electronics 21, No. 2, 244-249. (2010)
38. Xu, Z. S.: A method based on distance measure for interval-valued intuitionistic fuzzy group decision making. Information Sciences, Vol. 180, No. 1, 181-190. (2010)
39. Xu, Z. S.: A deviation-based approach to intuitionistic fuzzy multiple attribute group decision making. Group Decision and Negotiation, Vol. 19, No. 1, 57-76. (2010)
40. Xu, Z. S.; Cai, X.: Nonlinear optimization models for multiple attribute group decision making with intuitionistic fuzzy information. International Journal of Intelligent Systems, Vol. 25, No. 6, 489-513. (2010)

41. Xu, Z. S.; Hu, H.: Projection models for intuitionistic fuzzy multiple attribute decision making. International Journal of Information Technology and Decision Making, Vol. 9, No. 2, 267-280. (2010)
42. Zhang, R.; Liu, B.; Liu, S.: A multi-attribute auction model by dominance-based rough sets approach. Computer Science and Information Systems, Vol. 7, No. 4, 843-858. (2010)
43. Wei, G. W.: A method for multiple attribute group decision making based on the ET-WG and ET-OWG operators with 2-tuple linguistic information. Expert Systems with Applications, Vol. 37, No. 12, 7895-7900. (2010)
44. Zarghami, M.; Szidarovsky, F.: On the relation between compromise programming and ordered weighted averaging operator. Information Sciences, Vol. 180, No. 11, 2239-2248. (2010)
45. Zhou, L. G.; Chen, H. Y.: Generalized ordered weighted logarithm aggregation operators and their applications to group decision making. International Journal of Intelligent Systems, Vol. 25, No. 7, 683-707. (2010)
46. Merigó, J. M.; Gil-Lafuente, A. M.: Unification point in methods for the selection of financial products. Fuzzy Economic Review, Vol. 12, No. 1, 35-50. (2007)
47. Ahn, B. S., Park, H.: Parameterized OWA Operator Weights: An Extreme Point Approach. International Journal of Approximate Reasoning, Vol. 51, No. 7, 820-831. (2010)
48. Emrouznejad, A.; Amin, G. R.: Improving minimax disparity model to determine the OWA operator weights. Information Sciences, Vol. 180, No. 8, 1477-1485. (2010)
49. Yager, R. R.: Families of OWA operators. Fuzzy Sets and Systems, Vol. 59, No. 2, 125-148. (1993)
50. Sugeno, M.: Theory of fuzzy integrals and its applications. PhD Thesis, Tokyo Institute of Technology, Japan. (1974)
51. Merigó, J.M.: A unified model between the weighted average and the induced OWA operator. Expert Systems with Applications, Vol. 38, No. 9, 11560-11572. (2011)
52. Merigó, J. M.; Gil-Lafuente, A. M.: Fuzzy induced generalized aggregation operators and its application in multi-person decision making. Expert Systems with Applications. Vol. 38, No. 8, 9761-9772. (2011)
53. Merigó, J. M.; Gil-Lafuente, A. M.; Barcellos, L.: Uncertain induced generalized aggregation operators and its application in the theory of expertons. Fuzzy Economic Review, Vol. 15, No. 2, 25-42. (2010)

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