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SOME PROPERTIES AND CONNECTIONS BETWEEN SPECIAL CLASSES OF POLYNOMIALS

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A b s t r a c t. The aim of this paper is to give formulas and relations including special monic polynomials related to the Bernoulli, Euler, Apostol-Bernoulli, Apostol-Euler, Genocchi and Fibonacci type numbers and polynomais.

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This paper is dedicated to Professor Hari M. Srivastava

1. *Introduction*

In this paper, we provide a survey of the most important properties and connections between special classes of polynomials associated with Bernoulli, Euler, Apostol-Bernoulli, Apostol-Euler, Genocchi and Fibonacci numbers and polynomials, as well as some new results in that area.

1° The *Bernoulli polynomials* of higher order $B_d^{(h)}(x)$ are defined by

$$F_{Bh}(x,t;h) = \left(\frac{t\mathrm{e}^{xt}}{\mathrm{e}^{t}-1}\right)^{h} = \sum_{d=0}^{\infty} B_d^{(h)}(x)\,\frac{t^d}{d!}. \tag{1.1}$$

For $h = 1$, (1.4) reduces to the generating function of the classical Bernoulli polynomials, $B_d^{(1)}(x) = B_d(x)$. Furthermore, for $x = 0$, this gives the well known Bernoulli numbers $B_d = B_d(0)$. For details see [1]– [7], [13]– [26], [38].

2° The *Apostol-Bernoulli polynomials* were introduced in 1951 by Apostol [1]:

$$F_{AB}(x,t;\lambda) = \frac{t\mathrm{e}^{xt}}{\lambda \mathrm{e}^t - 1} = \sum_{d=0}^{\infty} \mathcal{B}_d(x;\lambda)\frac{t^d}{d!}, \tag{1.2}$$

where $|t + \log\lambda| < 2\pi$ (for details see [1]– [7], [13]– [26], [38]).

Substituting $x = 0$ in (1.2), for $\lambda \neq 1$, we get the Apostol-Bernoulli numbers $\mathcal{B}_d(\lambda)$,

$$\mathcal{B}_d(\lambda) = \mathcal{B}_d(0;\lambda), \tag{1.3}$$

and they can be expressed it terms of Stirling numbers of the second kind [1, Eq. (3.7)]. Setting $\lambda = 1$ in (1.2), we get the classical Bernoulli polynomials $B_d(x) = \mathcal{B}_d(x,1)$.

The *generalized Apostol–Bernoulli polynomials* $\mathcal{B}_k^{(h)}(x;\lambda)$ of order h are defined by

$$F_{AB}^{(h)}(x,t;\lambda) = \left(\frac{t}{\lambda \mathrm{e}^t - 1}\right)^h \mathrm{e}^{xt} = \sum_{k=0}^{\infty} \mathcal{B}_k^{(h)}(x;\lambda)\frac{t^k}{k!} \tag{1.4}$$

for an arbitrary real or complex parameter h, where $|t| < 2\pi$ when $\lambda = 1$; $|t| < |\log h|$ when $\lambda \neq 1$; $1^h := 1$ (cf. [18]). For $h = 1$, (1.4) reduces to the generating function of the *Apostol–Bernoulli polynomials* $\mathcal{B}_k(x;\lambda)$ given above by (1.2), and for $\lambda = 1$ to the *classical Bernoulli polynomials*, $B_k(x) = \mathcal{B}_k(x;1)$. Furthermore, for $x = 0$, this gives the well known Bernoulli numbers $B_k = B_k(0) = \mathcal{B}_k(0;1)$.

However, for $x = 0$, (1.4) gives the generalized Apostol-Bernoulli numbers of order h, $\mathcal{B}_k^{(h)}(0;\lambda)$, and in addition for $h = 1$, the numbers $\mathcal{B}_k^{(1)}(0;\lambda)$ reduce to the Apostol-Bernoulli numbers $\mathcal{B}_k(0;\lambda) = \mathcal{B}_k(\lambda)$ (see [1]). In fact, in this paper [1] the author gave a new proof of Lerch's functional equation [15] for the function $\phi(\xi,a,s)$, defined by analytic continuation of the series

$$\sum_{k=0}^{\infty} \frac{\mathrm{e}^{2k\pi\mathrm{i}\xi}}{(k+a)^s} \qquad \text{(cf. [18])}$$

and also derived a number of properties of the numbers $\mathcal{B}_k(\lambda)$, including an interesting connection with Stirling numbers of the second kind.

Several interesting properties, formulas and extensions have been also obtained by Srivastava [34] (see also the recent book [35]).

The *Stirling numbers of the first kind* $S_1(n,k)$ and *the second kind* $S_2(n,k)$ ($k \in \mathbb{N}_0$) are defined by means of the following generating functions

$$F_{S_1}(t,k) = \frac{(\log(1+t))^k}{k!} = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!} \tag{1.5}$$

and

$$F_{S_2}(t,k) = \frac{\left(\mathrm{e}^t - 1\right)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!}, \tag{1.6}$$

respectively (cf. [3, 29, 34]).

The numbers $S_2(n,k)$ can be generalized using the following generating function (cf. [3], [29], [34]):

$$F_{S_2}(t,k;\lambda) = \frac{\left(\lambda\mathrm{e}^t - 1\right)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k;\lambda)\frac{t^n}{n!}, \tag{1.7}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. For $\lambda = 1$, (1.7) reduces to (1.6), and we get the Stirling numbers of the second kind

$$S_2(n,k) = S_2(n,k;1) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n \tag{1.8}$$

(cf. [3, 29, 34]).

The previous mentioned connection of the Apostol-Bernoulli numbers $\mathcal{B}_k(\lambda)$ with Stirling numbers of the second kind was given by Apostol [1] (see also [18, Eq. (7)]):

Lemma 1.1.

$$\mathcal{B}_n(\alpha) = \frac{n\alpha}{(\alpha-1)^n}\sum_{s=0}^{n-1}(-1)^s\, s!\alpha^{s-1}\,(\alpha-1)^{n-1-s}\, S_2\,(n-1,s)\,. \tag{1.9}$$

The numbers $\mathcal{B}_n(\alpha)$ can be also written in terms of the Eulerian numbers.

A generalisation of Stirling numbers can be also defined by means of the following generating function (cf. [3]):

$$\frac{\left(\mathrm{e}^t - 1\right)^k}{k!}\mathrm{e}^{t\alpha} = \sum_{n=0}^{\infty} S^{(\alpha)}(n,k)\frac{t^n}{n!} \tag{1.10}$$

Several combinatorial properties of these polynomials have been proved in [3].

Simsek [29] has modified the generating function (1.10), defining the so-called λ-array polynomials $S_k^n(x;\lambda)$ by means of the following generating function

$$F_A(t,x,k;\lambda) = \frac{\left(\lambda \mathrm{e}^t - 1\right)^k}{k!}\mathrm{e}^{tx} = \sum_{n=0}^{\infty} S_k^n(x;\lambda)\frac{t^n}{n!}, \tag{1.11}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. Substituting $\lambda = 1$, the λ-array polynomials reduce to the array polynomials, $S^{(\alpha)}(n,k) = S_k^n(\alpha;1)$ (cf. [3,29]).

Srivastava and Todorov [37] proved an explicit formula for the generalized Bernoulli polynomials $\mathcal{B}_n^{(h)}(x) \equiv \mathcal{B}_n^{(h)}(x;1)$ in the form

$$\mathcal{B}_n^{(h)}(x) = \sum_{\nu=0}^{n}\binom{n}{\nu}\binom{h+\nu-1}{\nu}\frac{\nu!}{(2\nu)!}\sum_{j=0}^{\nu}(-1)^j\binom{\nu}{j}j^{2\nu}(x+j)^{n-\nu}$$

$$\times\, {}_2F_1\left[\nu-n,\nu-h;2\nu+1;\frac{j}{x+j}\right], \tag{1.12}$$

where the Gaussian hypergeometric function is defined by

$${}_2F_1[a,b;c;z] = \sum_{k=0}^{+\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}.$$

3° The *Apostol-Euler polynomials of the first kind* $\mathcal{E}_d(x,\lambda)$ are defined by means of the generating function

$$F_{AE}(x,t;\lambda) = \frac{2\mathrm{e}^{xt}}{\lambda\mathrm{e}^t+1} = \sum_{k=0}^{\infty}\mathcal{E}_k(x,\lambda)\frac{t^k}{k!}, \tag{1.13}$$

where $|2t+\log\lambda| < \pi$ (cf. [1]– [7], [26], [38]). For $\lambda \neq 1$, substituting $x = 1/2$ in (1.13) and making some arrangement, we obtain the Apostol-Euler numbers. Setting $\lambda = 1$ in (1.13), we get the first kind Euler polynomials $E_k(x) = \mathcal{E}_k\left(x,1\right)$.

4° The *Apostol-Euler polynomials of the second kind* are defined by means of the generating function

$$\frac{2}{\lambda\mathrm{e}^t+\lambda^{-1}\mathrm{e}^{-t}}\mathrm{e}^{tx} = \sum_{k=0}^{\infty}\mathcal{E}_k^*(x,\lambda)\frac{t^k}{k!} \tag{1.14}$$

(cf. [30]). A special kind of these polynomials for $\lambda = 1$ are denoted by $\mathcal{E}_k^*(x) = \mathcal{E}_k^*(x,1)$, and the corresponding numbers by $\mathcal{E}_k^* = \mathcal{E}_k^*(0)$. By using (1.13) and (1.14), for $x = 0$, we have the following relation

$$\mathcal{E}_k^*\left(0,\lambda\right) = 2^k\lambda\mathcal{E}_k\left(\frac{1}{2},\lambda^2\right).$$

The second kind Euler numbers E_k^* are defined by the special case of the first kind Euler polynomials, $E_k^* = 2^k E_k\left(1/2\right)$.

5° The *Euler polynomials of higher order* $E_k^{(h)}(x)$ are defined by means of the following generating function

$$F_{Eh}(x,t;h) = \left(\frac{2\mathrm{e}^{xt}}{\mathrm{e}^t+1}\right)^h = \sum_{k=0}^{\infty} E_k^{(h)}(x)\frac{t^k}{k!}, \tag{1.15}$$

so that, obviously, $E_k^{(1)}(x) = E_k(x)$.

6° The *Genocchi numbers and polynomials and their generalizations.* The Genocchi numbers G_d are defined by the generating function

$$F_g\left(t\right) = \frac{2t}{\mathrm{e}^t+1} = \sum_{k=0}^{\infty} G_k\frac{t^k}{k!}, \tag{1.16}$$

where $|t| < \pi$ (cf. [13], [20], [26], [38]).

In general, for these numbers we have $G_0 = 0$, $G_1 = 1$, and $G_{2k+1} = 0$ for $k \in \mathbb{N}$. Some relations between the Genocchi, Bernoulli and Euler numbers are given by $G_{2k} = 2\big(1-2^{2k}\big)B_{2k}$ and $G_{2k} = 2kE_{2k-1}$. The sequence of Genocchi numbers is

$$\{g_k\}_{k\geq 0} = \{0,1,-1,0,1,0,-3,0,17,0,-155,0,\ldots\}.$$

The Genocchi polynomials $G_k(x)$ are defined by the following generating function

$$F_g\left(x;t\right) = F_g\left(t\right)\mathrm{e}^{xt} = \sum_{k=0}^{\infty} G_k(x)\frac{t^k}{k!}, \tag{1.17}$$

where $|t| < \pi$. Using (1.17), it is easy to see that

$$G_k(x) = \sum_{\nu=0}^{k}\binom{k}{\nu}G_\nu x^{k-\nu}.$$

The Apostol-Genocchi polynomials $g_k(x,\lambda)$ are defined by the generating function

$$\frac{2t}{\lambda\mathrm{e}^t+1}\mathrm{e}^{xt} = \sum_{k=0}^{\infty} G_k(x,\lambda)\frac{t^k}{k!}, \tag{1.18}$$

where $|2t+\log\lambda| < \pi$. Setting $\lambda = 1$ in (1.18), we get the classical Genocchi polynomials $G_k(x) = G_k(x,1)$, which reduce to the classical Genocchi numbers $G_k = G_k(0)$ for $x = 0$.

Substituting $x = 0$ in (1.18), for $\lambda \neq 1$, we obtain the Apostol-Genocchi numbers $G_k(\lambda) = G_k(0, \lambda)$. For some detail, properties and other generalizations see [11], [13], [20], [26], [34], [35], [38].

Recently, Simsek [31, Eq. (2.13)] defined the following combinatorial numbers:

$$\frac{2}{\lambda\,(1+\lambda t)-1} = \sum_{n=0}^{\infty} Y_n(\lambda)\frac{t^n}{n!}. \tag{1.19}$$

Using (1.19), we have

$$Y_n(\lambda) = (-1)^n\,\frac{2^k n!\lambda^{2n}}{(\lambda-1)^{k+n}}. \tag{1.20}$$

(cf. [31, Theorem 14]).

A relation among the numbers $Y_n(\lambda)$, $S_1(m,n)$ and $\mathcal{B}_n(\lambda)$ is given as follows

$$2\lambda^m \sum_{n=0}^{m} \frac{\mathcal{B}_{n+1}(\lambda)S_1(m,n)}{n+1} = Y_m(\lambda). \tag{1.21}$$

1.1. Eulerian type monic polynomials

The Apostol-Bernoulli numbers can be expressed in terms of the monic polynomials $\varphi_k(\lambda)$:

$$\mathcal{B}_0(\lambda) = 0,\ \ \mathcal{B}_1(\lambda) = \frac{1}{\lambda-1},$$

and, in general for $k \geq 2$,

$$\mathcal{B}_k(\lambda) = (-1)^{k-1}\,\frac{k\lambda\,\varphi_{k-2}(\lambda)}{(\lambda-1)^k}. \tag{1.22}$$

where $\varphi_k(\lambda)$ are monic polynomials in λ and of degree k and $\varphi_k(0) = 1$ (cf. [25]). Using the generating function (1.4), for $h = 1$ and $x = 0$, and (1.22), it is easy to prove that the polynomials $\varphi_k(\lambda)$ are self-inversive (cf. [23, pp. 16–18]), i.e.,

$$\lambda^k \varphi_k\Big(\frac{1}{\lambda}\Big) \equiv \varphi_k(\lambda).$$

Ozdemir et al [25] proved that

$$\varphi_k(\lambda) = (1-\lambda)^k + \lambda\sum_{\nu=1}^{k}\binom{k+1}{\nu}(1-\lambda)^{\nu-1}\varphi_{k-\nu}(\lambda), \quad k \geq 1, \tag{1.23}$$

with $\varphi_0(\lambda) = 1$, as well as the following determinant form

$$\varphi_k(\lambda) = (-1)^k \lambda^k \begin{vmatrix} -1/\lambda & 0 & 0 & \cdots & 0 & 1 \\ \binom{2}{1} & -1/\lambda & 0 & \cdots & 0 & \xi \\ \binom{3}{1}\xi & \binom{3}{2} & -1/\lambda & \cdots & 0 & \xi^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{k}{1}\xi^{k-2} & \binom{k}{2}\xi^{k-3} & \binom{k}{3}\xi^{k-4} & \cdots & -1/\lambda & \xi^{k-1} \\ \binom{k+1}{1}\xi^{k-1} & \binom{k+1}{2}\xi^{k-2} & \binom{k+1}{3}\xi^{k-3} & \cdots & \binom{k+1}{k} & \xi^k \end{vmatrix},$$

where $\xi = 1 - \lambda$. For example, we have

$$\begin{aligned}
\varphi_0(\lambda) =&1, \\
\varphi_1(\lambda) =&\lambda + 1, \\
\varphi_2(\lambda) =&\lambda^2 + 4\lambda + 1, \\
\varphi_3(\lambda) =&\lambda^3 + 11\lambda^2 + 11\lambda + 1, \\
\varphi_4(\lambda) =&\lambda^4 + 26\lambda^3 + 66\lambda^2 + 26\lambda + 1, \\
\varphi_5(\lambda) =&\lambda^5 + 57\lambda^4 + 302\lambda^3 + 302\lambda^2 + 57\lambda + 1, \\
\varphi_6(\lambda) =&\lambda^6 + 120\lambda^5 + 1191\lambda^4 + 2416\lambda^3 + 1191\lambda^2 + 120\lambda + 1, \\
\varphi_7(\lambda) =&\lambda^7 + 247\lambda^6 + 4293\lambda^5 + 15619\lambda^4 + 15619\lambda^3 \\
&+ 4293\lambda^2 + 247\lambda + 1, \\
\varphi_8(\lambda) =&\lambda^8 + 502\lambda^7 + 14608\lambda^6 + 88234\lambda^5 + 156190\lambda^4 \\
&+ 88234\lambda^3 + 14608\lambda^2 + 502\lambda + 1 \\
\varphi_9(\lambda) =&512\lambda^9 + 52905\lambda^8 + 898068\lambda^7 + 4441476\lambda^6 + 7862124\lambda^5 \\
&+ 5296638\lambda^4 + 1305420\lambda^3 + 99732\lambda^2 + 1524\lambda + 1, \quad \text{etc.}
\end{aligned}$$

Using (1.23) for $\lambda = 1$, we have $\varphi_k(1) = (k+1)\varphi_{k-1}(1)$, and conclude that $\varphi_k(1) = (k+1)!$.

$$G(\lambda, t) = \frac{\mathrm{e}^{t(1-\lambda)} - 1}{1 - \lambda \mathrm{e}^{t(1-\lambda)}}. \tag{1.24}$$

Then

$$G(\lambda, t) = \sum_{k=1}^{\infty} \varphi_{k-1}(\lambda) \frac{t^k}{k!} \tag{1.25}$$

and

$$\frac{\partial}{\partial t} G(\lambda, t) = \frac{(\lambda - 1)^2 \mathrm{e}^{t(1+\lambda)}}{(\lambda \mathrm{e}^t - \mathrm{e}^{t\lambda})^2} = \sum_{k=0}^{\infty} \varphi_k(\lambda) \frac{t^k}{k!}. \tag{1.26}$$

Alternatively, by combining (1.24) with (1.2), we obtain the following functional equation

$$G(\lambda,t)=\frac{1}{t(1-\lambda)}\left[F_{AB}(0,t(1-\lambda);\lambda)-F_{AB}(1,t(1-\lambda);\lambda\right],$$

from which we again obtain

$$k\varphi_{k-2}(\lambda)=(1-\lambda)^{k-1}\left[\mathcal{B}_k(\lambda)-\mathcal{B}_k(1;\lambda)\right],$$

i.e.,

$$\varphi_k(\lambda)=\frac{(1-\lambda)^{k+1}}{k+2}\left[\mathcal{B}_{k+2}(\lambda)-\mathcal{B}_{k+2}(1;\lambda)\right],\qquad k\geq 0.$$

We can also prove that

$$\varphi_k(\lambda)=(k\lambda+1)\varphi_{k-1}(\lambda)+\lambda(1-\lambda)\varphi'_{k-1}(\lambda) \tag{1.27}$$

or equivalently

$$\varphi_k(\lambda)=(1-\lambda)^{k+2}\frac{\mathrm{d}}{\mathrm{d}\lambda}\left\{\frac{\lambda\varphi_{k-1}(\lambda)}{(1-\lambda)^{k+1}}\right\}. \tag{1.28}$$

The last formula (1.28) enables us to prove that $\varphi_k(\lambda)$ has k distinct real negative zeros (using Rolle's theorem).

Also, the following expression in terms of Stirling numbers of the second kind

$$\varphi_k(\lambda)=\sum_{\nu=0}^{k}(\nu+1)!S_2(k+1,\nu+1)\lambda^{\nu}(1-\lambda)^{k-\nu}.$$

holds.

To generalize the numbers $\varphi_k(\lambda)$ for $h\in\mathbb{N}$, we introduce the generating function $G^{[h]}(\lambda,t)$, so that $G(\lambda,t)$ reduces to (1.26):

Definition 1.1. *The numbers* $\varphi_k^{[h]}(\lambda)$ *of order* $h\in\mathbb{N}$ *are defined by*

$$G^{[h]}(\lambda,t)=\frac{(\lambda-1)^{h+1}\mathrm{e}^{t(h\lambda+1)}}{(\lambda\mathrm{e}^t-\mathrm{e}^{\lambda t})^{h+1}}=\sum_{k=0}^{\infty}\varphi_k^{[h]}(\lambda)\,\frac{t^k}{k!}.$$

Note that $\varphi_k^{[1]}(\lambda)\equiv\varphi_k(\lambda)$. Here, we list these numbers for $h=2$:

$$\begin{aligned}
\varphi_0^{[2]}(\lambda)&=1,\\
\varphi_1^{[2]}(\lambda)&=2\lambda+1,\\
\varphi_2^{[2]}(\lambda)&=4\lambda^2+7\lambda+1,
\end{aligned}$$

$$
\begin{aligned}
\varphi_3^{[2]}(\lambda) =&8\lambda^3+33\lambda^2+18\lambda+1,\\
\varphi_4^{[2]}(\lambda) =&16\lambda^4+131\lambda^3+171\lambda^2+41\lambda+1,\\
\varphi_5^{[2]}(\lambda) =&32\lambda^5+473\lambda^4+1208\lambda^3+718\lambda^2+88\lambda+1,\\
\varphi_6^{[2]}(\lambda) =&64\lambda^6+1611\lambda^5+7197\lambda^4+8422\lambda^3+2682\lambda^2+183\lambda+1,\\
\varphi_7^{[2]}(\lambda) =&128\lambda^7+5281\lambda^6+38454\lambda^5+78095\lambda^4+49780\lambda^3\\
&+9327\lambda^2+374\lambda+1,\\
\varphi_8^{[2]}(\lambda) =&256\lambda^8+16867\lambda^7+190783\lambda^6+621199\lambda^5+689155\lambda^4\\
&+264409\lambda^3+30973\lambda^2+757\lambda+1,\\
\varphi_9^{[2]}(\lambda) =&512\lambda^9+52905\lambda^8+898068\lambda^7+4441476\lambda^6+7862124\lambda^5\\
&+5296638\lambda^4+1305420\lambda^3+99732\lambda^2+1524\lambda+1, \quad \text{etc.}
\end{aligned}
$$

and for $h=3$:

$$
\begin{aligned}
\varphi_0^{[3]}(\lambda) =&1,\\
\varphi_1^{[3]}(\lambda) =&3\lambda+1,\\
\varphi_2^{[3]}(\lambda) =&9\lambda^2+10\lambda+1,\\
\varphi_3^{[3]}(\lambda) =&27\lambda^3+67\lambda^2+25\lambda+1,\\
\varphi_4^{[3]}(\lambda) =&81\lambda^4+376\lambda^3+326\lambda^2+56\lambda+1,\\
\varphi_5^{[3]}(\lambda) =&243\lambda^5+1909\lambda^4+3134\lambda^3+1314\lambda^2+119\lambda+1,\\
\varphi_6^{[3]}(\lambda) =&729\lambda^6+9094\lambda^5+25215\lambda^4+20420\lambda^3+4775\lambda^2+246\lambda+1,\\
\varphi_7^{[3]}(\lambda) =&2187\lambda^7+41479\lambda^6+180639\lambda^5+248595\lambda^4+115105\lambda^3\\
&+16293\lambda^2+501\lambda+1,\\
\varphi_8^{[3]}(\lambda) =&6561\lambda^8+183412\lambda^7+1193548\lambda^6+2575404\lambda^5+2048710\lambda^4\\
&+590764\lambda^3+53388\lambda^2+1012\lambda+1,\\
\varphi_9^{[3]}(\lambda) =&19683\lambda^9+792697\lambda^8+7435036\lambda^7+23807260\lambda^6+29793394\lambda^5\\
&+14969662\lambda^4+2843548\lambda^3+170284\lambda^2+2035\lambda+1, \quad \text{etc.}
\end{aligned}
$$

If $A_n(z)$ are Eulerian polynomials, then

$$\varphi_k(\lambda)=\frac{A_{k+1}(\lambda)}{\lambda}.$$

In this paper we give some new identities for the previous classes of polynomials and investigate some new properties of these polynomials. Moreover, by using their generating functions, we give some applications which are associated with the Fibonacci type polynomials of higher order in two variables.Now, we introduce the generating function for these new special polynomials in two variables $(x,y) \mapsto \mathbb{G}_d(x,y;k,m,n)$, $d \geq 0$, with the three free parameters k,m,n. The polynomials have been defined in [25].

Definition 1.2 (cf. [25]). *The polynomials $\mathbb{G}_d(x,y;k,m,n)$ are defined by means of the following generating function*

$$\begin{aligned}\mathbb{F}(z;x,y;k,m,n) &= \frac{1-x^k-y^m}{1-x^k\mathrm{e}^z-y^m\mathrm{e}^{z(m+n)}}\\ &= \sum_{d=0}^{\infty}\frac{\mathbb{G}_d(x,y;k,m,n)}{d!}\left(\frac{z}{1-x^k-y^m}\right)^d. \end{aligned} \tag{1.29}$$

A recurrence relation for the polynomials $\mathbb{G}_d(x,y;k,m,n)$ was proved in [25].

Theorem 1.1. *(cf. [25]) Let $\mathbb{G}_0(x,y;k,m,n)=1$ and d be a positive integer. Then we have*

$$\begin{aligned}\mathbb{G}_d(x,y;k,m,n) &= x^k\sum_{j=0}^{d}\binom{d}{j}\mathbb{G}_j(x,y;k,m,n)\left(1-x^k-y^m\right)^{d-j}\\ &\quad + y^m\sum_{j=0}^{d}\binom{d}{j}\mathbb{G}_j(x,y;k,m,n)(m+n)^{d-j}\left(1-x^k-y^m\right)^{d-j}.\end{aligned}$$

PROOF. By applying the umbral calculus methods to (1.29), we get

$$\begin{aligned}1-x^k-y^m &= \sum_{d=0}^{\infty}\mathbb{G}_d(x,y;k,m,n)\frac{z^d}{\left(1-x^k-y^m\right)^d d!}\\ &\quad - x^k\sum_{d=0}^{\infty}\left[\mathbb{G}(x,y;k,m,n)+1-x^k-y^m\right]^d\frac{z^d}{\left(1-x^k-y^m\right)^d d!}\\ &\quad - y^m\sum_{d=0}^{\infty}\left[\mathbb{G}(x,y;k,m,n)+(m+n)\left(1-x^k-y^m\right)\right]^d\frac{z^d}{\left(1-x^k-y^m\right)^d d!},\end{aligned}$$

with the usual convention of replacing $\mathbb{G}^d(x,y;k,m,n)$ by $\mathbb{G}_d(x,y;k,m,n)$. Comparing the coefficients of z^d on the both sides of the previous equality, we arrive at the desired result.

A few first polynomials are $\mathbb{G}_0(x,y;k,m,n)=1$, $\mathbb{G}_1(x,y;k,m,n)=x^k+(m+n)y^m$, $\mathbb{G}_2(x,y;k,m,n)=[x^k+(m+n)y^m]^2-(m+n-1)^2x^ky^m+x^k+(m+n)^2y^m$, etc.

The Bernstein basis functions are defined by means of the following generating functions:

$$\frac{(tx)^k}{k!}e^{(1-x)t} = \sum_{n=0}^{\infty} B_k^n(x)$$

where

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}, \tag{1.30}$$

$n, k \in \mathbb{N}_0$ with $0 \leq k \leq n$ (cf. [33], [28]).

2. *Relations between the polynomials $\varphi_d(\lambda)$ and special numbers and polynomials*

Theorem 2.1.

$$\varphi_k(\lambda) = \sum_{s=1}^{k+1} \sum_{j=0}^{s} (-1)^{s-j} \binom{s}{j} j^{k+1} \lambda^s (1-\lambda)^{k+1-s}. \tag{2.1}$$

PROOF. Using (1.22), we have

$$\varphi_k(\lambda) = (-1)^{k-1} \frac{(\lambda-1)^k}{\lambda} \mathcal{B}_{k+2}(\lambda).$$

Combining the above equation with (1.9) yields

$$\varphi_k(\lambda) = (-1)^{k-1} \sum_{s=1}^{k+1} (-1)^s s! \lambda^s (\lambda-1)^{k+1-s} S_2(k+1, s).$$

Thus, combining the above equation with (1.8) yields assertion of this theorem.

Combining (2.1) with (1.30), we get the following result:

Theorem 2.2.

$$\varphi_k(\lambda) = \sum_{s=1}^{k+1} \sum_{j=0}^{s} (-1)^{s-j} \frac{\binom{s}{j}}{\binom{k+1}{s}} j^{k+1} B_s^{k+1}(\lambda).$$

Theorem 2.3. *For $k \in \mathbb{N}$ and $|\lambda| < 1$, we have*

$$\sum_{n=0}^{\infty} \lambda^n (n+1)^k = \frac{\varphi_{k-1}(\lambda)}{(1-\lambda)^{k+1}}.$$

PROOF. Interpolation function of the Apostol-Bernoulli numbers is the Hurwitz-Lerch zeta function, which is given by

$$\Phi(\lambda, -k, 1) = -\frac{\mathcal{B}_{k+1}(\lambda)}{k+1}, \tag{2.2}$$

where $k \in \mathbb{N}_0$,

$$\Phi(z, s, a) = \sum_{k=0}^{\infty} \frac{z^n}{(n+a)^s}$$

($a \in \mathbb{C} \setminus \{0, -1, -2, -3, \ldots\}$, $s \in \mathbb{C}$ when $|z| < 1$, $\mathrm{Re}(s) > 1$ when $|z| = 1$ (cf. [1], [14], [35]). Combining (1.23) with (2.2), we get desired result.

The following theorem is a modification of (1.23):

Theorem 2.4. *We have*

$$\varphi_k(-\lambda) = \lambda \sum_{j=0}^{k} \binom{k+1}{j} \frac{1}{(\lambda+1)^{j-k}} \varphi_j(-\lambda).$$

PROOF. Combining the above well-known identity

$$\mathcal{E}_k(\lambda) = -\frac{2}{n+1} \mathcal{B}_{k+1}(-\lambda)$$

with (1.22) yields

$$\mathcal{E}_k(\lambda) = \frac{2\lambda}{(\lambda+1)^k} \varphi_{k-1}(-\lambda).$$

Combining the above equation with the following known identity

$$\mathcal{E}_k(\lambda) = \frac{\lambda}{1+\lambda} \sum_{j=0}^{k-1} \binom{k}{j} \mathcal{E}_j(\lambda)$$

(cf. [32], [35]), we get desired result.

The following theorem gives us a relation between the polynomial $\varphi_n(\lambda)$ and the Stirling numbers of the first kind $S_1(m, n)$.

Theorem 2.5. *We have*

$$\sum_{n=0}^{m} (-1)^{n-m} \frac{\varphi_{n-1}(\lambda) S_1(m, n)}{(\lambda-1)^n} = m! \frac{\lambda^{m-1}}{(\lambda-1)^m}$$

PROOF. Combining (1.22) with (1.21), we get

$$2\lambda^{m+1}\sum_{n=0}^{m}(-1)^n\frac{\varphi_{n-1}(\lambda)S_1(m,n)}{(\lambda-1)^{n+1}}=Y_m(\lambda).$$

Combining the above equation with (1.20), after some elementary calculations, we arrive at the desired result.

Combining the following well-known identity with

$$\mathcal{B}_k(-\lambda)=-\frac{1}{2}G_k(\lambda)$$

(1.22), we arrive at the following result:

Corollary 2.1.

$$\varphi_{k-2}(-\lambda)=-\frac{(\lambda+1)^k}{2k\lambda}G_k(\lambda).$$

By combining (1.25) with (1.2), we get the following functional equation

$$G(z,t)=\frac{1}{t(1-\lambda)}\big(F_{AB}(0,t(1-\lambda);\lambda)-F_{AB}(1,t(1-\lambda);\lambda)\big).$$

By using the above equation, we get

$$\sum_{k=1}^{\infty}\varphi_{k-1}(\lambda)\,\frac{t^k}{k!}=\frac{1}{t}\sum_{k=0}^{\infty}\left(\mathcal{B}_k(\lambda)-\mathcal{B}_k(1,\lambda)\right)(1-\lambda)^{k-1}\frac{t^k}{k!}.$$

After some calculations, we obtain

$$\sum_{k=2}^{\infty}k\varphi_{k-2}(\lambda)\,\frac{t^k}{k!}=\sum_{k=0}^{\infty}\left(\mathcal{B}_k(\lambda)-\mathcal{B}_k(1,\lambda)\right)(1-\lambda)^{k-1}\frac{t^k}{k!}.$$

Therefore

$$\sum_{k=2}^{\infty}k\varphi_{k-2}(\lambda)\,\frac{t^k}{k!}=\sum_{k=0}^{\infty}\left(\mathcal{B}_k(\lambda)-\mathcal{B}_k(1,\lambda)\right)(1-\lambda)^{k-1}\frac{t^k}{k!}$$
$$+\left(\mathcal{B}_0(\lambda)-\mathcal{B}_0\left(1,\lambda\right)\right)(1-\lambda)^{-1}+\left(\mathcal{B}_1\left(\lambda\right)-\mathcal{B}_1(1,\lambda)\right).$$

Since $\mathcal{B}_0(\lambda)=0$ and $\mathcal{B}_1(\lambda)=1/(\lambda-1)$, we have

$$\sum_{k=2}^{\infty}k\varphi_{k-2}(\lambda)\,\frac{t^k}{k!}=\sum_{k=2}^{\infty}\left(\mathcal{B}_k(\lambda)-\mathcal{B}_k(1,\lambda)\right)(1-\lambda)^{k-1}\frac{t^k}{k!}$$
$$+\mathcal{B}_1(\lambda)-\frac{1+\mathcal{B}_1(\lambda)}{\lambda},$$

i.e.,

$$\sum_{k=2}^{\infty} k\varphi_{k-2}(\lambda)\,\frac{t^k}{k!} = \sum_{k=2}^{\infty} \left(\mathcal{B}_k(\lambda) - \mathcal{B}_k(1,\lambda)\right)(1-\lambda)^{k-1}\frac{t^k}{k!}.$$

Comparing the coefficients of $\frac{t^k}{k!}$ on both sides of the above equality, we have

$$\varphi_{k-2}(\lambda) = \frac{\left(\mathcal{B}_k(\lambda) - \mathcal{B}_k(1,\lambda)\right)(1-\lambda)^{k-1}}{k}.$$

If we arrange the above equation for $k > 1$ according to the specific identity for the Apostol Bernoulli numbers and polynomials, the following results are achieved:

When $k \geq 2$, we have

$$\varphi_{k-2}(\lambda) = \frac{(1-\lambda)^{k-1}}{k}\mathcal{B}_k\left(\lambda\right) - \frac{(1-\lambda)^{k-1}}{k}\sum_{j=0}^{k}\binom{k}{j}\mathcal{B}_j(\lambda).$$

Consequently, we arrive at the following theorem, which gives us modification of the equation (1.22):

Theorem 2.6. *Let* $k \geq 2$. *Then we have*

$$\varphi_k(\lambda) = \frac{(1-\lambda)^{k-1}}{k}\sum_{j=0}^{k}\binom{k}{j}\mathcal{B}_j(\lambda).$$

Theorem 2.7. *Let* $\lambda \neq 1$. *The Apostol-Bernoulli polynomials* $x \mapsto \mathcal{B}_k^{[h]}(x,\lambda)$ *of degree* $k-h$ *and order* $h \in \mathbb{N}$ *are given by*

$$\mathcal{B}_k^{[h]}(x,\lambda) = 0, \quad 0 \leq k \leq h-1;$$

$$\mathcal{B}_k^{[h]}(x,\lambda) = \mathcal{B}_k^{[h]}(0,\lambda) = \frac{h!}{(\lambda-1)^k}, \quad k = h;$$

$$\mathcal{B}_k^{[h]}(x,\lambda) = \frac{h!}{(\lambda-1)^k}\binom{k}{h}\Bigg\{(\lambda-1)^{k-h}x^{k-h} + h\lambda\sum_{\nu=1}^{k-h}(-1)^{\nu}\binom{k-h}{\nu}(\lambda-1)^{k-h-\nu}\varphi_{\nu-1}^{[h]}(\lambda)x^{k-h-\nu}\Bigg\}, \; k \geq h+1.$$

3. *Zeros of the polynomials* $\varphi_k(\lambda)$

All zeros of $\varphi_k(t)$ are real, mutually different and negative! For polynomials of odd degree one zero is at -1. Other zeros are half of them are in $(-1,0)$, and others are reciprocal of the previous one (belong to $(-\infty,-1)$).

Thus, for polynomials $\varphi_k(t)$ of degree k we have

$$\varphi_k(t) = h_k(t) \prod_{\nu=1}^{[k/2]} \Big[t^2 - \big(\tau_\nu^{(k)} + 1/\tau_\nu^{(k)}\big)t + 1\Big] = h_k(t) \prod_{\nu=1}^{[k/2]} \big(t-\tau_\nu^{(k)}\big)\big(t-1/\tau_\nu^{(k)}\big),$$

where

$$h_k(t) = (t+1)^{k-2[k/2]} = \begin{cases} 1, & k \text{ is even}, \\ t+1, & k \text{ is odd}. \end{cases}$$

$$k=2: \quad \tau_1^{(2)} = -2+\sqrt{3} \approx -0.267949;$$

$$k=3: \quad \tau_1^{(3)} = -5+2\sqrt{6} \approx -0.101021;$$

$$k=4: \quad \tau_{1,2}^{(4)} = -\frac{1}{2}\left(13 \pm \sqrt{105} - \sqrt{2\left(135 \pm 13\sqrt{105}\right)}\right) \approx \begin{cases} -0.0430963, \\ -0.430575; \end{cases}$$

$$k=5: \quad \tau_{1,2}^{(5)} = -\left(14 \pm 3\sqrt{15} - \sqrt{6\left(55 \pm 14\sqrt{15}\right)}\right) \approx \begin{cases} -0.0195242, \\ -0.220171. \end{cases}$$

For example,

$$\varphi_4(t) = \big(t-\tau_1^{(4)}\big)\big(t-\tau_2^{(4)}\big)\big(t-1/\tau_1^{(4)}\big)\big(t-1/\tau_2^{(4)}\big),$$

$$\varphi_5(t) = (t+1)\big(t-\tau_1^{(5)}\big)\big(t-\tau_2^{(5)}\big)\big(t-1/\tau_1^{(5)}\big)\big(t-1/\tau_2^{(5)}\big).$$

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