

RELATIONS FOR BARNES' TYPE MULTIPLE BERNOULLI POLYNOMIALS AND GENERALIZED STIRLING NUMBERS VIA LAPLACE TRANSFORM AND GENERATING FUNCTIONS

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A b s t r a c t. The aim of this paper is not only to survey some properties of Barnes' multiple Changi q -Bernoulli polynomials with Barnes' zeta function, but also to derive some relations for Barnes' multiple Changi q -Bernoulli polynomials and Barnes' multiple Changi q -zeta functions using the q -integral method and modified generating functions. By applying the Laplace transform to these generating functions, we obtain infinite series and integral representation for these polynomials. Using the power series of the function $\sinh x$ and these generating functions, we obtain some new formulas for generalized Stirling numbers of the second kind. Finally, we give some special values of our general results.

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1. Introduction

Special numbers and polynomials with their generating functions and interpolation functions have been investigated by many researchers in many different applied sciences (cf. [1]– [35]).

The motivation of this paper deals with not only generating functions for special numbers and polynomials involving the Barnes' type multiple Bernoulli numbers,

the Barnes' type multiple Changhee q -Bernoulli polynomials, Stirling numbers of the second kind, but also the Barnes zeta function and the Hurwitz zeta function.

We can use the following definitions and notations. Let k tuples

$$\vec{u}_k = (u_1, u_2, \dots, u_k), \quad \vec{w}_k = (w_1, w_2, \dots, w_k),$$

with $u_1, u_2, \dots, u_k, w_1, w_2, \dots, w_k \in \mathbb{R}$. In [30], we constructed the following generating function

$$\begin{aligned} F_2(t, k; \vec{u}_k, \vec{w}_k) &= \frac{t^k}{h(k; \vec{u}_k, \vec{w}_k)} \\ &= \sum_{j=0}^{\infty} \theta_j(k; \vec{u}_k, \vec{w}_k) t^j, \end{aligned} \quad (1.1)$$

where the function $h(v; \vec{u}_v, \vec{w}_v)$ can be expanded as convergent (power) series.

Putting

$$h(k; \vec{u}_k, \vec{w}_k) = \prod_{j=1}^k (e^{tu_j} - 1)^{w_j}$$

in (1.1), we get

$$\sum_{j=0}^{\infty} \theta_j(k; \vec{u}_k, \vec{w}_k) t^j = \frac{t^{w_1+w_2+\dots+w_k}}{\prod_{j=1}^k (e^{tu_j} - 1)^{w_j}}, \quad (1.2)$$

which is also known as the Barnes' type multiple Bernoulli numbers. These numbers also come from extend the Bernoulli numbers of higher order. These numbers are generated by functions associated with the following Barnes zeta function, which is a generalization of the Riemann zeta function and also constructed by British mathematician and scientist Barnes (1901):

$$\zeta_k(z, x; \vec{u}_k) = \sum_{\ell_1, \ell_2, \dots, \ell_k=0}^{\infty} \frac{1}{(x + \ell_1 u_1 + \ell_2 u_2 + \dots + u_k \ell_k)^z},$$

where x and u_1, u_2, \dots, u_k have positive real part and z has real part greater than k (cf. [2]–[4], [5]–[6], [12], [21], [23, 31, 32] [33], [34]).

Due to Theorem 3 in [30, p.1147], with the aid of the Cauchy product rule, Simsek showed that

$$\sum_{j=0}^{\infty} \theta_j(k; \vec{u}_k, \vec{w}_k) t^j = \sum_{j=0}^{\infty} \sum_{\ell_1+\ell_2+\dots+\ell_k=j} \prod_{v=1}^k \frac{u_v^{\ell_v} B_{\ell_v}^{(w_v)}}{\ell_v!} t^j. \quad (1.3)$$

Equating coefficients t^j in the above equation, we have

$$\theta_j(k; \vec{u}_k, \vec{w}_k) = \sum_{\ell_1+\ell_2+\dots+\ell_k=j} \prod_{v=1}^k \frac{u_v^{\ell_v} B_{\ell_v}^{(w_v)}}{\ell_v!}.$$

For $\vec{u}_k = \vec{1} = (1, 1, \dots, 1)$, by combining (1.2) with the following generating function for Bernoulli numbers of order α :

$$\frac{t^\alpha}{(e^t - 1)^\alpha} = \sum_{j=0}^{\infty} B_j^{(\alpha)} \frac{t^j}{j!}$$

(cf. [10], [28], [33], [34]), we get

$$\begin{aligned} \sum_{j=0}^{\infty} \theta_j(k; \vec{1}, \vec{w}_k) t^j &= \frac{t^{w_1+w_2+\dots+w_k}}{(e^t - 1)^{w_1+w_2+\dots+w_k}} \\ &= \sum_{j=0}^{\infty} B_j^{(w_1+w_2+\dots+w_k)} \frac{t^j}{j!}. \end{aligned}$$

Equating coefficients t^j in the above equation, we have

$$\theta_j(k; \vec{1}, \vec{w}_k) = \frac{1}{j!} B_j^{(w_1+w_2+\dots+w_k)},$$

where $j \in \mathbb{N}_0$.

In the book of Phillips [19], some properties of the q -integers are given as follows:

If q is a real parameters, then, we assume that $q \in (-1, 1]$. If q is a complex parameters, then, we assume that $|q| < 1$.

$$[n, q] = [n] = \begin{cases} \frac{1 - q^n}{1 - q}, & \text{if } q \rightarrow 1, \\ n, & \text{if } q \rightarrow 1, \end{cases} \tag{1.4}$$

$$[n]! = [n][n - 1] \cdots [2][1], \quad [0]! = 1, \quad \binom{n}{k}_q = \frac{[n]!}{[n - k]![k]!}$$

see also, [31, 33].

The Stirling numbers of the second kind $S(n, k)$ are defined by means of the generating function:

$$F_s(t, v) = \frac{1}{v!} (e^t - 1)^v = \sum_{j=0}^{\infty} S(j, v) \frac{t^j}{j!}, \tag{1.5}$$

where $v \in \mathbb{N}_0$ (cf. [7], [9], [22], [25], [28], [31], [34]).

Cakic and Milovanovic [7] gave the following generating function for the numbers $S^{(\alpha)}(n, k)$, which is known as the Stirling numbers of the second kind:

$$F_\alpha(t, v) = e^{\alpha t} F_s(t, v),$$

i.e.,

$$F_\alpha(t, v) = \sum_{n=0}^{\infty} S^{(\alpha)}(n, v) \frac{t^n}{n!}. \quad (1.6)$$

By combining (1.5) and (1.6), we have

$$\sum_{n=0}^{\infty} S^{(\alpha)}(n, v) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \alpha^n \frac{t^n}{n!} \sum_{j=0}^{\infty} S(j, v) \frac{t^j}{j!}$$

By using the Cauchy product rule in the above equation yields

$$\sum_{n=0}^{\infty} S^{(\alpha)}(n, v) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^n \binom{n}{j} \alpha^{n-j} S(j, v) \frac{t^n}{n!}.$$

Equating coefficients t^j in the above equation, we have

$$S^{(\alpha)}(n, v) = \sum_{j=0}^n \binom{n}{j} \alpha^{n-j} S(j, v).$$

For detail results see also (cf. [7], [18], [26], [28], [34]).

Barnes' type multiple Changhee q -Bernoulli polynomials are defined by means of the follows generating function: For $k \in \mathbb{N}$,

$$\begin{aligned} G_q^{(k)}(w, t \mid w_1, w_2, \dots, w_k) &= (-t)^k \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots \sum_{n_k=0}^{\infty} q^{w+n_1w_1+\dots+n_kw_k} e^{[w+n_1w_1+\dots+n_kw_k]t} \prod_{j=1}^k w_j \\ &= \sum_{n=0}^{\infty} \frac{B_n^{(k)}(w : q \mid w_1, w_2, \dots, w_k) t^n}{n!} \end{aligned} \quad (1.7)$$

(see for detail cf. [33, Theorem 11]),

$$\lim_{q \rightarrow 1} B_n^{(k)}(w : q \mid w_1, w_2, \dots, w_k) = B_n^{(k)}(w \mid w_1, w_2, \dots, w_k),$$

which gives us the Barnes' type multiple Bernoulli numbers (cf. [18], [31], [34]).

The sections with their results are briefly summarized as follows:

In Section 1, by using q -integres method, we derive some results and relations for the Barnes' type multiple Changhee q -Bernoulli polynomials and the Barnes' type Changhee multiple q -zeta functions

In Section 2, by applying the Laplace transform to the modified generating function, we derive infinite series and integral representation for the polynomials $B_n^{(k)}(w \mid w_1, w_2, \dots, w_k)$.

In Section 3, by using power series of the function $\sinh x$, and generating function, we give some new formulas for the numbers $Y(n, c; k \mid w_1, w_2, \dots, w_k)$.

Finally, we conclude this paper with the Conclusion section.

2. New Formulas and Relations for Barnes' Type Multiple Changhee q -Bernoulli Polynomials and Barnes' Type Changhee Multiple q -Zeta Functions

In this section, we modified generating functions for special numbers and polynomials involving the Barnes' type multiple Bernoulli numbers, the Barnes' type multiple Changhee q -Bernoulli polynomials, which were given by Ozden et al [18]. By using these new generating functions, we derive many new formulas and relations involving the Barnes' type multiple Bernoulli numbers, the Barnes' type multiple Changhee q -Bernoulli polynomials, the Stirling numbers of the second kind and also the Barnes zeta function and the Hurwitz zeta function.

By the help of the Taylor series of e^{tx} , (1.7) gives us

$$\sum_{n=0}^{\infty} \frac{B_n^{(k)}(w : q \mid w_1, w_2, \dots, w_k)t^n}{n!} = (-1)^k \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots \sum_{n_k=0}^{\infty} \left(\prod_{j=1}^k w_j \right) \times \sum_{n=0}^{\infty} \frac{q^{w+n_1w_1+n_2w_2+\dots+n_kw_k} [w + n_1w_1 + n_2w_2 + \dots + n_kw_k] t^{n+k}}{n!}.$$

Thus,

$$\sum_{n=0}^{\infty} \frac{B_n^{(k)}(w : q \mid w_1, w_2, \dots, w_k)t^n}{n!} = (-1)^k \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots \sum_{n_k=0}^{\infty} \left(\prod_{j=1}^k w_j \right) \times \sum_{n=0}^{\infty} \binom{n}{k} k! \frac{q^{w+n_1w_1+n_2w_2+\dots+n_kw_k} [w + n_1w_1 + n_2w_2 + \dots + n_kw_k] t^n}{n!}.$$

By equating the coefficients t^n found on both sides of the previous equation, and after performing the necessary calculations, the following result is obtained

$$\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots \sum_{n_k=0}^{\infty} q^{w+n_1w_1+\dots+n_kw_k} [w + n_1w_1 + \dots + n_kw_k]^{n-k} = \frac{(-1)^k}{\left(\prod_{j=1}^k w_j \right) \binom{n}{k} k!} B_n^{(k)}(w : q \mid w_1, w_2, \dots, w_k). \tag{2.1}$$

In order to give the values of the polynomials $B_n^{(k)}(w : q \mid w_1, w_2, \dots, w_k)$, in the left-hand side of the above equation $|q| < 1$. Due to Definition 5 and Theorem 12 in the work of Srivastava, Kim and Simsek [33], equation (2.1) gives us interpolation

function for the polynomials $B_n^{(k)}(w : q \mid w_1, w_2, \dots, w_k)$. It is known as the Barnes' type Changhee multiple q -zeta functions. That is

$$\begin{aligned} & \zeta_{q,k}(s, w : q \mid w_1, w_2, \dots, w_k) \\ &= \left(\prod_{j=1}^k w_j \right) \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \frac{q^{w+n_1w_1+n_2w_2+\cdots+n_kw_k}}{[w+n_1w_1+n_2w_2+\cdots+n_kw_k]^s}, \end{aligned}$$

where $\operatorname{Re}(w) > 0$ and $\operatorname{Re}(s) > 1$ and $|q| < 1$ [33, Definition 5]. Therefore,

$$\begin{aligned} & \zeta_{q,k}(s, w : q \mid w_1, w_2, \dots, w_k) \\ &= (1-q)^s \left(\prod_{j=1}^k w_j \right) \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \frac{q^{w+n_1w_1+n_2w_2+\cdots+n_kw_k}}{(1-q^{w+n_1w_1+n_2w_2+\cdots+n_kw_k})^s}. \end{aligned}$$

Since $|q| < 1$, with the aid of binomial theorem, we get

$$\begin{aligned} \zeta_{q,k}(s, w : q \mid w_1, \dots, w_k) &= (1-q)^s \left(\prod_{j=1}^k w_j \right) \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \\ &\quad \times \sum_{m=0}^{\infty} (-1)^m \frac{s(s+1) \cdots (s+m-1)}{m!} q^{(m+1)(w+n_1w_1+\cdots+n_kw_k)}. \end{aligned}$$

The Lah numbers

$$s(s+1) \cdots (s+m-1) = \sum_{j=1}^m |L(m, j)| s(s-1) \cdots (s-j+1),$$

(cf. [8], [9], [28]), we obtain the following alternative form for the function $\zeta_{q,k}(s, w : q \mid w_1, \dots, w_k)$:

$$\begin{aligned} \zeta_{q,k}(s, w : q \mid w_1, \dots, w_k) &= \left(\prod_{j=1}^k w_j \right) \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (1-q)^s}{m!} \\ &\quad \times \sum_{j=1}^m |L(m, j)| s(s-1) \cdots (s-j+1) q^{(m+1)(w+n_1w_1+\cdots+n_kw_k)}. \end{aligned}$$

Since

$$[x+y] = [x] + q^x [y],$$

we get the following alternative form for the function $\zeta_{q,k}(s, w : q \mid w_1, w_2, \dots, w_k)$ as follows:

$$\begin{aligned} &\zeta_{q,k}(s, w : q \mid w_1, \dots, w_k) \\ &= q^{-ws} \left(\prod_{j=1}^k w_j \right) \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \frac{q^{w+n_1w_1+\dots+n_kw_k}}{\left(\frac{[w]}{q^w} + [n_1w_1 + \dots + n_kw_k] \right)^s}. \end{aligned}$$

When $q \rightarrow 1$, equation (1.7) reduces to the following generating function for the polynomials $B_n^{(k)}(w \mid w_1, w_2, \dots, w_k)$:

$$F(w, t \mid w_1, w_2, \dots, w_k) = \frac{t^k e^{t\left(\frac{2w+w_1+w_2+\dots+w_k}{2}\right)} \prod_{j=1}^k w_j}{\left(e^{t\frac{w_1}{2}} - e^{-t\frac{w_1}{2}} \right) \left(e^{t\frac{w_2}{2}} - e^{-t\frac{w_2}{2}} \right) \cdots \left(e^{t\frac{w_k}{2}} - e^{-t\frac{w_k}{2}} \right)}.$$

Combining the above equation with

$$2 \sinh(\alpha x) = e^{\alpha x} - e^{-\alpha x},$$

we get

$$F(w, t \mid w_1, \dots, w_k) = \left(\frac{t}{2} \right)^k e^{t\left(\frac{2w+w_1+\dots+w_k}{2}\right)} \prod_{j=1}^k \frac{w_j}{\sinh\left(t\frac{w_j}{2}\right)}$$

Therefore, we get

$$\left(\frac{t}{2} \right)^k e^{t\left(\frac{2w+w_1+\dots+w_k}{2}\right)} \prod_{j=1}^k \frac{w_j}{\sinh\left(t\frac{w_j}{2}\right)} = \sum_{n=0}^{\infty} B_n^{(k)}(w \mid w_1, \dots, w_k) \frac{t^n}{n!}, \quad (2.2)$$

which yields

$$\begin{aligned} &t^k e^{-t\left(\frac{2w-w_1-\dots-w_k}{2}\right)} \prod_{j=1}^k w_j \\ &= \sum_{n=0}^{\infty} \frac{2^k B_n^{(k)}(-w \mid w_1, \dots, w_k)}{n!} t^n \prod_{j=1}^k \sinh\left(t\frac{w_j}{2}\right). \quad (2.3) \end{aligned}$$

Putting $w = 0$ and $w_1 = w_2 = \dots = w_k = 1$ in (2.3), one has

$$\frac{t^k}{(e^t - 1)^k} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}$$

(cf. [18], [22], [31], [34]).

By the help of the equation (2.3), one has

$$\begin{aligned} F_B^{(n)}(t | w_1, w_2, \dots, w_k) &= \frac{t^{kn} \prod_{j=1}^k w_j^n}{\prod_{j=1}^k (e^{tw_j} - 1)^n} \\ &= \sum_{n=0}^{\infty} B_n^{(k,n)}(0 | w_1, w_2, \dots, w_k) \frac{t^n}{n!}. \end{aligned} \quad (2.4)$$

If $n = 1$, then we see that

$$B_n^{(k)}(0 | w_1, w_2, \dots, w_k) := B_n^{(k,n)}(0 | w_1, w_2, \dots, w_k)$$

(cf. [18]).

3. Infinite Series and Integral Representation for the Polynomials

$$B_n^{(k)}(w | w_1, w_2, \dots, w_k)$$

Here, by applying the Laplace transform to the both sides of equation (2.3), we derive infinite series and integral representation for the polynomials $B_n^{(k)}(w | w_1, w_2, \dots, w_k)$.

By applying the Laplace transform to the both sides of (2.3), and using the following known formula

$$\int_0^{\infty} u^k e^{-u} du = k!$$

and

$$\begin{aligned} \mathcal{L}(t^k e^{-t\alpha}) &= \int_0^{\infty} t^k e^{-t(\alpha+s)} dt \\ &= \frac{k!}{(\alpha + s)^{k+1}} \end{aligned} \quad (3.1)$$

we get the following theorem:

Theorem 3.1. *We have*

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{B_n^{(k)}(-w | w_1, w_2, \dots, w_k)}{n!} \int_0^{\infty} t^n e^{-st} \prod_{j=1}^k \sinh\left(t \frac{w_j}{2}\right) dt \\ = \frac{2k! \prod_{j=1}^k w_j}{(s + 2w - w_1 - w_2 - \dots - w_k)^{k+1}}, \end{aligned} \quad (3.2)$$

where

$$s \neq 2w - w_1 - w_2 - \dots - w_k.$$

Gun and Simsek [11] gave the following interesting formula

$$\sum_{j=0}^{\infty} (-1)^j B_j^{(k)}(x-w) \sum_{d=0}^k (-1)^d \binom{k}{d} \frac{1}{(w+d)^{j+1}} = \frac{k!}{x^{k+1}},$$

where

$$x \neq 0, \quad w \neq 0, \quad x \neq w, \quad \text{and} \quad \left| \frac{x-w}{w+d} \right| < 1, \quad d \in \{0, 1, 2, \dots, k\}.$$

On the other hand, Polat and Simsek [20, Theorem 6.5] gave the following interesting formula with the aid of the application Laplace transform to generating function for the Bernoulli polynomials

$$\zeta(2+k, y) = \sum_{n=0}^{\infty} (-1)^{n+k+1} \frac{1}{(k+1)!x^{n+1}} \frac{\partial^k}{\partial y^k} \{B_n(y-x)\}$$

where $k \geq 0$ and $y \notin \{0, -1, -2, -3, \dots\}$.

Substituting $k = 1$ into (3.2), we get

$$\sum_{n=0}^{\infty} \frac{B_n(-w | w_1)}{n!} \int_0^{\infty} t^n \sinh\left(t \frac{w_1}{2}\right) e^{-st} dt = \frac{2w_1}{(s+2w-w_1)^2}.$$

Combining the left-hand side of the above equation with (3.1) and

$$\begin{aligned} \int_0^{\infty} t^n \sinh\left(t \frac{w_1}{2}\right) e^{-st} dt &= \frac{1}{2} \int_0^{\infty} t^n e^{-t(s-\frac{w_1}{2})} dt - \frac{1}{2} \int_0^{\infty} t^n e^{-t(s+\frac{w_1}{2})} dt \\ &= \frac{n!}{2} \left(\frac{1}{(s-\frac{w_1}{2})^{n+1}} - \frac{1}{(s+\frac{w_1}{2})^{n+1}} \right) \end{aligned}$$

yields

$$\sum_{n=0}^{\infty} B_n(-w | w_1) \left(\frac{1}{(s-\frac{w_1}{2})^{n+1}} - \frac{1}{(s+\frac{w_1}{2})^{n+1}} \right) = \frac{4w_1}{(s+2w-w_1)^2}.$$

Substituting $k = 2$ into (3.2), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{B_n^{(2)}(-w | w_1, w_2)}{n!} \int_0^{\infty} t^n \sinh\left(t \frac{w_1}{2}\right) \sinh\left(t \frac{w_2}{2}\right) e^{-st} dt \\ = \frac{4!w_1w_2}{(s+2w-w_1-w_2)^3}. \end{aligned}$$

4. Formulas for the Numbers $Y(n, c; k \mid w_1, w_2, \dots, w_k)$

In this section, with the aid of power series of the function $\sinh x$, and generating function method, we derive some novel formulas for the numbers $Y(n, c; k \mid w_1, w_2, \dots, w_k)$.

For $c \in \mathbb{N}_0$, we modify Eq. (14) in [18] as follows

$$\begin{aligned} F_S(t \mid w_1, w_2, \dots, w_k; c) &= \left(\frac{1}{c!}\right)^c (e^{tw_1} - 1)^c (e^{tw_2} - 1)^c \dots (e^{tw_k} - 1)^c \\ &= \sum_{n=0}^{\infty} Y(n, c; k \mid w_1, w_2, \dots, w_k) \frac{t^n}{n!}. \end{aligned} \quad (4.1)$$

By combining (1.5) with (4.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} Y(n, c \mid w_1, w_2, \dots, w_k) \frac{t^n}{n!} \\ = \sum_{n=0}^{\infty} S(n, c) \frac{w_1^n t^n}{n!} \sum_{n=0}^{\infty} S(n, c) \frac{w_2^n t^n}{n!} \dots \sum_{n=0}^{\infty} S(n, c) \frac{w_k^n t^n}{n!}. \end{aligned}$$

By combining multinomial theorem in [24, Theorem 10 and Theorem 11] and using the Cauchy product rule in the above equation yields

$$\sum_{n=0}^{\infty} Y(n, c; k \mid w_1, \dots, w_k) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{\Phi(n, \ell, k)} \binom{n}{\ell_1, \dots, \ell_k} \prod_{j=1}^k S(\ell_j, c) w_j^{\ell_j} \frac{t^n}{n!},$$

where

$$\sum_{\Phi(n, \ell, k)} = \sum_{\substack{\ell_1, \ell_2, \dots, \ell_k \geq 0 \\ \ell_1 + \ell_2 + \dots + \ell_k = n}}$$

and

$$\binom{n}{\ell_1, \ell_2, \dots, \ell_k} = \frac{n!}{\ell_1! \ell_2! \dots \ell_k!}$$

with $\ell_1 + \ell_2 + \dots + \ell_k = n$. By comparing coefficients $t^n/n!$ both sides in the above equation, in light of the notation $\sum_{\Phi(n, \ell, k)}$ given above, we arrive at the following theorem:

Theorem 4.1. For $n, k \in \mathbb{N}$ we have

$$Y(n, c; k \mid w_1, w_2, \dots, w_k) = \sum_{\Phi(n, \ell, k)} \binom{n}{\ell_1, \ell_2, \dots, \ell_k} \prod_{j=1}^k S(\ell_j, c) w_j^{\ell_j}. \quad (4.2)$$

Substituting $k = 2$ into (4.2), we get

$$Y(n, c; 2 | w_1, w_2) = \sum_{\Phi(n, \ell, 2)} \binom{n}{\ell_1, \ell_2} S(\ell_1, c) S(\ell_2, c) w_1^{\ell_1} w_2^{\ell_2},$$

which yields

$$Y(n, c | w_1, w_2) = \sum_{\ell_1=0}^n \binom{n}{\ell_1} S(\ell_1, c) S(n - \ell_1, c) w_1^{\ell_1} w_2^{n-\ell_1}.$$

Substituting $w_1 = w_2 = \dots = w_k = 1$ into (4.1), we obtain

$$\frac{1}{(c!)^k} (e^t - 1)^{kc} = \sum_{n=0}^{\infty} Y(n, c; k | 1, 1, \dots, 1) \frac{t^n}{n!}.$$

Combining the above equation with (1.5) yields

$$\frac{(kc)!}{(c!)^k} \sum_{n=0}^{\infty} S(n, kc) \frac{t^n}{n!} = \sum_{n=0}^{\infty} Y(n, c; k | 1, 1, \dots, 1) \frac{t^n}{n!}.$$

By comparing coefficients $t^n/n!$ both sides in the above equation, we get

$$Y(n, c; k | 1, 1, \dots, 1) = \frac{(kc)!}{(c!)^k} S(n, kc).$$

Putting $c = 1$ in (4.2), we have

$$(e^{tw_1} - 1)(e^{tw_2} - 1) \dots (e^{tw_k} - 1) = \sum_{n=0}^{\infty} Y(n, 1; k | w_1, w_2, \dots, w_k) \frac{t^n}{n!}.$$

Thus,

$$e^{t\left(\frac{w_1+w_2+\dots+w_k}{2}\right)} \prod_{j=1}^k \left(e^{t\frac{w_j}{2}} - e^{-t\frac{w_j}{2}} \right) = \sum_{n=0}^{\infty} Y(n, 1; k | w_1, w_2, \dots, w_k) \frac{t^n}{n!},$$

which yields

$$e^{t\left(\frac{w_1+w_2+\dots+w_k}{2}\right)} \prod_{j=1}^k \sinh\left(t\frac{w_j}{2}\right) = \sum_{n=0}^{\infty} Y(n, 1; k | w_1, w_2, \dots, w_k) \frac{t^n}{n!}.$$

Hence

$$\begin{aligned} & \prod_{j=1}^k \sum_{n=0}^{\infty} \left(t \frac{w_j}{2} \right)^{2n+1} \frac{1}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} Y(n, 1; k \mid w_1, w_2, \dots, w_k) \frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{\left(\frac{w_1+w_2+\dots+w_k}{2} \right)^n (-t)^n}{n!}. \end{aligned}$$

By combining multinomial theorem and using the Cauchy product rule in the above equation yields

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{\Phi(n, \ell, k)} \prod_{j=1}^k w_j^{2\ell_j+1} \frac{t^{2n+k}}{(2\ell_j+1)!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^n (-1)^j \binom{n}{j} \left(\frac{w_1 + w_2 + \dots + w_k}{2} \right)^j Y(n-j, 1; k \mid w_1, w_2, \dots, w_k) \frac{t^n}{n!}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{\Phi(n, \ell, k)} \prod_{j=1}^k w_j^{2\ell_j+1} \frac{t^{2n}}{(2\ell_j+1)!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^n (-1)^j \binom{n}{j} \left(\frac{w_1 + \dots + w_k}{2} \right)^j Y(n-j, 1; k \mid w_1, \dots, w_k) \frac{t^{n-k}}{n!}. \end{aligned}$$

Combining the above equation with the following series relation

$$\sum_{n=0}^{\infty} \frac{y_n}{n!} t^{n+k} = \sum_{n=0}^{\infty} \frac{s_n}{n!} t^n,$$

which yields

$$\sum_{n=0}^{\infty} \frac{y_{n-k}}{(n-k)!} t^n = \sum_{n=0}^{\infty} \frac{s_n}{n!} t^n,$$

so that

$$\frac{y_{n-k}}{(n-k)!} = \frac{s_n}{n!}$$

or

$$y_n = \frac{(n-k)! s_{n+k}}{(n+k)!},$$

after some calculations, we obtain

$$\sum_{n=0}^{\infty} \sum_{\Phi(n,\ell,k)} \prod_{j=1}^k w_j^{2\ell_j+1} \frac{t^{2n}}{(2\ell_j+1)!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n+k} (-1)^j \frac{\binom{n+k}{j}}{\binom{n+k}{k!} k!} \times \left(\frac{w_1 + w_2 + \dots + w_k}{2} \right)^j Y(n+k-j, 1; k \mid w_1, w_2, \dots, w_k) \frac{t^n}{n!}.$$

By comparing coefficients t^{2n} both sides in the above equation, we get the following result.

Theorem 4.2. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{\Phi(n,\ell,k)} \prod_{j=1}^k w_j^{2\ell_j+1} \frac{1}{(2\ell_j+1)!} = \sum_{j=0}^{2n+k} (-1)^j \frac{\binom{2n+k}{j}}{\binom{n+k}{k} k!} \times \left(\frac{w_1 + w_2 + \dots + w_k}{2} \right)^j Y(2n+k-j, 1; k \mid w_1, w_2, \dots, w_k) \quad (4.3)$$

and

$$\sum_{j=0}^{2n+k+1} (-1)^j \binom{2n+k+1}{j} \left(\frac{w_1 + w_2 + \dots + w_k}{2} \right)^j \times Y(2n+k+1-j, 1; k \mid w_1, w_2, \dots, w_k) = 0. \quad (4.4)$$

Substituting $k = 2$ into (4.3) and (4.4), we get

$$\sum_{\Phi(n,\ell,2)} \prod_{j=1}^2 w_j^{2\ell_j+1} \frac{1}{(2\ell_j+1)!} = \sum_{j=0}^{2n+2} (-1)^j \frac{\binom{2n+3}{j}}{2 \binom{n+2}{2}} \left(\frac{w_1 + w_2}{2} \right)^j Y(n+2-j, 1; 2 \mid w_1, w_2),$$

respectively, which yields

$$(n+1) \sum_{\ell_1=0}^n \binom{2n+1}{2\ell_1} w_1^{2\ell_1+1} w_2^{2n-2\ell_1+1} = \sum_{j=0}^{2n+2} \binom{2n+3}{j} \left(-\frac{w_1 + w_2}{2} \right)^j Y(2n+k-j, 1; k \mid w_1, w_2),$$

and

$$\sum_{j=0}^{2n+3} (-1)^j \binom{2n+3}{j} \left(\frac{w_1 + w_2}{2} \right)^j Y(2n+3-j, 1; 2 | w_1, w_2) = 0.$$

5. Conclusion

This article consists of five sections. The first section briefly outlines the fundamental notations, definitions, and relationships used throughout the paper before moving on to the main sections. By using q -integres method, we gave some results and relations for the Barnes' type multiple Changhee q -Bernoulli polynomials and the Barnes' type Changhee multiple q -zeta functions. We also derived the following modified generating function

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{j=0}^n \binom{n}{j} \left(\frac{2w + w_1 + w_2 + \dots + w_k}{2} \right)^{n-j} B_j^{(k)}(w | w_1, w_2, \dots, w_k) \frac{t^n}{n!} \\ = \prod_{j=1}^k \frac{tw_j}{2 \sinh\left(t \frac{w_j}{2}\right)} \end{aligned}$$

and

$$e^{t \left(\frac{w_1 + w_2 + \dots + w_k}{2} \right)} \prod_{j=1}^k 2 \sinh\left(t \frac{w_j}{2}\right) = \sum_{n=0}^{\infty} Y(n, 1; k | w_1, w_2, \dots, w_k) \frac{t^n}{n!}.$$

By applying the Laplace transform to the above function, we showed that the polynomials $B_n^{(k)}(w | w_1, w_2, \dots, w_k)$ have the following infinite series and integral representation

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{B_n^{(k)}(-w | w_1, w_2, \dots, w_k)}{n!} \int_0^{\infty} t^n e^{-st} \prod_{j=1}^k \frac{e^{\frac{tw_j}{2}} - e^{-\frac{tw_j}{2}}}{2} dt \\ = \frac{2k! w_1 w_2 \dots w_k}{(s + 2w - w_1 - w_2 - \dots - w_k)^{k+1}}. \quad (5.1) \end{aligned}$$

Moreover, we proved that the numbers $Y(n, c; k | w_1, w_2, \dots, w_k)$ can be written in terms of the Stirling numbers of the second kind. That is,

$$\sum_{\Phi(n, \ell, k)} \binom{n}{\ell_1, \ell_2, \dots, \ell_k} \prod_{j=1}^k S(\ell_j, c) w_j^{\ell_j} = Y(n, c; k | w_1, w_2, \dots, w_k).$$

The specific values of our generalizations formulas and relations were also separately and examined individually.

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