Bulletin T.CXLVIII de l'Académie serbe des sciences et des arts − 2025
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 50

DYNAMICS OF n-PARAMETER OPERATOR SEMIGROUPS ON FINITE-DIMENSIONAL SPACES

MARKO KOSTIĆ

(Accepted at the 9th Meeting, held on December 26, 2025)

A b s t r a c t. In this paper, we investigate S*-hypercyclic and* S*-chaotic strongly continuous n-parameter semigroups of operators on finite-dimensional spaces, where $n \in \mathbb{N}$,* S *is closed subset of field $\mathbb{K} \in \{\mathbb{R},\ \mathbb{C}\}$ and $\mathrm{S} \setminus \{0\} \neq \emptyset$. We also provide extension of Oxtoby–Ulam theorem for multiparameter* S*-hypercyclic strongly continuous semigroups of operators on arbitrary Fréchet spaces.*

AMS Mathematics Subject Classification (2000): 47A16, 26A33, 47D99
Key Words: n-parameter semigroups of operators, finite-dimensional spaces, tuples of operators, hypercyclicity and chaos.

1. *Introduction and preliminaries*

If E is a separable Fréchet space over the field $\mathbb{K} \in \{\mathbb{R},\ \mathbb{C}\}$ and $L(E)$ denotes the space of all linear continuous mappings from E into E, then a mapping $T \in L(E)$ is said to be hypercyclic if there exists an element $x \in E$ whose orbit $\mathrm{Orb}(x, T) := \{T^k x : k \in \mathbb{N}_0\}$ is dense in E, while T is said to be topologically transitive if for any pair of open non-empty subsets U, V of E there exists $k \in \mathbb{N}$ such that $T^k(U) \cap V \neq \emptyset$. The Baire cathegory theorem implies that T

* This research is partially supported by Ministry of Science and Technological Development, Republic of Serbia.

is hypercyclic if and only if T is topologically transitive; in this case, E must be infinite-dimensional. For more details about linear topological dynamics, we refer the reader to the research monographs [1] by F. Bayart, E. Matheron, [6] by K.-G. Grosse-Erdmann, A. Peris and [17] by M. Kostić.

The hypercyclic tuples of operators were considered by L. Kérchy in [15] (2005) and N. Feldman in [7] (2008). Recall, a tuple $(T_1, ..., T_n)$ of commuting linear continuous operators on E, where $n \in \mathbb{N} \setminus \{1\}$, is said to be hypercyclic if there exists $x \in E$ such that the orbit $\{T_1^{k_1} \cdots T_n^{k_n} x : k_1 \in \mathbb{N}_0, ..., k_n \in \mathbb{N}_0\}$ is dense in E. A tuple $(T_1, ..., T_n)$ of commuting linear continuous operators on E is hypercyclic if and only if $(T_1, ..., T_n)$ is topologically transitive, i.e., for any pair of open non-empty subsets U, V of E there exist integers $k_1 \in \mathbb{N}_0, ..., k_n \in \mathbb{N}_0$ such that $T_1^{k_1} \cdots T_n^{k_n}(U) \cap V \neq \emptyset$. The hypercyclic tuples of operators exist on finite-dimensional spaces, as well; see, e.g, [7, Example 2.7], the research article [20] by S. Shkarin and the research article [3] by G. Costakis, D. Hadjiloukas, A. Manoussos.

On the other hand, the hypercyclic and chaotic strongly continuous one-parameter semigroups of linear operators were systematically analyzed by W. Desch, W. Schappacher and G. F. Webb in [5] (1997). The hypercyclic multiparameter strongly continuous semigroups of linear operators were introduced and analyzed by M. Janfada and A. N. Baghan in [14] (2019). Further on, in a joint work [19] with H. C. Koyuncuoğlu and D. Velinov, the author of this paper has recently investigated various dynamical properties of operator semigroups over topological monoids. If $(M, +, \tau)$ is a topological monoid and $(T(t))_{t \in M}$ is a strongly continuous semigroup of operators on E, then the notion introduced in [19] provides a general unification concept in the existing literature. This notion extends:

(i) the notion of dynamical properties of usually considered linear continuous operators on Fréchet spaces;

(ii) the notion of dynamical properties of tuples of commuting linear continuous operators on Fréchet spaces;

(iii) the notion of dynamical properties of strongly continuous semigroups defined on complex sectors, cf. the list of references quouted in [19] and [16, Chapter 3];

(iv) the notion of dynamical properties of multiparameter strongly continuous semigroups.

The notion of a semigroup over topological monoid naturally generalizes the notion of usually considered one-parameter strongly continuous semigroup of bounded linear operators. This broad class of semigroups includes the semigroups defined on the monoid $[0,+\infty)^n$, which are usually called multiparameter semigroups or n-parameter semigroups (this class of semigroups was introduced by E. Hille in 1944; see [2] and [12]). By a multiparameter semigroup on E, we mean any operator-valued function $T : [0,+\infty)^n \to L(E)$ such that $T(0) = I$, the identity operator on E, and $T(t+s) = T(t)T(s)$ for all $t,\ s \in [0,+\infty)^n$. We say that a semigroup $T : [0,+\infty)^n \to L(E)$ is strongly continuous if, for every $x \in E$, the mapping $t \mapsto T(t)x$, $t \in [0,+\infty)^n$ is strongly continuous at $t = 0$. A strongly continuous semigroup $T : [0,+\infty)^n \to L(E)$ will be also denoted by $(T(t))_{t\in[0,+\infty)^n}$ henceforth.

Suppose now that $(T(t))_{t\in[0,+\infty)^n}$ is strongly continuous. If $(e_1, ..., e_n)$ is the standard basis of $\mathbb{R}^n$, then we set $T_i(t_i) := T(t_ie_i)$, $t_i \geq 0$; then $(T_i(t_i))_{t_i\geq 0}$ is a strongly continuous semigroup on E ($i \in \mathbb{N}_n \equiv \{1, 2, ..., n\}$). If $T_i(\cdot)$ is generated by A_i ($i \in \mathbb{N}_n$), then

$$T\big(t_1, ..., t_n\big) = T_1\big(t_1\big) \cdots T_n\big(t_n\big), \quad \big(t_1, ..., t_n\big) \in [0,+\infty)^n, \tag{1.1}$$

and $(A_1, A_2, \ldots, A_n)$ is said to be the infinitesimal generator of $(T(t))_{t\in[0,+\infty)^n}$. We can simply prove the following result:

Theorem 1.1. *Suppose that A_i is the infinitesimal generator of a strongly continuous semigroup $(T_i(t))_{t\geq 0}$ for $1 \leq i \leq n$. If $(T(t))_{t\in[0,+\infty)^n}$ is defined through* (1.1)*, then $(A_1, A_2, \cdots, A_n)$ is the infinitesimal generator of multiparameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^n}$ if and only if $T_i(t_i)T_j(t_j) = T_j(t_j)T_i(t_i)$ for all $1 \leq i, j \leq n$, $t_i \geq 0$ and $t_j \geq 0$.*

The well-posedness of n-parameter abstract Cauchy problem

$$(ACP): \begin{cases} u \in C([0,+\infty)^n : E) \cap C^1((0,+\infty)^n : E), \\ u_{t_i}(t) = A_iu(t), \quad t \in (0,+\infty)^n,\ 1 \leq i \leq n, \\ u(0) = x, \quad x \in \bigcap_{i\in\mathbb{N}_n} D\big(A_i\big), \end{cases}$$

has been analyzed by many authors; for example, M. Janfada and A. Niknam have proved, in [13, Theorem 2.1], that if $(A_1, A_2, \ldots, A_n)$ is the infinitesimal generator of multiparameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^n}$ and $x \in \bigcap_{i\in\mathbb{N}_n} D(A_i)$, then the function $u(t) := T(t)x$, $t \in [0,+\infty)^n$ is a unique strong solution of the above problem.

In this paper, we analyze dynamical properties of multiparameter strongly continuous semigroups of operators on finite-dimensional spaces. If $(T(t))_{t\in[0,+\infty)^n}$ is a multiparameter strongly continuous semigroup on the space $\mathbb{K}^k$, where $n,\ k \in \mathbb{K}$, then Theorem 1.1 implies that there exist matrices $A_1, ..., A_n \in \mathbb{K}^{n,n}$, the algebra of all $\mathbb{K}$-matrices of format $n \times n$, such that $T(t_1, ..., t_n) = \exp(t_1A_1 + \cdots + t_nA_n)$, $(t_1, ..., t_n) \in [0,+\infty)^n$. The semigroup $(T(t))_{t\in[0,+\infty)^n}$ of this form can be always extended to the multiparameter strongly continuous group of operators $(T(t))_{t\in\mathbb{R}^n}$ by $T(t_1,...,t_n)=\exp(t_1A_1+\cdots+t_nA_n)$, $(t_1,...,t_n)\in\mathbb{R}^n$, so that $T(t+s)=T(t)T(s)$ for all $t,\ s\in\mathbb{R}^n$ and, for every $x\in\mathbb{K}^k$, the mapping $t\mapsto T(t)x$, $t\in\mathbb{R}^n$ is strongly continuous at $t=0$; furthermore, for every $x\in\mathbb{K}^k$, the function $u(t):=T(t)x$, $t\in\mathbb{R}^n$ is a unique strong solution of problem

$$(ACP)_G:\begin{cases} u\in C^1(\mathbb{R}^n:E), \\ u_{t_i}(t)=A_iu(t),\ t\in\mathbb{R}^n,\ 1\leq i\leq n, \\ u(0)=x; \end{cases}$$

cf. [19] for more details.

Before proceeding any further, it would be necessary to recall the basic properties of matrix exponentials and logarithms. We know that for a complex matrix $A\in\mathbb{C}^{n,n}$ there exists a complex matrix $B\in\mathbb{C}^{n,n}$ such that

$$A=e^B:=\sum_{k=0}^{+\infty}\frac{B^k}{k!}$$

if and only if A is invertible. The logarithm $B=\log(A)$ is not unique, but if A has no negative real eigenvalues, then the logarithm $\log(A)$ is unique and all its eigenvalues belong to the strip $\{z\in\mathbb{C}:-\pi<\Im z<\pi\}$. The question of existence of a logarithm of a real matrix $A\in\mathbb{R}^{n,n}$ is a little bit complicated. We know that there exists a real matrix $B\in\mathbb{R}^{n,n}$ such that $A=e^B$ if and only if A is invertible and each elementary divisor (Jordan block) belonging to a negative eigenvalue of A occurs an even number of times. If this is the case, the real logarithm of A need not be unique; cf. W. J. Culver [4, Theorem 1] and S. Shkarin [20, Lemma 3.3] for more details in this direction. Finally, let us mention that, if a scalar $\lambda\in\mathbb{K}$ is an eigenvalue of matrix $A\in\mathbb{K}^{n,n}$ of the algebraic multiplicity $j\in\mathbb{N}$, then e^λ is an eigenvalue of matrix e^A of the same algebraic multiplicity; cf. [8, p. 158].

The layout of this paper can be briefly described as follows. In Subsection 1.1, we recall the basic dynamical properties of operator families indexed with arbitrary sets. Here, we furnish an extension of the Oxtoby-Ulam theorem for multiparameter S-hypercyclic strongly continuous semigroups of operators which satisfies condition (C) clarified below; cf. Theorem 1.2. With the exception of this result, all original contributions of ours are formulated in Section 2, where we essentially use the properties of matrix exponentials and logarithms. In Theorem 2.1, we prove that for each natural number $n\in\mathbb{N}$ there exists a hypercyclic $(n+1)$-parameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^{n+1}}\subseteq L(\mathbb{C}^n)$ consisting of diagonal matrices, while in Theorem 2.2, we prove that for each integer $n\in\mathbb{N}$ there is no hypercyclic n-parameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^n}\subseteq L(\mathbb{C}^n)$ such that condition (C) holds with $\mathrm{S}=\{1\}$, for some $x\in\mathrm{HC}(T)$, where $\mathrm{HC}(T)$ denotes the set of all hypercyclic vectors of $(T(t))_{t\in[0,+\infty)^n}$.

Following N. Feldman [7], in Theorem 2.3 we prove that for each natural number $n \in \mathbb{N}$ and a closed subset S of $\mathbb{K}$ such that $\mathrm{S} \setminus \{0\} \neq \emptyset$ and $\mathrm{S} \cdot [0, +\infty)^n$ is not dense in $\mathbb{R}^n$, then there is no S-hypercyclic n-parameter strongly continuous semigroup $(T(t))_{t \in [0,+\infty)^n} \subseteq L(\mathbb{C}^n)$ consisting of diagonal matrices. In Theorem 2.4, we prove that for each natural number $n \in \mathbb{N}$ there exists a hypercyclic $(n+2)$-parameter strongly continuous semigroup $(T(t))_{t \in [0,+\infty)^{n+2}} \subseteq L(\mathbb{C}^n)$ which do not consist merely of diagonal matrices; moreover, we will prove that the number $n+2$ is the minimal cardinality of a parameter $k \in \mathbb{N}$ such that there is a hypercyclic k-parameter strongly continuous semigroup $(T(t))_{t \in [0,+\infty)^k} \subseteq L(\mathbb{C}^n)$ which do not consist merely of diagonal matrices and which satisfies condition (C) with $\mathrm{S} = \{1\}$ and the number n replaced therein with the number $n+2$, for some $x \in \mathrm{HC}(T)$ (cf. the research article [20] by S. Shkarin for the discrete versions of Theorem 2.2, Theorem 2.4 Theorem 2.5 and Theorem 2.6; in the proofs, we also use the extended Oxtoby-Ulam theorem).

After that, we provide some observations on the proofs of structural results [14, Proposition 3.1, Proposition 3.2] by M. Janfada and A. N. Baghan. The main aim of Theorem 2.5 is to show that, for every even natural number $n = 2m \geq 2$, there exists a hypercyclic $(m+1)$-parameter strongly continuous semigroup $(T(t))_{t \in [0,+\infty)^{m+1}} \subseteq L(\mathbb{R}^n)$ as well as that, for every odd natural number $n = 2m+1 \geq 1$, there exists an S-hypercyclic $(m+2)$-parameter strongly continuous semigroup $(T(t))_{t \in [0,+\infty)^{m+2}} \subseteq L(\mathbb{R}^n)$, where $\mathrm{S} = \{\pm 1\}$. The main aim of Theorem 2.6 is to prove that, for every even number $n = 2m \geq 2$, there is no hypercyclic m-parameter strongly continuous semigroup $(T(t))_{t \in [0,+\infty)^m} \subseteq L(\mathbb{R}^n)$ such that condition (C) holds with $\mathrm{S} = \{1\}$ and the number n replaced therein with the number m, for some $x \in \mathrm{HC}(T)$, as well as that, for every odd number $n = 2m+1 \geq 1$, there is no hypercyclic $(m+1)$-parameter strongly continuous semigroup $(T(t))_{t \in [0,+\infty)^{m+1}} \subseteq L(\mathbb{R}^n)$ such that condition (C) holds with $\mathrm{S} = \{1\}$ and the number n replaced therein with the number $m+1$, for some $x \in \mathrm{HC}(T)$. In connection with Theorem 2.5 and Theorem 2.6, we propose two open problems (see Question 1 and Question 2).

Chaotic tuples and para-chaotic tuples of commuting linear continuous operators have been investigated by M. Habibi [10], M. Habibi, F. Safari [11] and B. Yousefi, J. Izadi [21]. Subsection 2. investigates the S-chaotic multiparameter operator semigroups on finite-dimensional spaces (for the sake of brevity, we will not consider here the S-chaoticity of tuples of commuting linear continuous operators on finite-dimensional spaces, which is also a very unexplored topic).

Our first result in Subsection 2.1 is Proposition 2.2, which states that if $k \in \mathbb{N}$ is the minimal cardinality of a parameter for which there exists a k-parameter S-hypercyclic strongly continuous semigroup on $\mathbb{K}^n$, then the minimal cardinality of a parameter $j \in \mathbb{N}$ for which there exists a j-parameter S-chaotic strongly continuous semigroup on $\mathbb{K}^n$ is k or $k+1$; cf. also Example 2.1 and the open problems proposed in Question 3, Question 4 and Question 5. If $(T(t))_{t\in[0,+\infty)^k}$ is a strongly continuous semigroup on $\mathbb{K}^n$ with the infinitesimal generator $(A_1,...,A_k)$ and if there exists a dense subset P of $\mathbb{K}^n$ such that, for every $x \in P$, there exists a tuple $(t_1,...,t_k) \in [0,+\infty)^k \setminus \{0\}$ such that $(t_1A_1 + \cdots + t_kA_k)x = 0$, then the set of all periodic points of $(T(t))_{t\in[0,+\infty)^k}$ is dense in $\mathbb{K}^n$; cf. Proposition 2.3.

Further on, if $(T(t))_{t\in\mathbb{R}^k}$ is a k-parameter group of operators on the space $\mathbb{R}^n$ and $k < n$, then the set of all periodic points of $(T(t))_{t\in\mathbb{R}^k}$ is equal to the whole space $\mathbb{R}^n$; cf. Theorem 2.7. The main purpose of Theorem 2.8 is to show that, for every even number $n = 2m \geq 4$, there exists a chaotic $(m+1)$-parameter strongly continuous group $(T(t))_{t\in\mathbb{R}^{m+1}} \subseteq L(\mathbb{R}^n)$ as well as that, for every odd number $n = 2m+1 \geq 5$, there exists a $\{\pm 1\}$-chaotic $(m+2)$-parameter strongly continuous group $(T(t))_{t\in\mathbb{R}^{m+2}} \subseteq L(\mathbb{R}^n)$.

1.1. Notation and terminology

In this subsection, we recall the notions of various dynamical properties of operator families indexed with arbitrary sets. Unless stated otherwise, we will always assume here that M is an infinite set, S is a closed subset of $\mathbb{K}$ such that $\mathrm{S} \setminus \{0\} \neq \emptyset$, and $(R(t))_{t\in M} \subseteq L(E)$ is an operator family on a Fréchet space E over the field $\mathbb{K}$, which can be finite-dimensional or not; cf. the research article [9] by K.-G. Grosse-Erdmann for the first results in this direction.

It is said that $(R(t))_{t\in M}$ is:

(i) hypercyclic, if there exists an element $x \in E$ whose orbit $\mathrm{Orb}(x,R) := \{R(t)x : t \in M\}$ is dense in E. Such an element x is called a hypercyclic vector for $(R(t))_{t\in M}$; $\mathrm{HC}(T)$ denotes the set of all hypercyclic vectors for $(R(t))_{t\in M}$,

(ii) chaotic, if $(R(t))_{t\in M}$ is hypercyclic and the set of periodic points of $(R(t))_{t\in M}$, defined by $\{x \in E : R(t_0)x = x \text{ for some } t_0 \in M \setminus \{0\}\}$, is dense in E,

(iii) topologically transitive, if for every pair of open non-empty sets U, V of E, there exists $t \in M$ such that $R(t)U \cap V \neq \emptyset$,

(iv) S-hypercyclic, if there exists an element $x \in E$ such that its S-projective orbit $\{cR(t)x : c \in \mathrm{S},\ t \in M\}$ is dense in E; $\mathrm{HC}_{\mathrm{S}}(R)$ denotes the set of all elements $x \in E$ whose S-projective orbit is dense in E,

(v) S-chaotic, if $(R(t))_{t\in M}$ is S-hypercyclic and the set of periodic points of $(R(t))_{t\in M}$ is dense in E,

(vi) S-topologically transitive, if for every pair of open non-empty sets U, V of E, there exist $c \in \mathrm{S}$ and $t \in M$ such that $cR(t)U \cap V \neq \emptyset$.

We know that $(R(t))_{t\in M}$ is S-topologically transitive if and only if $(R(t))_{t\in M}$ is S-hypercyclic and the set $\mathrm{HC}_{\mathrm{S}}(R)$ is a dense G_δ-subset of E; cf. [19, Theorem 3.3]. The S-Hypercyclicity Criterion can be formulated for arbitrary operator family $(R(t))_{t\in M}$; if the requirements of this criterion hold, then the operator family $(R_m(t))_{t\in M}$, given by $R_m(t) := (R\oplus \cdots \oplus R)(t)$, $t\in M$, is S-topologically transitive on E^m (here, the direct sum $\oplus$ occur exactly m-times in the above expression, cf. [19, Definition 3.4, Theorem 3.5] for more details).

Suppose that S is a compact subset of $\mathbb{K}$ such that $\mathrm{S}\setminus\{0\}\neq\emptyset$. In [18, Theorem 2(i)], we have proved that, if $(T(t))_{t\geq 0}\subseteq L(E)$ is an S-hypercyclic strongly continuous semigroup of operators and $x\in \mathrm{HC}_{\mathrm{S}}(T)$, then for each real number $s>0$ the set $\{cT(t)x : c\in \mathrm{S},\ t\geq s\}$ is dense in E; this statement is no longer true in the multi-dimensional setting, when we can only prove that the set $\{cT(t)x : c\in \mathrm{S},\ |t|\geq s\}$ is dense in E (cf. [19, Proposition 4.4] and condition (C)). Keeping in mind these observations, it readily follows that the following result extends the Oxtoby-Ulam theorem (cf. [6, Theorem 7.22]):

Theorem 1.2. *Suppose that* S *is a compact subset of* $\mathbb{K}$ *such that* $\mathrm{S}\setminus\{0\}\neq\emptyset$, *and* $(T(t))_{t\in[0,+\infty)^n}\subseteq L(E)$ *is a strongly continuous semigroup of operators given by* (1.1). *If* $(T(t))_{t\in[0,+\infty)^n}$ *is* S*-hypercyclic,* $x\in HC_{\mathrm{S}}(T)$ *and* (C) *holds, where:*

(C) *the set* $\{cT(t_1,...,t_n)x : c\in \mathrm{S},\ t_1\geq s_1,...,t_n\geq s_n\}$ *is dense in* E *for every* $(s_1,...,s_n)\in(0,+\infty)^n$,

then there exists a dense G_δ*-set* $J\subseteq(0,+\infty)^n$ *such that, for every* $t=(t_1,...,t_n)\in J$, *the tuple* $(T_1(t_1),...,T_n(t_n))$ *is* S*-hypercyclic with* x *being its* S*-hypercyclic vector, i.e., the set*

$$\{cT_1(k_1t_1)\cdots T_n(k_nt_n)x : c\in \mathrm{S},\ (k_1,...,k_n)\in\mathbb{N}_0^n\}$$

is dense in E.

PROOF. By the prescribed assumption, the set

$$\big\{cT(t)x : c\in \mathrm{S},\ t\in[0,+\infty)^n\big\}$$

is dense in E. Let $(U_k)_{k\in\mathbb{N}}$ be a countable base of nonempty open sets in E. Set

$$J_k := \Big\{t=\big(t_1,...,t_n\big)\in(0,+\infty)^n : cT_1\big(k_1t_1\big)\ \cdots\ T_n\big(k_nt_n\big)x\in U_k \text{ for some } c\in \mathrm{S} \text{ and } (k_1,...,k_n)\in\mathbb{N}_0^n\Big\},\ k\in\mathbb{N}.$$

By the strong continuity of $(T(t))_{t\in[0,+\infty)^n}$, the set J_k is open for all $k\in\mathbb{N}$. Let us prove that the set J_k is also dense in E for all $k\in\mathbb{N}$. We will do that in the two-dimensional setting; the proof in general case can be given in a completely analogous way. Clearly, it suffices to show that, for every fixed integer $k \in \mathbb{N}$ and for every positive real numbers $0 < a_1 < b_1$ and $0 < a_2 < b_2$, the rectangle $P = (a_1, b_1) \times (a_2, b_2)$ contains a point $(t_1, t_2) \in U_k$. Towards this end, let us first observe that there exist two integers $k_{0,1} \in \mathbb{N}$ and $k_{0,2} \in \mathbb{N}$ such that $\bigcup_{k \geq k_{0,1}} (ka_1, kb_1) = (k_{0,1}a_1, +\infty)$ and $\bigcup_{k \geq k_{0,2}} (ka_2, kb_2) = (k_{0,2}a_2, +\infty)$; cf. also the proof of [6, Theorem 7.22]. Our assumption that the set

$$\{cT(t_1, t_2)x : c \in \mathrm{S},\ t_1 \geq r_1,\ t_2 \geq r_2\}$$

is dense in E for every $(r_1, r_2) \in (0, +\infty)^2$, there exists $s = (s_1, s_2) \in (0, +\infty)^n$ such that $cT(s)x \in U_k$ and $s_1 > k_{0,1}a_1$ and $s_2 > k_{0,2}a_2$. Then there exist an integer $k_1 \geq k_{0,1}$, an integer $k_2 \geq k_{0,2}$ and a point $(t_1, t_2) \in P$ such that $s_1 = k_1 t_1$ and $s_2 = k_2 t_2$. Hence, $(t_1, t_2) \in P \cap J_k$ and J_k is dense in E. Therefore, the set $J := \bigcap_{k \in \mathbb{N}} J_k$ is a dense G_δ-subset of $(0, +\infty)^n$ due to the Baire category theorem, which simply implies the final conclusion.

If the requirements of Theorem 1.2 hold, then it is a serious problem to prove or disapprove that for each point $t = (t_1, ..., t_n) \in (0, +\infty)^n$ the tuple

$$(T_1(t_1), \ldots, T_n(t_n))$$

is S-hypercyclic with x being its S-hypercyclic vector; this is always true if $\mathrm{S} = \{1\}$ and $n = 1$ ([6]). We will not discuss this question henceforth.

2. *Dynamics of multiparameter operator semigroups on finite-dimensional spaces*

We open this section by stating the following result:

Proposition 2.1. *Suppose that $(T(t))_{t \in [0,+\infty)^k} \subseteq L(\mathbb{K}^n)$ is a strongly continuous semigroup, where $k,\ n \in \mathbb{N}$. If S is a closed subset of $\mathbb{K}$ such that $\mathrm{S} \setminus \{0\} \neq \emptyset$, $S' := \mathrm{S} \cdot [0, +\infty)$, $(T(t))_{t \in [0,+\infty)^k}$ is S-hypercyclic and $(A_1, ..., A_k)$ is the infinitesimal generator of $(T(t))_{t \in [0,+\infty)^k}$, then the tuple $(A_1, ..., A_k)$ is S'-hypercyclic, i.e., there exists a vector $x \in \mathbb{K}^n$ such that the set*

$$\{c' A_1^{l_1} \cdots A_k^{l_k} x : c' \in \mathrm{S}',\ (l_1, \ldots, l_k) \in \mathbb{N}_0^k\}$$

is dense in $\mathbb{K}^n$.

PROOF. By the prescribed assumption, there exists a vector $x \in \mathbb{K}^n$ such that the set

$$\{c \exp(t_1 A_1 + \cdots + t_k A_k)x : c \in \mathrm{S},\ (t_1, ..., t_k) \in [0, +\infty)^k\}$$

is dense in $\mathbb{K}^n$.

Then the final conclusion simply follows from the next computation involving the polynomial formula:

$$c\exp(t_1A_1+\cdots+t_kA_k)x = c\sum_{s=0}^{+\infty}\frac{1}{s!}\Big(t_1A_1+\cdots+t_kA_k\Big)^s x$$
$$= c\sum_{s=0}^{+\infty}\frac{1}{s!}\sum\frac{(t_1A_1)^{s_1}\cdots(t_kA_k)^{s_k}}{s_1!\cdots s_k!}x,$$

where the second sum is taken with respect to all tuples $(s_1,...,s_k)\in\mathbb{N}_0^k$ such that $s_1+\cdots+s_k=s$.

A similar statement can be formulated for a k-parameter strongly continuous group $(T(t))_{t\in\mathbb{R}^k}$: if $(T(t))_{t\in\mathbb{R}^k}$ is S-hypercyclic and $(A_1,...,A_k)$ is the infinitesimal generator of $(T(t))_{t\in[0,+\infty)^k}$, then the tuple $(A_1,...,A_k)$ is S$'$-hypercyclic with S$'$ $=$ S $\cdot\,\mathbb{R}$. Further on, in [7, Theorem 3.4], N. Feldman has proved that for each $n\in\mathbb{N}$ there exists a hypercyclic $(n+1)$-tuple of diagonal matrices on $\mathbb{C}^n$. Now we will state and prove the continuous analogue of this result:

Theorem 2.1. *For each $n\in\mathbb{N}$, there exists a hypercyclic $(n+1)$-parameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^{n+1}}\subseteq L(\mathbb{C}^n)$ consisting merely of diagonal matrices.*

PROOF. Let $(T_1,...,T_{n+1})$ be a hypercyclic $(n+1)$-tuple of diagonal matrices on $\mathbb{C}^n$, and let $T_j=\operatorname{diag}(a_{11;j},...,a_{nn;j})$ for $1\le j\le n+1$. If $z=(z_1,...,z_n)\in\mathbb{C}^n$ is a hypercyclic vector of the tuple $(T_1,...,T_{n+1})$, then the set

$$\Big\{(a_{11;1}^{k_1}\cdots a_{11;n+1}^{k_{n+1}}z_1,...,a_{n1;1}^{k_1}\cdots a_{nn;n+1}^{k_{n+1}}z_n) : k_1\in\mathbb{N}_0,...,k_n\in\mathbb{N}_0\Big\}$$

is dense in $\mathbb{C}^n$. In particular, the above implies that for each $i\in\mathbb{N}_n$ and $j\in\mathbb{N}_{n+1}$ we have $a_{ii;j}\neq 0$ so that there exists $b_{ii;j}\in\mathbb{C}$ such that $a_{ii;j}=\exp(b_{ii;j})$. Define now $B_j:=\operatorname{diag}(b_{11;j},...,b_{nn;j})$ for $1\le j\le n+1$. Then we have $T_j^{k_j}=\exp(k_jB_j)$ for all $j\in\mathbb{N}_{n+1}$ and $k_j\in\mathbb{N}$. If we set

$$T\big(t_1,...,t_{n+1}\big):=e^{t_1B_1}\cdots e^{t_{n+1}B_{n+1}},\ \ t=\big(t_1,...,t_{n+1}\big)\in[0,+\infty)^{n+1}, \quad (2.1)$$

then $(T(t))_{t\in[0,+\infty)^{n+1}}$ is a strongly continuous semigroup on $\mathbb{C}^n$ consisting merely of diagonal matrices; furthermore, $(T(t))_{t\in[0,+\infty)^{n+1}}$ is hypercyclic and z is a hypercylic vector for $(T(t))_{t\in[0,+\infty)^{n+1}}$.

Keeeping in mind Theorem 1.2 and [20, Corollary 1.4], we can immediately clarify the following result:

Theorem 2.2. *For each $n \in \mathbb{N}$, there is no hypercyclic n-parameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^n} \subseteq L(\mathbb{C}^n)$ such that condition* (C) *holds with* $\mathrm{S} = \{1\}$*, for some $x \in HC(T)$.*

Taking the absolute values of entries in the matrices $T_j = \mathrm{diag}(a_{11;j}, ..., a_{nn;j})$ for $1 \leq j \leq n+1$, N. Feldman has proved, in [7, Theorem 4.4], that for each $n \in \mathbb{N}$ there exists an $(n+1)$-tuple of diagonal matrices on $\mathbb{R}^n$ whose orbit is dense in $[0,+\infty)^n$. Keeping in mind the proof of Theorem 2.1, it folows that for each $n \in \mathbb{N}$ there exists an $(n+1)$-parameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^{n+1}} \subseteq L(\mathbb{R}^n)$ consisting merely of diagonal matrices, which satisfies that its orbit is dense in $[0,+\infty)^n$. Arguing as in the proof of [7, Theorem 3.6], we can prove the following result (cf. also [7, Theorem 4.4]):

Theorem 2.3. *If $n \in \mathbb{N}$ and $\mathrm{S} \cdot [0,+\infty)^n$ is not dense in $\mathbb{R}^n$, then there is no* S-*hypercyclic n-parameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^n} \subseteq L(\mathbb{C}^n)$ consisting merely of diagonal matrices.*

Further on, in [20, Corollary 1.4], S. Shkarin has proved that for each $n \in \mathbb{N}$ the minimal cardinality of a non-diagonalizabe hypercyclic tuple of matrices on $\mathbb{C}^n$ is $n+2$. Concerning this result, we will prove the following

Theorem 2.4. *For each $n \in \mathbb{N}$, there exists a hypercyclic $(n+2)$-parameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^{n+2}} \subseteq L(\mathbb{C}^n)$ which do not consist merely of diagonal matrices. Furthermore, the number $n+2$ is the minimal cardinality of a parameter $k \in \mathbb{N}$ such that there is a hypercyclic k-parameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^k} \subseteq L(\mathbb{C}^n)$ which do not consist merely of diagonal matrices and which satisfies condition* (C) *with* $\mathrm{S} = \{1\}$ *and the number n replaced therein with the number $n+2$, for some $x \in HC(T)$.*

PROOF. Let $(T_1, ..., T_{n+2})$ be a non-diagonalizabe hypercyclic tuple of matrices on $\mathbb{C}^n$ and let $z = (z_1, ..., z_n) \in \mathbb{C}^n$ be a hypercyclic vector of this tuple. Then all matrices $T_1, ..., T_{n+2}$ must be regular since they commute and the set

$$Orb := \left\{T_1^{k_1} \cdots T_{n+2}^{k_{n+2}} x : k_1 \in \mathbb{N}_0, \dots, k_{n+2} \in \mathbb{N}_0\right\}$$

is dense in E; in actual fact, if T_j is not a regular matrix for some $j \in \mathbb{N}_{n+2}$, then $Orb \subseteq R(T_j)$, which is a closed non-trivial linear subspace of $\mathbb{C}^n$. Therefore, any matrix T_j has a logarithm B_j so that $T_j = \exp(B_j)$ for $1 \leq j \leq n+2$; it is also clear that B_j cannot be a diagonal matrix if T_j is not. Now, if we define $(T(t))_{t\in[0,+\infty)^{n+2}}$ by replacing the number $n+1$ in (2.1) by the number $n+2$, then $(T(t))_{t\in[0,+\infty)^{n+2}} \subseteq L(\mathbb{C}^n)$ is a hypercyclic $(n+2)$-parameter strongly continuous semigroup on $\mathbb{C}^n$ and it does not consist merely of diagonal matrices. The last statement readily follows from Theorem 1.2 and [20, Corollary 1.4].

Further on, it is very simple to transfer all counterexamples given in [7, Example 4.3] to strongly continuous multiparamater semigroups of operators. In particular, for each $n \in \mathbb{N}$ there exist non-hypercyclic $(2n)$-parameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^{2n}}$ of operators on $\mathbb{R}^n$ and elements $x_1, ..., x_{2^n}$ of $\mathbb{R}^n$ such that the set $\{T(t)x_j : t \in [0, +\infty)^{2n},\ 1 \leq j \leq 2^n\}$ is dense in $\mathbb{R}^n$. In actual fact, any matrix considered in [7, Example 4.3] has a real logarithm since its eigenvalues are non-negative real numbers.

We continue with the following observations:

Remark 2.1. (i) In [14, Proposition 3.1], the authors have proved that there exists a hypercyclic $(2n)$-parameter strongly continuous semigroup on the space $\mathbb{R}^n$. Unfortunately, the proof of this result given on p. 106 of [14] is not correct. Speaking-matter-of-factly, if $t = (t_1, ..., t_n) \in [0, +\infty)^n,\ s = (s_1, ..., s_n) \in [0, +\infty)^n$ and there exists an integer $j \in \mathbb{N}_n$ such that $t_j \neq s_j$, then the operator

$$W\big(t_1, ..., t_n, s_1, ..., s_n\big)\big(x_1, ..., x_n\big) := \big(x_1 + t_1 - s_1, ..., x_n + t_n - s_n\big),$$

for any $x = (x_1, ..., x_n) \in \mathbb{R}^n$, is not a linear operator on the space $\mathbb{R}^n$ since

$$W\big(t_1, ..., t_n, s_1, ..., s_n\big)(0, ..., 0) = \big(t_1 - s_1, ..., t_n - s_n\big) \neq (0, ..., 0).$$

(ii) In [14, Proposition 3.2], the authors have proved that for each even number $n \in \mathbb{N}$, resp. for each odd number $n \in \mathbb{N}$, there exists a hypercyclic n-parameter stongly continuous semigroup on $\mathbb{R}^n$, there exists a hypercyclic $(n+1)$-parameter stongly continuous semigroup on $\mathbb{R}^n$. We feel it is our duty to say that the proof of this result is wrong if n is an odd number; furthermore, we will get a much better result for even numbers n in Theorem 2.5. First of all, if $n = 1$, then there is no $k \in \mathbb{N}$ such that there exists a k-parameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^k}$ on $\mathbb{R}$. Indeed, if we assume the contrary, then there exist real numbers $a_1 \in \mathbb{R}, ..., a_k \in \mathbb{R}$ and $x \in \mathbb{R}$ such that the set $D = \{\exp(a_1t_1 + ... + a_kt_k)x : t_1 \geq 0, ..., t_k \geq 0\}$ is dense in $\mathbb{R}$. This is impossible since we always have $D = \{0\}$, $D \subseteq (0, +\infty)$ or $D \subseteq (-\infty, 0)$. If $n = 2m + 1$ for some integer $m \in \mathbb{N}$, then the proof is also not correct since the operator $W_{m+1}(s, t)$ is not a linear operator on $\mathbb{R}^n$ if $t \neq s$ and $(t, s) \in [0, +\infty)^2$ since $W_{m+1}(s, t)0 = (s - t)e_n \neq 0$; cf. [14, p. 107, l. 4–l. 6].

Motivated by [20, Corollary 1.5], we will prove the following result:

Theorem 2.5. *We have*

(i) *For every even number $n = 2m \geq 2$, there exists a hypercyclic $(m + 1)$-parameter strongly continuous semigroup*

$$(T(t))_{t\in[0,+\infty)^{m+1}} \subseteq L(\mathbb{R}^n).$$

(ii) *Suppose that* $\mathrm{S} = \{\pm 1\}$. *Then for every odd number* $n = 2m + 1 \geq 1$, *there exists an* S-*hypercyclic* $(m + 2)$-*parameter strongly continuous semigroup* $(T(t))_{t \in [0,+\infty)^{m+2}} \subseteq L(\mathbb{R}^n)$.

PROOF. Suppose first that $n = 2m \geq 2$. By the proof of [20, Corollary 1.5], there exists an $(m+1)$-tuple of invertible real matrices $(A_1, ..., A_{m+1})$ in $\mathbb{R}^{n,n}$ such that the set $\{A_1^{k_1} \cdots A_{m+1}^{k_{m+1}} x : k_1 \in \mathbb{N}_0, ..., k_{m+1} \in \mathbb{N}_0\}$ is dense in $\mathbb{R}^n$ for some $x \in \mathbb{R}^n$; moreover, we can assume that any matrix A_j has the form

$$A_j = \begin{bmatrix} A_{j;1} & 0 & 0 & \cdots & 0 \\ 0 & A_{j;2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & A_{j;m} \end{bmatrix},$$

where

$$A_{j;k} = \begin{bmatrix} a_{j;k} & b_{j;k} \\ -b_{j;k} & a_{j;k} \end{bmatrix} \quad \Big(a_{j;k},\ b_{j;k} \in \mathbb{R}\Big),$$

for $1 \leq k \leq m$ and $1 \leq j \leq m + 1$. The eigenvalues of matrix A_j are $a_{j;k} \pm i b_{j;k}$ for $1 \leq k \leq m$, so that its characteristic polynomial is

$$\det\big(\lambda I - A_j\big) = \prod_{k=1}^{m} \Big(\big(\lambda - a_{j;k}\big)^2 + b_{j;k}^2\Big), \quad \lambda \in \mathbb{R} \ \big(1 \leq j \leq m + 1\big).$$

Therefore, the matrix A_j has a negative real value λ' if and only if there exists $k \in \mathbb{N}_m$ such that $\lambda' = a_{j;k} < 0$ and $b_{j;k} = 0$ $(1 \leq j \leq m + 1)$. It is clear that, for every negative eigenvalue λ' of A_j, each elementary divisor belonging to it occurs an even number of times so that $\log(A_j)$ is well-defined as a real matrix. Now we can proceed as in the proof of Theorem 2.1 to complete the proof of (i). Suppose now that $\mathrm{S} = \{\pm 1\}$ and $n = 2m + 1 \geq 1$ is an odd number. By the proof of [20, Corollary 1.5], there exists an $(m+2)$-tuple of invertible real matrices $(A_1, ..., A_{m+2})$ in $\mathbb{R}^{n,n}$ such that the set

$$\Big\{A_1^{k_1} \cdots A_{m+2}^{k_{m+2}} x : k_1 \in \mathbb{N}_0, ..., k_{m+2} \in \mathbb{N}_0\Big\}$$

is dense in $\mathbb{R}^n$ for some $x \in \mathbb{R}^n$; moreover, we can assume that any matrix A_j has the form

$$A_j = \begin{bmatrix} D_j & 0 & 0 & \cdots & 0 \\ 0 & A_{j;1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & A_{j;m} \end{bmatrix},$$

where $D_j = [d_j] \in \mathbb{R}^{1,1}$ and all matrices $A_{j;k}$ has the form as in the first part of proof. The characteristic polynomial of matrix A_j is equal to

$$\det\big(\lambda I - A_j\big) = \big(\lambda - d_j\big) \prod_{k=1}^{m} \Big(\big(\lambda - a_{j;k}\big)^2 + b_{j;k}^2\Big), \quad \lambda \in \mathbb{R} \ \big(1 \leq j \leq m+2\big),$$

so that $\log(A_j)$ does not exist in $\mathbb{R}^{n,n}$ if $d_j < 0$; but, if this is the case, it readily follows that the $\log(-A_j)$ exists in $\mathbb{R}^{n,n}$ $(1 \leq j \leq m+2)$. Set $B_j := \log(A_j)$, if exist in $\mathbb{R}^{n,n}$, and $B_j := \log(-A_j)$, otherwise. Then the set

$$\Big\{A_1^{k_1} \cdots A_{m+2}^{k_{m+2}} x : k_1 \in \mathbb{N}_0, ..., k_{m+2} \in \mathbb{N}_0\Big\}$$

is a subset of the set

$$\Big\{\pm e^{k_1 B_1} \cdots e^{k_{m+2} B_{m+2}} x : k_1 \in \mathbb{N}_0, ..., k_{m+2} \in \mathbb{N}_0\Big\},$$

which completes the proof of theorem in a routine manner.

As an immediate consequence of Theorem 1.2 and [20, Corollary 1.5], we can state the following result:

Theorem 2.6. *We have*

(i) *For every even number $n = 2m \geq 2$, there is no hypercyclic m-parameter strongly continuous semigroup $(T(t))_{t \in [0,+\infty)^m} \subseteq L(\mathbb{R}^n)$ such that condition* (C) *holds with* $\mathrm{S} = \{1\}$ *and the number n replaced therein with the number m, for some $x \in \mathrm{HC}(T)$.*

(ii) *For every odd number $n = 2m+1 \geq 1$, there is no hypercyclic $(m+1)$-parameter strongly continuous semigroup $(T(t))_{t \in [0,+\infty)^{m+1}} \subseteq L(\mathbb{R}^n)$ such that condition* (C) *holds with* $\mathrm{S} = \{1\}$ *and the number n replaced therein with the number $m+1$, for some $x \in \mathrm{HC}(T)$.*

The first example of a hypercyclic tuple (A_1, A_2, A_3) of invertible matrices in $\mathbb{R}^{3,3}$ has been given by G. Costakis, D. Hadjiloukas and A. Manoussos in [3, Proposition 3.3]. Unfortunately, the matrix A_1 or the matrix A_3 constructed in the proof of this result does not have a real logarithm and we therefore cannot so simply prove that there exists a hypercyclic 3-parameter strongly continuous operator semigroup $(T(t_1,t_2,t_3))_{(t_1,t_2,t_3)\in[0,+\infty)^3}$ on $\mathbb{R}^3$. Now we would like to ask the following questions which is closely connected with the above observation and the results established in Theorem 2.5(ii) and Theorem 2.6(ii):

Question 1. Suppose that $n = 2m+1 \geq 3$. Does there exist a hypercyclic $(m+2)$-parameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^{m+2}} \subseteq L(\mathbb{R}^n)$?

Question 2. Suppose that $n = 2m+1 \geq 3$ and $\mathrm{S} = \{\pm 1\}$. Is it true that there is no S-hypercyclic $(m+1)$-parameter strongly continuous semigroup

$$(T(t))_{t\in[0,+\infty)^{m+1}} \subseteq L(\mathbb{R}^n)\,?$$

We close this section with the following note: Since every invertible Jordan block of format 2×2 has a real logarithm which can be explicitely calculated as it has been done on p. 1148 of [4], it is straightforward to formulate the continuous analogues of [3, Theorem 1.2, Theorem 1.3] for hypercyclic multiparameter strongly continuous semigroups on $\mathbb{R}^2$.

2.1. S-*Chaotic multiparameter operator semigroups on finite-dimensional spaces*

In this subsection, we will provide a few noteworthy results on the S-chaotic multiparameter operator semigroups on finite-dimensional spaces. We start by stating the following result:

Proposition 2.2. *Suppose that $(T(t))_{t\in[0,+\infty)^k}$ is an* S*-hypercyclic strongly continuous semigroup on $\mathbb{K}^n$, where $k,\ n \in \mathbb{N}$. Let $A_1, ..., A_k$ be mutually commuting $\mathbb{K}$-matrices of format $n \times n$ and let*

$$T(t_1, ..., t_k) = \exp(t_1A_1 + \cdots + t_kA_k)$$

for all $(t_1, ..., t_k) \in [0,+\infty)^k$. If $A_{k+1} \in \mathbb{K}^{n,n}$ and there exists a tuple

$$(t_1, ..., t_{k+1}) \in [0,+\infty)^{k+1} \setminus \{0\}$$

such that $t_1A_1+...+t_{k+1}A_{k+1} = 0$, then the $(k+1)$-parameter strongly continuous semigroup $(S(t))_{t\in[0,+\infty)^{k+1}}$, given by

$$S\big(t_1, ..., t_{k+1}\big) := e^{t_1A_1+\cdots+t_{k+1}A_{k+1}}, \quad \big(t_1, ..., t_{k+1}\big) \in [0,+\infty)^{k+1},$$

is S*-chaotic.*

PROOF. It is clear that any S-hypercyclic vector of $(T(t))_{t\in[0,+\infty)^k}$ is likewise an S-hypercyclic vector of $(S(t))_{t\in[0,+\infty)^{k+1}}$. Since $t_1A_1 + \cdots + t_{k+1}A_{k+1} = 0$, we have $\exp(t_1A_1 + \cdots + t_{k+1}A_{k+1})x = x$ for all $x \in \mathbb{K}^n$, which clearly implies that $(S(t))_{t\in[0,+\infty)^{k+1}}$ is S-chaotic.

Therefore, if $k \in \mathbb{N}$ is the minimal cardinality of a parameter for which there exists a k-parameter S-hypercyclic strongly continuous semigroup on $\mathbb{K}^n$, then the minimal cardinality of a parameter $j \in \mathbb{N}$ for which there exists a j-parameter S-chaotic strongly continuous semigroup on $\mathbb{K}^n$ can be k or $k+1$. The following illustrative example indicates that we can have $j = k$:

Example 2.1. Suppose first that $n = 1$, $\mathbb{K} = \mathbb{C}$, $a_1 = 2 + i$ and $a_2 = -1 - i$. Let us consider now the strongly continuous semigroup $(T(t_1, t_2))_{(t_1,t_2)\in[0,+\infty)^2} \subseteq L(\mathbb{C})$ given by

$$T(t_1, t_2)z := e^{a_1 t_1 + a_2 t_2} z, \quad z \in \mathbb{C}, \quad (t_1, t_2) \in [0, +\infty)^2.$$

Then every complex number z is a periodic point of $(T(t_1, t_2))_{(t_1,t_2)\in[0,+\infty)^2}$ since

$$-2\pi i\mathbb{N} \subseteq D \equiv \{a_1 t_1 + a_2 t_2 : t_1 \geq 0,\ t_2 \geq 0\}.$$

Furthermore, it is simple to prove that the exponential function maps D onto $\mathbb{C}\setminus\{0\}$, which implies that every non-zero complex number z is a hypercyclic vector of $(T(t_1, t_2))_{(t_1,t_2)\in[0,+\infty)^2}$. Hence, $(T(t_1, t_2))_{(t_1,t_2)\in[0,+\infty)^2}$ is chaotic.

Suppose now that $n \in \mathbb{N}$, $\mathbb{K} = \mathbb{C}$,

$$A_1 := \mathrm{diag}(a_1, ..., a_1) \in \mathbb{C}^{n,n} \quad \text{and} \quad A_2 := \mathrm{diag}(a_2, ..., a_2) \in \mathbb{C}^{n,n}.$$

Then any vector $z \in \mathbb{C}^n$ is a periodic point of the strongly continuous semigroup of operators $(\exp(t_1 A_1 + t_2 A_2))_{(t_1,t_2)\in[0,+\infty)^2}$, as easily approved.

In connection with Theorem 2.1, Theorem 2.5, Proposition 2.2 and Example 2.1, we would like to raise the following issues:

Question 3. Suppose that $n \in \mathbb{N}\setminus\{1\}$. Does there exist a chaotic $(n+1)$-parameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^{n+1}} \subseteq L(\mathbb{C}^n)$ consisting merely of diagonal matrices?

Question 4. Suppose that $n = 2m \geq 2$ is an even number. Is it true that there exists a chaotic $(m+1)$-parameter strongly continuous semigroup

$$(T(t))_{t\in[0,+\infty)^{m+1}} \subseteq L(\mathbb{R}^n)?$$

Question 5. Suppose that $\mathrm{S} = \{\pm 1\}$ and $n = 2m + 1 \geq 1$ is an odd number. Is it true that there exists an S-chaotic $(m+2)$-parameter strongly continuous semigroup $(T(t))_{t\in[0,+\infty)^{m+2}} \subseteq L(\mathbb{R}^n)$?

Concerning Question 3, we would like to make the following observation: Let the invertible complex matrices $A,\ B_1, ...,\ B_n$ be given as in the proof of [7, Theorem 3.4, p. 89, eq. (3)], with the number p replaced by the number n therein. Define

$D := \log(A)$ and $D_i := \log(B_i)$, $i \in \mathbb{N}_n$. Then, by the proof of Theorem 2.1, we know that the strongly continuous semigroup

$$(\exp(t_1 D + t_2 D_1 + \cdots + t_{n+1} D_n))_{(t_1,\ldots,t_{n+1})\in[0,+\infty)^{n+1}}$$

is hypercyclic on $\mathbb{C}^n$. Furthermore, a simple computation involving the definition of numbers a_i and b_i given in [7, eq. (2), p. 87] shows that any vector $z \in \mathbb{C}^n$ with all non-zero coordinates will be a periodic point of the above semigroup if the numbers $x_1, ..., x_{2n} \in (0,1)^n$ determined by the property that the set

$$\Big\{ \big(10^s x_1 - k_1, 10^s x_2 + m_1, ..., 10^s x_{2n-1} - k_n, 10^s x_{2n} + m_n\big) : \\ s,\ k_1, ...,\ k_n \in \mathbb{N}; m_1, ...,\ m_n \in \mathbb{Z} \Big\}$$

is dense in $\mathbb{R}^{2n}$ can be chosen by requiring the extra assumption that $x_2/x_j \in \mathbb{Q}$ for $3 \leq j \leq 2n$. If this can be done, then the semigroup

$$\big(\exp(t_1 D + t_2 D_1 + \cdots + t_{n+1} D_n)\big)_{(t_1,\ldots,t_{n+1})\in[0,+\infty)^{n+1}}$$

will be chaotic on $\mathbb{C}^n$.

We continue with the following simple result:

Proposition 2.3. *Suppose that $(T(t))_{t\in[0,+\infty)^k}$ is a strongly continuous semigroup on $\mathbb{K}^n$, where $k,\ n \in \mathbb{N}$. Let $A_1, ..., A_k$ be mutually commuting $\mathbb{K}$-matrices of format $n \times n$ and let $T(t_1, ..., t_k) = \exp(t_1 A_1 + \cdots + t_k A_k)$ for all $(t_1, ..., t_k) \in [0,+\infty)^k$. If there exists a dense subset P of $\mathbb{K}^n$ such that, for every $x \in P$, there exists a tuple $(t_1, ..., t_k) \in [0,+\infty)^k \setminus \{0\}$ such that $(t_1 A_1 + \cdots + t_k A_k)x = 0$, then the set of all periodic points of $(T(t))_{t\in[0,+\infty)^k}$ is dense in $\mathbb{K}^n$.*

PROOF. If $x \in P$, then there exists a tuple $(t_1, ..., t_k) \in [0,+\infty)^k \setminus \{0\}$ such that $(t_1 A_1 + \cdots + t_k A_k)x = x$, so that $\exp(t_1 A_1 + \cdots + t_k A_k)x = x$ and each point from P is a periodic point of $(T(t))_{t\in[0,+\infty)^k}$. This yields the required conclusion.

In the following theorem, we analyze the set of periodic points of multiparameter (semi-)groups of operators on the space $E = \mathbb{R}^n$:

Theorem 2.7. *If $(T(t))_{t\in\mathbb{R}^k}$ is a k-parameter group of operators on the space $\mathbb{R}^n$ and $k < n$, then the set of all periodic points of $(T(t))_{t\in\mathbb{R}^k}$ is equal to the whole space $\mathbb{R}^n$.*

PROOF. Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$ be given. We will prove that x is a periodic point of $(T(t))_{t\in\mathbb{R}^k}$. Suppose that $T(t_1, ..., t_k) = \exp(t_1 A_1 + ... + t_k A_k)$, $(t_1, ..., t_k) \in \mathbb{R}^k$ for some real matrices $A_1 = [a_{ij;1}]_{n\times n}, ..., A_k = [a_{ij;k}]_{n\times n}$. It suffices to show that there exists a vector $(t_1, ..., t_k) \in \mathbb{R}^k \setminus \{0\}$ such that

$(t_1 A_1 + \cdots + t_k A_k)x = 0$. But, the last condition is equivalent with the system of linear equations

$$(S): \begin{cases} \big(a_{11;1}x_1 + \cdots + a_{1n;1}x_n\big)t_1 + \cdots + \big(a_{11;k}x_1 + \cdots + a_{1n;k}x_n\big)t_k & = 0, \\ & \vdots \\ \big(a_{n1;1}x_1 + \cdots + a_{nn;1}x_n\big)t_1 + \cdots + \big(a_{n1;k}x_1 + \cdots + a_{nn;k}x_n\big)t_k & = 0. \end{cases}$$

Since $k < n$, the system (S) always have a non-trivial solution in $\mathbb{R}^n$, finishing the proof of theorem.

Keeping in mind Theorem 2.5 and Theorem 2.7, we immediately get the following result:

Theorem 2.8. (i) *For every even number* $n = 2m \geq 4$, *there exists a chaotic* $(m+1)$*-parameter strongly continuous group* $(T(t))_{t\in\mathbb{R}^{m+1}} \subseteq L(\mathbb{R}^n)$.

(ii) *Suppose that* $\mathrm{S} = \{\pm 1\}$. *Then for every odd number* $n = 2m+1 \geq 5$, *there exists an* S*-chaotic* $(m+2)$*-parameter strongly continuous group* $(T(t))_{t\in\mathbb{R}^{m+2}} \subseteq L(\mathbb{R}^n)$.

REFERENCES

[1] F. Bayart, E. Matheron, *Dynamics of Linear Operators,* vol. **179** of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, UK, 1 edition, 2009.

[2] P. L. Butzer, H. Berens, *Semi-Groups of Operators and Approximation,* Springer-Verlag, New York, 1967.

[3] G. Costakis, D. Hadjiloukas, A. Manoussos, *Dynamics of tuple of matrices,* Proc. Amer. Math. Soc. **137** (2009), 1025–1034.

[4] W. J. Culver, *On the existence and uniqueness of the real logarithm of a matrix,* Proc. Amer. Math. Soc. **17** (1966), 1146–1151.

[5] W. Desch, W. Schappacher, G. F. Webb, *Hypercyclic and chaotic semigroups of linear operators,* Ergodic Theory Dynam. Systems **17** (1997), 1–27.

[6] K.-G. Grosse-Erdmann, A. P. Manguillot, *Linear Chaos,* Springer-Verlag, Berlin, 2011.

[7] N. Feldman, *Hypercyclic tuples of operators and somewhere dense orbits,* J. Math. Anal. Appl. **346** (2008), 82–98.

[8] F. R. Gantmacher, *The Theory of Matrices,* Vol. I, Chelsea, New York, 1959.

[9] K.-G. Grosse-Erdmann, *Universal families and hypercyclic operators,* Bull. Amer. Math. Soc. **36** (1999), 345–381.

[10] M. Habibi, *n-Tuples and chaoticity,* Int. J. Math. Anal. **6** (2012), 651–657.

[11] M. Habibi, F. Safari, *Chaoticity of a pair of operators,* Int. J. Appl. Math. **24** (2011), 155–160.

[12] E. Hille, R. S. Phillips, *Functional Analysis and Semi-Groups,* American Math. Society, New York, 1948.

[13] M. Janfada, A. Niknam, *On the n-parameter abstract Cauchy problem,* Bull. Austr. Math. Soc. **69** (2004), 383–394.

[14] M. Janfada, A. N. Baghan, *Hypercyclic tuple C_0-semigroups of operators,* U.P.B. Sci. Bull., Series A **81** (2019), 103–110.

[15] L. Kérchy, *Cyclic properties and stability of commuting power bounded operators,* Acta Sci. Math. (Szeged) **7** (2005), 299–312.

[16] M. Kostić, *Abstract Volterra Integro-Differential Equations,* CRC Press, Boca Raton, Fl., 2015.

[17] M. Kostić, *Chaos for Linear Operators and Abstract Differential Equations,* Nova Science Publishers, New York, 2020.

[18] M. Kostić, *S-hypercyclic, mixing and chaotic strongly continuous semigroups,* Funct. Anal. Appl. Comp. **1:1** (2009), 49–86.

[19] M. Kostić, H. C. Koyuncuoğlu, D. Velinov, *Dynamics of operator semigroups over topological monoids,* submitted. https://www.researchgate.net/publication/397547025.

[20] S. Shkarin, *Hypercyclic tuples of operators on $\mathbb{C}^n$ and $\mathbb{R}^n$,* Linear Multilinear Algebra **60** (2012), 885–896.

[21] B. Yousefi, J. Izadi, *Para-chaotic tuples of operators,* The Australian J. Math. Anal. Appl. **9** (2012), 1–8.

Faculty of Technical Sciences
University of Novi Sad
Trg Dositeja Obradovića 6
Novi Sad 21125, Serbia
e-mail: marco.s@verat.net