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FRÉCHET AND GF FRAMES AND INVERTIBILITY OF FRAME MULTIPLIERS ON BANACH AND FRÉCHET FRAMES

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Dedicated to the 100th anniversary of the birth of Academician Bogoljub Stanković

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A b s t r a c t. In this paper we indicate some of our earlier results concerning localization of Fréchet frames and invertibility of the frame operator for such frames, as well as some new results in this direction concerning the more general case of frame multipliers and including two versions of GS-frames which we introduce for the first time in our investigations.

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1. *Introduction*

We recall in Section 1 the definitions of a frame for a Hilbert space and a frame multiplier, as well as the definition of a Fréchet frame on countable projective limits of Banach spaces, spaces of rapidly decreasing functions (Schwartz space $\mathcal{S}(\mathbb{R}^d)$) and the space of ultradifferentiable functions $\Sigma^{p!^s}$, $s > 1/2$. In this paper we consider a spacial class of frames, the localized frames introduced by Groechening [21], and our focus is on invertibility of frame multipliers for such frames. In Section 2 we review some known results from [21, 37] that concern invertibility of frame operators on Banach spaces and on the important Fréchet spaces $\mathcal{S}(\mathbb{R}^d)$ and $\Sigma^{p!^s}$, $s > 1/2$. These results serve as a basis for our investigations in the field of frame multipliers for the distribution space $\mathcal{S}(\mathbb{R}^d)$ and the space of ultradifferentiable functions $\Sigma^{p!^s}$, $s > 1/2$. Let us note that in the analysis of spaces of ultradifferentiable functions of any kind we follow the approach of Komatsu [26], where one can find an excellent introduction to the spaces of ultradifferentiable functions and ultradistributions. In Section 3 we review our known results which serve as a basis for our investigations in the field of frame multipliers in Fréchet frames, as well in the Frames on locally convex spaces with special properties concerning the seminorms which define such a space. Then, in Section 4, we give two new definitions of GS-frames for the spaces of Gelfand Shilow type which are countable inductive limit of Hilbert spaces. This is a novelty since we present these spaces as projective limits of uncountable many Hilbert spaces.

1.1. Frames and related notions and notation

A sequence $(e_n)_{n=1}^{\infty}$ with elements from a Hilbert space $\mathcal{H}$ is called a *frame for* $\mathcal{H}$ [14] if there are positive constants A and B satisfying

$$(\forall x \in \mathcal{H}) \qquad A\|x\|_{\mathcal{H}} \leq \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \leq B\|x\|^2.$$

Given a frame $(e_n)_{n=1}^{\infty}$ for $\mathcal{H}$, there always exists a frame $(u_n)_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty}\langle x, e_n\rangle u_n = \sum_{n=1}^{\infty}\langle x, u_n\rangle e_n, \forall x \in \mathcal{H}$, called a *dual frame* of $(e_n)_{n=1}^{\infty}$. Let $E = (e_n)_{n=1}^{\infty}$ denote a frame for $\mathcal{H}$. The so called *frame operator* S_E determined by $S_E x = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$ is a bounded bijection from $\mathcal{H}$ onto $\mathcal{H}$ [14]. The sequence $(S_E^{-1}x_n)_{n=1}^{\infty}$ is also a frame for $\mathcal{H}$, called the *canonical dual of* E, and it will be denoted as $(\widetilde{e_n})_{n=1}^{\infty}$.

A frame for $\mathcal{H}$ which is at the same time a Schauder basis for $\mathcal{H}$ is called a *Riesz basis for* $\mathcal{H}$ (the notion of a Riesz basis was introduced by Bari in [3, 4] in another but equivalent way). A frame for $\mathcal{H}$ which is not a Schauder basis for $\mathcal{H}$ is called an *overcomplete* or *redundant frame*. A Riesz basis for $\mathcal{H}$ has unique dual frame (the canonical dual), while overcomplete frames have infinitely many dual frames. The canonical dual of a frame has some special properties (e.g. it provides frame expansions with minimal ℓ^2-norm of the coefficients [14]; in the case of Gabor frames, the canonical dual is also with a Gabor structure (see, e.g., [7, Section 12.3]) and the canonical dual window belongs to the Schwartz space $\mathcal{S}$ when the original window is in $\mathcal{S}$ [25, 16]). However, it may fail some other important properties (e.g. its computation might not be efficient; it may fail to have compact support or desired smoothness in the Gabor case, and other) and in such cases other dual frames might be more useful/appropriate (see, e.g., [6, 8, 10, 11, 27, 32]). That is why dual frames different from the canonical one are also of importance for investigation.

1.2. Frame multipliers

A frame multiplier is an operator determined via two frames and one scalar sequence as follows: given frames $\Phi = (\phi_n)_{n=1}^\infty$ and $\Psi = (\psi_n)_{n=1}^\infty$ for $\mathcal{H}$ and a sequence $m = (m_n)$ of complex numbers, the operator $M_{m,\Phi,\Psi}$ determined by $M_{m,\Phi,\Psi} f = \sum_{n=1}^\infty m_n \langle f, \psi_n \rangle \phi_n$ is called a *frame multiplier* and for bounded m it is well defined and bounded operator on $\mathcal{H}$; m is called the *symbol* of the multiplier. While multipliers have been implicitly used in applications for a long time, the first systematic theoretical study on Gabor multipliers was done in [17]. Not much later, multipliers in a more general setting (multipliers for Bessel sequences) were considered in [1] and a deep investigation of inversion of multipliers was initiated [28, 29, 30, 2, 31].

As it was mentioned above, in certain cases dual frames different from the canonical one have some desired useful properties and are of interest for further investigation. The next theorem from [2] determines a specific dual frame which has a special role in representation of the inverse of an invertible frame multiplier. It shows that for quite general class of inverible frame multipliers, the inverse operator can be represented as a multiplier using the reciprocal symbol and dual frames of the initial frames - more precisely, using a unique special dual frame of one of the initial ones such that the dual of the other initial frame can be arbitrarily chosen:

Theorem 1.1 ([2]). *Let Φ and Ψ be frames for $\mathcal{H}$ and let the scalar sequence m satisfy $0 < \inf_n |m_n| \leq \sup_n |m_n| < \infty$. Assume that the multiplier $M_{m,\Phi,\Psi}$ is invertible. Then the following holds:*

(i) *there exists a unique dual frame $\Phi^\dagger$ of Φ, so that*

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m}, \Psi^d, \Phi^\dagger \text{ for any dual frame } \Psi^d \text{ of } \Psi;$$

(ii) *there exists a unique dual frame $\Psi^\dagger$ of Ψ, so that*

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m}, \Psi^\dagger, \Phi^d \text{ for any dual frame } \Phi^d \text{ of } \Phi.$$

Note that the dual frame $\Phi^\dagger$ (resp. $\Psi^\dagger$) can be written as $((M_{m,\Phi,\Psi}^{-1})^*(\overline{m_n}\psi_n))_{n=1}^\infty$ (resp. $((M_{m,\Phi,\Psi}^{-1})^*(m_n\phi_n))_{n=1}^\infty$) [2]. In Section 3 we present some localization results for these special dual frames $\Phi^\dagger$ and $\Psi^\dagger$.

1.3. Localized frames

In this paper we consider three types of localized frames, namely, polynomially and exponentially localized frames as defined in [21] and sub-exponentially localized frames as defined in [37]. Let G be a Riesz basis for the Hilbert space $\mathcal{H}$. A frame E for $\mathcal{H}$ is called:

– *polynomially localized with respect to G with decay $\gamma > 0$* (in short, *γ-localized wrt $(g_n)_{n=1}^\infty$*) if there is a constant $C_\gamma > 0$ so that

$$\max\{|\langle e_m, g_n\rangle|, |\langle e_m, \widetilde{g_n}\rangle|\} \leq C_\gamma(1+|m-n|)^{-\gamma},\ m,n\in\mathbb{N};$$

– *exponentially localized with respect to G* if for some $\gamma > 0$ there is a constant $C_\gamma > 0$ so that

$$\max\{|\langle e_m, g_n\rangle|, |\langle e_m, \widetilde{g_n}\rangle|\} \leq C_\gamma \mathrm{e}^{-\gamma|m-n|},\ m,n\in\mathbb{N}; \tag{1.1}$$

– *β-sub-exponentially localized with respect to G* (for $\beta \in (0,1)$) if for some $\gamma > 0$ there is $C_\gamma > 0$ so that

$$\max\{|\langle e_m, g_n\rangle|, |\langle e_m, \widetilde{g_n}\rangle|\} \leq C_\gamma \mathrm{e}^{-\gamma|m-n|^\beta},\ m,n\in\mathbb{N}. \tag{1.2}$$

1.4. Banach and Fréchet spaces under consideration

A sequence space is called a BK-space if the coordinate functionals are continuous. Given a BK-space $(\Theta, |||\cdot|||_\Theta)$ and a Riesz basis $G = (g_n)_{n=1}^\infty$ for $\mathcal{H}$ with $\widetilde{G} = (\widetilde{g}_n)_{n=1}^\infty$ denoting its canonical dual, one associates to Θ the Banach space

$$\mathcal{H}_G^\Theta := \left\{ f\in\mathcal{H}\ :\ (\langle f,\widetilde{g}_n\rangle)_{n=1}^\infty \in \Theta,\ \|f\|_{\mathcal{H}_G^\Theta} := |||(\langle f,\widetilde{g}_n\rangle)_{n=1}^\infty|||_\Theta \right\},$$

which can also be determined as

$$\mathcal{H}_G^\Theta := \left\{ f\in\mathcal{H}\ :\ f=\sum_{n=1}^\infty c_n g_n \text{ with } (c_n)_{n=1}^\infty\in\Theta,\ \|f\|_{\mathcal{H}_G^\Theta} := |||(c_n)_{n=1}^\infty|||_\Theta \right\}.$$

Recall that the well known Schwartz space $\mathcal{S}$ is the intersection of Banach spaces

$$\mathcal{S}_k(\mathbb{R}) := \left\{ f\in L^2(\mathbb{R}) : ||f||_k = \sum_{m=0}^k ||(1+|\cdot|^2)^{k/2} f^{(m)}||_{L^2(\mathbb{R})} \right\},\ \ k\in\mathbb{N}.$$

The space of sub-exponentially decreasing functions of order $1/\alpha$, $\alpha > 1/2$, is $\Sigma^\alpha := X_F = \cap_{k\in\mathbb{N}_0}\Sigma^{k,\alpha}$ where $\Sigma^{k,\alpha}$ are Banach spaces of L^2- functions with finite norms

$$||f||_k^\alpha = \sup_{n\in\mathbb{N}_0}\left\| \frac{k^n e^{k|x|^{1/\alpha}}|f^{(n)}(x)|}{n!^\alpha}\right\|_{L^2(\mathbb{R})} < \infty,\quad k\in\mathbb{N}.$$

In this paper we review some results on invertibility of the frame operator on certain Banach spaces of type $\mathcal{H}_G^\Theta$ and on the spaces $\mathcal{S}$ and Σ^α, $\alpha > 1/2$ (in Section 2) and present some new results on invertibility of the more general concept of frame multipliers (in Section 3).

1.5. General Fréchet frames

Recall the notion of a General Fréchet Frame for a Fréchet space. For this purpose, consider a sequence $\{Y_s, |\cdot|_s\}_{s\in\mathbb{N}_0}$ of separable Banach spaces such that

$$\{\mathbf{0}\} \neq \cap_{k\in\mathbb{N}_0} Y_k \subseteq \ldots \subseteq Y_2 \subseteq Y_1 \subseteq Y_0, \tag{1.3}$$

$$|\cdot|_0 \leq |\cdot|_1 \leq |\cdot|_2 \leq \cdots, \tag{1.4}$$

$$Y_F := \cap_{k\in\mathbb{N}_0} Y_k \text{ is dense in } Y_s, \quad s \in \mathbb{N}_0. \tag{1.5}$$

Then Y_F is a Fréchet space with the sequence of norms $|\cdot|_s$, $s \in \mathbb{N}_0$. We use the above type of sequences Y_s in two cases:

1. $Y_s = X_s$ with norm $\|\cdot\|_s, s \in \mathbb{N}_0$;
2. $Y_s = \Theta_s$ with norm $|||\cdot|||_s, s \in \mathbb{N}_0$.

Definition 1.1 ([35]). Let X_F be a Fréchet space determined by the separable Banach spaces X_s, $s \in \mathbb{N}_0$, satisfying (1.3)–(1.5), and let Θ_F be a Fréchet space determined by the BK-spaces Θ_s, $s \in \mathbb{N}_0$, satisfying (1.3)–(1.5). A sequence $(g_n)_{n=1}^\infty \in (X_F^*)^{\mathbb{N}}$ is called a *General pre-Fréchet frame* (in short, *General pre-F-frame*) *for X_F with respect to* Θ_F if there exist sequences $\{\widetilde{s}_k\}_{k\in\mathbb{N}_0} \subseteq \mathbb{N}_0, \{s_k\}_{k\in\mathbb{N}_0} \subseteq \mathbb{N}_0$ which increase to ∞ with the property $s_k \leq \widetilde{s}_k$, $k \in \mathbb{N}_0$, and there exist constants $B_k, A_k > 0$, $k\in\mathbb{N}_0$, satisfying $\{g_i(f)\}_{i=1}^\infty \in \Theta_F$ and

$$A_k\|f\|_{s_k} \leq |||\{g_i(f)\}_{i=1}^\infty|||_k \leq B_k\|f\|_{\widetilde{s}_k}, \quad f\in X_F,\ k \in \mathbb{N}_0. \tag{1.6}$$

The sequence $(g_n)_{n=1}^\infty \in (X_F^*)^{\mathbb{N}}$ is called a *General Fréchet frame* (in short, *General F-frame*) *for X_F with respect to* Θ_F if $(g_n)_{n=1}^\infty$ is a General pre-F-frame for X_F with respect to Θ_F and there exists a continuous operator $V : \Theta_F \to X_F$ so that $V\{g_i(f)\}_{i=1}^\infty = f$ for all $f\in X_F$.

2. *Invertibility of the frame operator on Banach and Fréchet spaces*

Recall that the frame operator associated to a given frame for a Hilbert space $\mathcal{H}$ is always invertible on $\mathcal{H}$. Introducing localized frames (of polynomial and exponential type), Groechenig [21] showed that the frame operator of localized frames of such types extends its invertibility property to entire class of appropriately determined Banach spaces and furthermore, the canonical dual of such frames has the same type of localization as the given frame. Motivated by [21], in [37] the authors expanded the investigation considering Frechet spaces and localization of sub-exponential type.

In this section we briefly review some statements which concern extension of the invertibility property of the (Hilbert) frame operator to Banach and Fréchet spaces and the localization property of the canonical dual. These statements motivated our results in Section 3, where we consider the more general case of frame multipliers and dual frames that may differ from the canonical ones.

To shorten the writing, let us introduce the following condition:

Definition 2.1. Given $p \in [1, \infty)$, we will say that a function μ on $\mathbb{R}$ satisfies the *condition L_p* if $\mu(n) \geq 1$ for every $n \in \mathbb{N}$ and $\ell^p_\mu \subset \ell^2$ with continuous embedding.

Let us first recall the main results that concern localized frames, the canonical dual, and the extension of the invertibility property of the frame operator to Banach spaces. The following statements related to the polynomial and exponential localization are from [21] and the ones related to the sub-exponential localization are from [37].

Theorem 2.1 ([21, 37]). *Let $p \in [1, \infty)$, $G = (g_n)_{n=1}^\infty$ be a Riesz basis for $\mathcal{H}$, and $E = (e_n)_{n=1}^\infty$ be a frame for $\mathcal{H}$. Assume that one of the following three conditions hold:*

- (P) *μ is a k-moderate weight satisfying L_p and the frame E is $(k+1+\varepsilon)$-localized for some $\varepsilon > 0$;*
- (E) *μ is a sub-exponential weight satisfying L_p and E is exponentially localized with respect to G;*
- (SE) *μ is a β_μ-sub-exponential weight satisfying L_p and E is β-sub-exponentially localized with respect to G and such that $\beta_\mu < \beta < 1$.*

Then the following statements hold:

- (i) *The frame operator S_E is invertible on $\mathcal{H}_G^{\ell^p_\mu}$.*
- (ii) *The canonical dual $(\widetilde{e_n})_{n=1}^\infty$ of E has the same type of localization as E, i.e., if E is polynomially, exponentially, sub-exponentially localized, respectively, then $(\widetilde{e_n})_{n=1}^\infty$ is also polynomially, exponentially, sub-exponentially localized, respectively (but the constant determining the level of the localization decay might differ).*

PROOF. For convenience of the readers, we include sketch of the proofs from [21, 37].

Assume that (P), (E), or (SE) holds. Consider the matrix $(A_{m,n})_{m,n\in\mathbb{N}}$ determined by:

$A_{m,n} = (1 + |m-n|)^{-(k+1+\varepsilon)}$ for $m, n \in \mathbb{N}$, in case (P) holds;

$A_{m,n} = e^{-\gamma|m-n|}$ for $m, n \in \mathbb{N}$, in case (E) holds, where γ is with meaning from the level of the exponential localization, see (1.1);

$A_{m,n} = e^{-\gamma|m-n|^\beta}$ for $m, n \in \mathbb{N}$, in case (SE) holds, where γ is with meaning from the level of the sub-exponential localization, see (1.2).

(i) Let $f \in \mathcal{H}_G^{\ell_\mu^p}$. Using [21, Lemma 3] in the cases (P) and (E), and [37, Lemma 5.1] in the case (SE) with traces back to [15, 22], one concludes that $\mathcal{A}(|\langle f, \widetilde{g_n}\rangle|)_{n=1}^\infty$ belongs to ℓ_μ^p, where $\mathcal{A}$ denotes the matrix type operator that corresponds to the matrix $(A_{m,n})_{m,n\in\mathbb{N}}$. Furthermore, representing f as $\sum_{n=1}^\infty \langle f, \widetilde{g_n}\rangle g_n$, one can conclude that for some constant C,

$$(\forall m \in \mathbb{N}) \quad |\langle f, e_m\rangle| \le C \sum_{n=1}^\infty A_{m,n} |\langle f, \widetilde{g_n}\rangle|,$$

implying that $(\langle f, e_m\rangle)_{m=1}^\infty$ belongs to ℓ_μ^p and

$$\|(\langle f, e_m\rangle)_{m=1}^\infty\|_{\ell_\mu^p} \le C\|\mathcal{A}\| \cdot \|f\|_{\mathcal{H}_G^{\ell_\mu^p}}.$$

Therefore, the analysis operator U_E (determined by $U_E f = (\langle f, e_m\rangle)_{m=1}^\infty$) maps boundedly $\mathcal{H}_G^{\ell_\mu^p}$ into ℓ_μ^p.

Now consider the synthesis operator T_E determined by $T_E(c_n)_{n=1}^\infty = \sum_{n=1}^\infty c_n e_n$. Fix an arbitrary $c = (c_n)_{n=1}^\infty$ from $\ell_\mu^p (\subseteq \ell^2)$ and let y denote the sum of the convergent series $\sum_{n=1}^\infty c_n e_n$ in $\mathcal{H}$. Using the above mentioned lemmas from [21, 37] according to the localization case, we have that $\mathcal{A}(|c_n|)_{n=1}^\infty \in \ell_\mu^p$. Since there is a constant C so that

$$|\langle y, \widetilde{g}_m\rangle| \le C \sum_{n=1}^\infty A_{m,n}|c_n| \quad \text{for every } m \in \mathbb{N},$$

it follows that $(\langle y, \widetilde{g}_m\rangle)_{m=1}^\infty \in \ell_\mu^p$ and therefore y belongs to $\mathcal{H}_G^{\ell_\mu^p}$. Hence, T_E maps ℓ_μ^p into $\mathcal{H}_G^{\ell_\mu^p}$ and furthermore,

$$\|T_E c\|_{\mathcal{H}_G^{\ell_\mu^p}} = \|(\langle y, \widetilde{g}_m\rangle)_{m=1}^\infty\|_{\ell_\mu^p} \le C\|\mathcal{A}\|\|c\|_{\ell_\mu^p}.$$

By the above, the frame operator $S_E = T_E U_E$ maps boundedly $\mathcal{H}_G^{\ell_\mu^p}$ into $\mathcal{H}_G^{\ell_\mu^p}$. It remains to show the bijectivity of S_E on $\mathcal{H}_G^{\ell_\mu^p}$. Denote $\mathcal{V} := U_{\widetilde{G}} S_E T_G$. The operator $\mathcal{V}$ is invertible on ℓ^2 and it maps boundedly ℓ_μ^p into ℓ_μ^p. Consider the inverse operator $\mathcal{V}^{-1}$ on ℓ^2. Using again the above mentioned lemmas from [21, 37], as well as the Jaffard's Theorem (see [21, Theorems 5 and 6], [37, Theorem 5.2], and the first sources [24, 39]), one proves that $\mathcal{V}^{-1}$ maps boundedly ℓ_μ^p into ℓ_μ^p. Therefore, $\mathcal{V}$ is a bounded bijection of ℓ_μ^p onto ℓ_μ^p. Finally, using the representation $S_E = T_G \mathcal{V} U_{\widetilde{G}}$, it follows that S_E is a bounded bijection of $\mathcal{H}_G^{\ell_\mu^p}$ onto $\mathcal{H}_G^{\ell_\mu^p}$.

(ii) To get conclusions for the localization of the canonical dual of E, consider the operator $\mathcal{V}^{-1}$, determined in (i), and the corresponding matrix $(V^{-1})_{j,n\in\mathbb{N}}$. Observe that

$$\langle \widetilde{e_m}, g_n\rangle = \sum_{j=1}^\infty \langle e_m, g_j\rangle \overline{(V^{-1})_{jn}} \quad \text{and} \quad \langle \widetilde{e_m}, \widetilde{g_n}\rangle = \sum_{j=1}^\infty \langle e_m, \widetilde{g_j}\rangle \overline{(V^{-1})_{jn}}$$

for $m, n \in \mathbb{N}$. Using the Jaffard's theorem and the above mentioned lemmas from [21, 37], in accordance with the localization case, one comes to the desired conclusion for $(\widetilde{e_n})_{n=1}^{\infty}$. $\square$

Now we consider the case of some important Fréchet spaces, namely, the Schwartz space $\mathcal{S}(\mathbb{R})$ and the space Σ^α ($\alpha > 1/2$). The following was proved in [37]:

Theorem 2.2 ([37]). *Assume that the sequence $E = (e_n)_{n=1}^{\infty}$ with elements from $\mathcal{S}(\mathbb{R})$ is a frame for $L^2(\mathbb{R})$ which is polynomially localized with respect to the Hermite basis $(h_n)_{n=1}^{\infty}$ with decay γ for every $\gamma \in \mathbb{N}$. Then the frame operator S_E is bijective from $\mathcal{S}$ onto $\mathcal{S}$ and the elements of the canonical dual of E belong to $\mathcal{S}$, i.e., $\widetilde{e_n} \in \mathcal{S}$, $n \in \mathbb{N}$.*

Theorem 2.3 ([37]). *Assume that the sequence $E = (e_n)_{n=1}^{\infty}$ with elements from Σ^α ($\alpha > 1/2$) is a frame for $L^2(\mathbb{R})$ which is β-sub-exponentially (with $\beta = 1/(2\alpha)$) or exponentially localized with respect to the Hermite basis $(h_n)_{n=1}^{\infty}$. Then the frame operator S_E is bijective from Σ^α onto Σ^α and $\widetilde{e_n} \in \Sigma^\alpha$, $n \in \mathbb{N}$.*

Actually the spaces $\mathcal{S}(\mathbb{R}^d)$ and $\Sigma^\alpha(\mathbb{R}^d)$ can be constructed as projective limits of Hilbert spaces. We will show in Section 4 how it can be done for the GS-frames in the inductive limit type space, Gelfand–Shilov space which has also a structure of a projective limit of uncountable Hilbert spaces.

3. *Extension of invertible Hilbert frame multipliers to Banach and Fréchet spaces*

In this section we expand the consideration from Section 2 to the more general case of frame multipliers and announce briefly new results in this direction. Proofs and further results will appear in a forthcoming paper.

Assuming invertibility of a frame multiplier on a given Hilbert space (for classes of such multipliers see e.g., [28]), we derive invertibility on certain Banach and Fréchet spaces. Let us begin with the Banach case.

Theorem 3.1. *Let the assumptions of Theorem 2.1 hold and let a frame F for $\mathcal{H}$ has the same type and level of localization as the frame E. Assume that m is bounded. Then the following holds:*

(i) *The frame multiplier $M_{m,E,F}$ is a bounded map from $\mathcal{H}_G^{\ell^p_\mu}$ into $\mathcal{H}_G^{\ell^p_\mu}$.*

(ii) *If $M_{m,E,F}$ is invertible on $\mathcal{H}$, then $M_{m,E,F}$ is invertible on $\mathcal{H}_G^{\ell^p_\mu}$.*

Now we consider localization properties of dual frames. In Section 2 we recalled statements from [21, 37] that concern the canonical dual of a given frame – given localized frame, its canonical dual has the same type of localization. As mentioned in the introduction, other dual frames are also of interest and sometimes they might be better for specific purposes in comparison to the canonical dual. Here we consider the special dual frames determined by Theorem 1.1 and show that they also preserve the type of localization of the given frame.

Theorem 3.2. *Let the assumptions of Theorem 3.1. Assume in addition that m is semi-normalized and $M_{m,E,F}$ invertible on $\mathcal{H}$. Then the following statements hold:*

(i) *The dual frame $E^\dagger = ((M^{-1}_{m,E,F})^*(\overline{m_n} f_n))_{n=1}^\infty$ of E induced by the invertible frame multiplier $M_{m,E,F}$ has the same type of localization as the localization of E and F (more precisely, if E and F are polynomially, exponentially, sub-exponentially localized, resp., then $E^\dagger$ is also polynomially, exponentially, sub-exponentially localized, resp., but the constant determining the level of the localization decay of $E^\dagger$ might be different from the one of E and F).*

(ii) *The dual frame $F^\dagger = (M^{-1}_{m,E,F}(m_n e_n))_{n=1}^\infty$ of F induced by the invertible frame multiplier $M_{m,E,F}$ has the same type of localization as the localization of E and F.*

Now we proceed with the Fréchet case. The next statements concern invertibility of multipliers on the spaces $\mathcal{S}$ and Σ^α, $\alpha > 1/2$.

Theorem 3.3. *Assume that the sequences $E = (e_n)_{n=1}^\infty$ and $F = (f_n)_{n=1}^\infty$ with elements from $\mathcal{S}(\mathbb{R})$ are frames for $L^2(\mathbb{R})$ which are polynomially localized with respect to the Hermite basis $(h_n)_{n=1}^\infty$ with decay γ for every $\gamma \in \mathbb{N}$. Let m be semi-normalized and let the frame multiplier $M_{m,E,F}$ be invertible on $\mathcal{H}$. Then $M_{m,E,F}$ is continuous and bijective from $\mathcal{S}$ onto $\mathcal{S}$ and the elements of the dual frames $E^\dagger$ and $F^\dagger$ belong to $\mathcal{S}$.*

Theorem 3.4. *Assume that the sequence $(e_n)_{n=1}^\infty$ with elements from Σ^α ($\alpha > 1/2$) is a frame for $L^2(\mathbb{R})$ which is β-sub-exponentially (with $\beta = 1/(2\alpha)$) or exponentially localized with respect to the Hermite basis $(h_n)_{n=1}^\infty$. Let m be semi-normalized and let the frame multiplier $M_{m,E,F}$ be invertible on $\mathcal{H}$. Then $M_{m,E,F}$ is continuous and bijective from Σ^α onto Σ^α and the elements of the dual frames $E^\dagger$ and $F^\dagger$ belong to Σ^α.*

4. *Frames in Gelfan Shilov space* $S^{\{p!^s\}(\mathbb{R}^d)}$, $s \geq 1/2$

Tempered ultradistributions (Beurling version) are introduced in [23] and [33], [34] while Gelfand Shilov spaces (Roumieu version) are introduced in [19]. These spaces are still popular cf. monograph [13] and papers [12], [5], [18], [20], for example. Let us denote by $\mathfrak{R}$ the set of all positive monotonically increasing sequences r_i, $i \in \mathbb{N}$, in the sequel denoted as (r_i), such that $r_i \to \infty$, as $i \to \infty$. With the partial order relation $(r_i) \leq (k_i)$ if $r_i \leq k_i$, $\forall i \geq i_0$ for some $i_0 \in \mathbb{N}$, $(\mathfrak{R}, \leq)$ becomes a directed set (both upwards and downwards directed).

For $(r_i) \in \mathfrak{R}$, consider the sequence $N_p = p!^s \prod_{i=1}^{p} r_i$, $p \in \mathbb{N}$. The associated function corresponding to the sequence $p!^s$, $M(\rho)$ is defined as $M(\rho) = \sup_{p\in\mathbb{N}} \ln_+(\rho^p/p!^s)$, $\rho > 0$, while the associated function for (N_p) is denoted by $N_{(r_i)}$ and determined by $N_{(r_i)}(\rho) = \sup_{p\in\mathbb{N}} \ln_+(\rho^p/p!^s \prod_{i=1}^{p} r_i)$, $\rho > 0$. We recall that $M(\rho) = const.e^{\rho^{1/s}}, \rho > 0$, and the relation between $M(\rho)$ and $N_{(r_i)}(\rho)$ is given by $N_{(r_i)}(\rho) = M(h(\rho)), \rho > 0$, where $h(\rho)$ is so called subordinate function, the one which is non-negative,increasing to ∞ so that $h(\rho)/\rho \to 0$ as $\rho \to \infty$. Note,

$$\forall (r_i) \in \mathfrak{R},\ \forall k > 0,\ \exists \rho_0 > 0,\ \ N_{(r_i)}(\rho) \leq kM(\rho),\ \rho > \rho_0.$$

Moreover,

$$\forall (r_i) \in \mathfrak{R},\ \forall c_1, c_2 > 0,\ \exists (\tilde{r}_i) \in \mathfrak{R},\ \ N_{(r_i)}(c_1\rho) \leq c_2 N_{(\tilde{r}_i)}(\rho),\ \rho > 0.$$

We denote by $\mathcal{S}^{p!^s,(r_i)}$, $s \geq 1/2, r_i \in \mathfrak{R}$, the Hilbert space of all $\varphi \in C^\infty(\mathbb{R}^d)$ with the scalar product

$$(\varphi, \vartheta) = \left(\sum_{\alpha\in\mathbb{N}_0^d} \frac{e^{N_{r_i}(|\cdot|)}}{\alpha!^s \Pi_{j=1}^{|\alpha|} r_j} \int_{\mathbb{R}^d} D^\alpha \varphi(t) \overline{D^\alpha \vartheta(t)} dt \right)^{1/2}$$

and with the finite norm

$$|\varphi|_{s,r_i} = \left(\sum_{\alpha\in\mathbb{N}^d} \frac{e^{N_{r_i}(|\cdot|)}}{\alpha!^s \Pi_{j=1}^{|\alpha|} r_j} \int_{\mathbb{R}^d} |D^\alpha \varphi(t)|^2 dt \right)^{1/2}.$$

We define the space of Roumieu tempered ultradifferentiable functions as

$$\mathcal{S}^{\{p!^s\}}(\mathbb{R}^d) = \text{proj} \lim_{(r_i)\in\mathfrak{R}} \mathcal{S}^{p!^s,(r_i)}.$$

We also consider the sequence space

$$\Theta^{\{p!^s\}} = \text{proj} \lim_{(r_i)\in\mathfrak{R}} \Theta^{p!^s,(r_i)},$$

where $\Theta^{p!^s,(r_i)}$ is the Hilbert space of sequences

$$\Theta^{p!^s,(r_i)} = \left\{ (c_n)_n : |||(c_n)_n|||_{s,(r_i)} = \left(\sum_{n=1}^{\infty} |c_n|^2 e^{2N_{(r_i)}(n)} \right)^{1/2} < \infty \right\},$$

supplied by the scalar product

$$((c_n)_n, (b_n)_n) = \sum_{n=1}^{\infty} |c_n \overline{b_n}| e^{2N_{(r_i)}(n)}$$

Now we give two definitions of GS-frames GF$_w$-frame (weak GF-frame) and GF$_s$-frame (strong GF-frame) . I our further investigations we will analyze both type of frames.

A sequence $(\phi_n)_{n=1}^{\infty}$ with elements from $\mathcal{S}^{\{p!^s\}}$ is called a GS$_w$- frame, respectively, GS$_s$- frame for $\mathcal{S}^{\{p!^s\}}(\mathbb{R}^d)$ with respect to $\Theta^{\{p!^s\}}$ if

$$\exists (m_i^k), (r_i^k), (s_i^k) \in \mathfrak{R},$$

respectively,

$$\forall (r_i^k), \exists (m_i^k), \exists (s_i^k) \in \mathfrak{R},$$
$$m_i^k, \leq m_i^{k+1},\ r_i^k, \leq r_i^{k+1},\ s_i^k, \leq s_i^{k+1},\ i, k \in \mathbb{N},$$

with the property that there exist constants $0 < A_k \leq B_k < \infty$, $k \in \mathbb{N}$, satisfying, for any $f \in \mathcal{S}^{\{p!^s\}}$,

$$A_k \|f\|_{(m_i^k)} \leq |||\{\phi_n(f)\}_{n=1}^{\infty}|||_{(r_i^k)} \leq B_k \|f\|_{(s_i^k)}, \quad k \in \mathbb{N}, \tag{4.1}$$

where

$$(\phi_n(f))_{n=1}^{\infty} \in \Theta^{\{p!^s\}}, \tag{4.2}$$

and there exists a continuous operator $V : \Theta^{\{p!^s\}} \to \mathcal{S}^{\{p!^s\}}$ so that $V((\phi_n(f))_{n=1}^{\infty})$ $= f$ for every $f \in \mathcal{S}^{\{p!^s\}}$.

Since the Hermite functions form an orthonormal basis for any $\mathcal{S}^{p!^s, r_i}$, one can easily see that one can simply construct frames in $\mathcal{S}^{\{p!^s\}}$.

Now it is an easy task to transfer our results from our papers [37] and[38] for Fréchet frames to both type of GS-frames. A more general definitions than given ones will be given in our next paper where the essential relations between them will be studied.

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