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NORM ESTIMATES FOR QUASI-BANACH MODULATION SPACES AND WIENER-LEBESGUE SPACES

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Dedicated to the 100th anniversary of the birth of Academician Bogoljub Stanković

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A b s t r a c t. *We investigate the relationship between quasi-Banach modulation spaces and Wiener-Lebesgue amalgam spaces. More precisely, we establish norm equivalence for these spaces across the full range of the Lebesgue parameters. Our main result unifies and complements the findings known so far.*

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1. *Introduction*

The Lebegue spaces $L^p(\mathbb{R}^d)$, $p \in [1,\infty]$, defined by the means of integrability conditions, are ubiquitous in mathematical analysis. However, a shortcoming of the usual Lebesgue spaces is that they do not allow for a distinction between local and global properties of their elements. A notable attempt to address this issue can be traced back to Norbert Wiener, who introduced the spaces $\mathsf{W}^1(L^p)$, $\mathsf{W}^p(L^1)$, with $p = 2$ or $p = \infty$, in the context of Tauberian-type theorems, [29].

These spaces are examples of "standard" or "classical" amalgams $\mathsf{W}^r(L^p, L^q)$ where local properties are defined by L^p-norm over the unit cubes, which is combined or "amalgamated" with a global condition related to the L^q space. More precisely, if f is a measurable function on $\mathbb{R}$ then

$$f \in \mathsf{W}(L^p, L^q) \quad \Leftrightarrow \quad \|f\|_{\mathsf{W}(L^p,L^q)} = \left(\sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} |f(t)|^p \, dt \right)^{q/p} \right)^{1/q} < \infty.$$

In the early 1980s Hans Feichtinger extended the concept of Wiener amalgams to a broad range of Banach spaces of functions or distributions acting as local or global components, and leading to the general Wiener amalgam spaces $\mathsf{W}(B, C)$, [5]. Here the spaces B, which are used for local measurements, and C, which describe the global property, satisfy certain admissibility conditions. It turns out that the Wiener amalgam spaces are valuable in various areas of harmonic analysis and its applications. For a recent survey of different aspects of Wiener amalgam spaces, including relevant applications, we refer to [8].

A central role in time-frequency analysis is played by the space $\mathsf{W}(\mathscr{F}L^1, L^1)$, also known as the Feichtinger or Segal algebra, often denoted by S_0. In the Feichtinger algebra, localized pieces have absolutely convergent Fourier series expansions, and the local norm is transformed into a global norm as a global L^1-amalgam. Among other features of S_0, it is the minimal Banach space that is isometrically invariant under both translations and modulations. A recent survey [18] is devoted to detailed properties of S_0 and its applications.

In addition, one may consider $\mathsf{W}(\mathscr{F}L^p, L^q)$ and define function spaces by taking the inverse Fourier transform, [6, 7]. These spaces are called modulation spaces and are denoted by $M^{p,q}(\mathbb{R}^d)$. In particular, $\mathsf{W}(\mathscr{F}L^p, L^p) = M^{p,p}(\mathbb{R}^d)$, which are Fourier transform invariant spaces.

Instead of being introduced as specific Wiener amalgam spaces on the Fourier transform side, modulation spaces $M^{p,q}_{(\omega)}(\mathbb{R}^d)$, $p, q \in (0, \infty)$, $\omega \in \mathscr{P}_E(\mathbb{R}^d)$, are commonly defined in terms of weighted mixed-norm conditions on the short-time Fourier transform of their elements, see Definition 3.1. Here $\mathscr{P}_E(\mathbb{R}^d)$ denotes the set of all moderate weights on $\mathbb{R}^d$, allowing to work within the framework of Gelfand-Shilov spaces of functions and their distribution spaces (see Subsection 2.2 for details). We note that the choice of the Lebesgue parameters p and q extends the usual Banach space setting to the more challenging situation of quasi-Banach spaces when $p, q \in (0, 1)$.

In this paper, we complement investigations of the relationship between $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ and Wiener-Lebesgue amalgam spaces $\mathsf{W}^r(\omega, \ell^{p,q})$, and establish certain norm equivalences between them. The Wiener-Lebesgue amalgam spaces $\mathsf{W}^r(\omega, \ell^{p,q})$ under consideration are introduced in Subsection 3.2. More precisely, it was proved that $f \in M^{p,q}_{(\omega)}(\mathbb{R}^d)$, if and only if $V_\phi f \in \mathsf{W}^r(\omega, \ell^{p,q})$, where $V_\phi f$ denotes the short-time Fourier transform, see Subsection 2.2. It was shown in [13] for $r = \infty$, and $p, q \in [1, \infty]$, and extended in [12, 23] to $p, q \in (0, \infty]$. We refer to [25, 26] for further extensions to $r \in (0, \infty]$. Our main result, Theorem 4.1 extends a general result formulated in [25, Proposition 2′] by relaxing the conditions on window functions. We note that here we use techniques different than the ones from those in the aforementioned contributions.

Mixed norm spaces of Lebesgue type, used in the definitions of modulation and Wiener amalgam spaces, can be replaced by more general function spaces. This gives rise to a broad class of modulation and Wiener amalgam spaces, denoted by $M(\omega,\mathscr{B})$ and $\mathsf{W}^r(\omega,\mathscr{B})$, respectively, where $\mathscr{B}$ is a normal quasi-Banach function space, and $\omega \in \mathscr{P}_E(\mathbb{R}^d)$. To prove that f belongs to $M(\omega,\mathscr{B})$, iff $V_\phi f \in \mathsf{W}^r(\omega,\mathscr{B})$, and that the corresponding norms, $\|f\|_{M(\omega,\mathscr{B})}$ and $\|V_\phi f\|_{\mathsf{W}^r(\omega,\mathscr{B})}$ are equivalent, more advanced techniques than those presented here are required. For further details, we refer to [28]. Additionally, the norm estimate results presented there are used to extend certain continuity properties in pseudo-differential calculus, see also Remark 4.1.

The paper is structured as follows. In Section 2 we set the stage by presenting the main notions that will be used in the sequel. In Section 3 we introduce the quasi-Banach modulation spaces and Wiener-Lebesgue amalgam spaces and outline their main properties. The main result of the paper, Theorem 4.1, along with several auxiliary results, is proved in Section 4.

2. *Preliminaries*

In this section we recall some basic facts on weight functions, the short-time Fourier transform, Gelfand-Shilov spaces, and mixed-norm Lebesgue spaces which will be useful in subsequent sections.

2.1. Weight functions

A *weight* or *weight function* on $\mathbb{R}^d$ is a positive function $\omega \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ such that $1/\omega \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$. If there is a weight v on $\mathbb{R}^d$ and a constant $C \geq 1$ such that

$$\omega(x+y) \leq C\omega(x)v(y), \qquad x,y \in \mathbb{R}^d, \tag{2.1}$$

then the weight ω is called *moderate*, or v*-moderate*. By (2.1) we have

$$C^{-1}v(-x)^{-1} \leq \omega(x) \leq Cv(x), \quad x \in \mathbb{R}^d. \tag{2.2}$$

We let $\mathscr{P}_E(\mathbb{R}^d)$ be the set of all moderate weights on $\mathbb{R}^d$.

We say that a weight v is *submultiplicative* if

$$v(x+y) \leq v(x)v(y) \quad \text{and} \quad v(-x) = v(x), \qquad x,y \in \mathbb{R}^d. \tag{2.3}$$

If v is positive and locally bounded and satisfies the inequality in (2.3), then $v(x) \leq C_0 e^{r_0|x|}$ for some positive constants C_0 and r_0, cf. [14].

Therefore, if $\omega \in \mathscr{P}_E(\mathbb{R}^d)$, then

$$\omega(x+y) \lesssim \omega(x)e^{r_0|y|}, \quad x,y \in \mathbb{R}^d,$$

for some $r_0 > 0$. In particular, (2.2) shows that for any $\omega \in \mathscr{P}_E(\mathbb{R}^d)$, there is a constant $r_0 > 0$ such that

$$e^{-r_0|x|} \lesssim \omega(x) \lesssim e^{r_0|x|}, \quad x \in \mathbb{R}^d.$$

Here $g_1 \lesssim g_2$ means that $g_1(\theta) \le C \cdot g_2(\theta)$ holds uniformly for all θ in the intersection of the domains of g_1 and g_2 for some constant $C > 0$, and we write $g_1 \asymp g_2$ when $g_1 \lesssim g_2 \lesssim g_1$.

We observe that given a v-moderate weight ω, one can find a continuous v-moderate weight ω_0 such that $\omega \asymp \omega_0$. In addition, a moderate weight ω is also moderated by a submultiplicative weight, cf. [21]. In the sequel, v and v_j for $j \ge 0$, always stand for submultiplicative weights if nothing else is stated. We refer to [4, 13, 14, 16, 22] for more facts about weights in time-frequency analysis.

2.2. Gelfand-Shilov spaces and the short-time Fourier transform

In what follows we let $\mathscr{F}$ be the Fourier transform which takes the form

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x)e^{-i\langle x,\xi\rangle}\,dx, \quad \xi \in \mathbb{R}^d,$$

when $f \in L^1(\mathbb{R}^d)$. Here $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the usual scalar product on $\mathbb{R}^d$. The map $\mathscr{F}$ extends uniquely to a homeomorphism on the space of tempered distributions $\mathscr{S}'(\mathbb{R}^d)$, to a unitary operator on $L^2(\mathbb{R}^d)$ and restricts to a homeomorphism on the Schwartz space of smooth rapidly decreasing functions $\mathscr{S}(\mathbb{R}^d)$. We observe that with our choice of the Fourier transform, the usual convolution identity for the Fourier transform takes the forms

$$\mathscr{F}(fg) = (2\pi)^{-\frac{d}{2}}\widehat{f} * \widehat{g} \quad \text{and} \quad \mathscr{F}(f*g) = (2\pi)^{\frac{d}{2}}\widehat{f}\cdot\widehat{g},$$

when $f,g \in \mathscr{S}(\mathbb{R}^d)$.

Since we are interested in general weights $\omega \in \mathscr{P}_E(\mathbb{R}^d)$, instead of the framework of tempered distributions $\mathscr{S}'(\mathbb{R}^d)$, which is natural when dealing with weights of polynomial growth, we consider Gelfand-Shilov spaces $\Sigma_s(\mathbb{R}^d)$ and $\mathcal{S}_s(\mathbb{R}^d)$ and their dual spaces of (ultra-)distributions $\Sigma_s'(\mathbb{R}^d)$ and $\mathcal{S}_s'(\mathbb{R}^d)$, $s \ge 1$.

In order to avoid technical issues related to the usual definition of the spaces of Gelfand-Shilov type and their distribution spaces, we introduce $\Sigma_s(\mathbb{R}^d)$ and $\mathcal{S}_s(\mathbb{R}^d)$ in terms of decay estimates of the functions and their Fourier transforms. More precisely, if $f \in \mathscr{S}'(\mathbb{R}^d)$ and $s \ge 1$, then f belongs to the *Gelfand-Shilov space*

$\Sigma_s(\mathbb{R}^d)$ *of Beurling type* (the *Gelfand-Shilov space* $\mathcal{S}_s(\mathbb{R}^d)$ *of Roumieu type*), if and only if

$$|f(x)| \lesssim e^{-r|x|^{\frac{1}{s}}} \quad \text{and} \quad |\mathscr{F} f(\xi)| \lesssim e^{-r|\xi|^{\frac{1}{s}}}, \quad x, \xi \in \mathbb{R}^d, \tag{2.4}$$

for every $r > 0$ (for some $r > 0$), cf. [1, 3].

Then the *Gelfand-Shilov distribution space* $\Sigma_s'(\mathbb{R}^d)$ (of *Beurling type*) can be introduced as the (strong) dual to $\Sigma_s(\mathbb{R}^d)$, and the *Gelfand-Shilov distribution space* $\mathcal{S}_s'(\mathbb{R}^d)$ (of *Roumieu type*) is the (strong) dual to $\mathcal{S}_s(\mathbb{R}^d)$, $s \geq 1$.

By (2.4), when $1 \leq s_1 \leq s_2$ we have (continuous and dense) embeddings

$$\begin{aligned} \Sigma_{s_1}(\mathbb{R}^d) \hookrightarrow \mathcal{S}_{s_1}(\mathbb{R}^d) \hookrightarrow \Sigma_{s_2}(\mathbb{R}^d) \hookrightarrow \mathcal{S}_{s_2}(\mathbb{R}^d) & \\ \hookrightarrow \mathcal{S}_{s_2}'(\mathbb{R}^d) \hookrightarrow \Sigma_{s_2}'(\mathbb{R}^d) \hookrightarrow \mathcal{S}_{s_1}'(\mathbb{R}^d) \hookrightarrow \Sigma_{s_1}'(\mathbb{R}^d). & \end{aligned}$$

If $A \subseteq B$ with continuous inclusion we write $A \hookrightarrow B$. For simplicity, throughout this subsection we consider $\Sigma_s(\mathbb{R}^d)$ and $\Sigma_s'(\mathbb{R}^d)$, $s \geq 1$.

The reader should keep in mind that all results and comments given there hold true when $\Sigma_s(\mathbb{R}^d)$ and $\Sigma_s'(\mathbb{R}^d)$ are replaced by $\mathcal{S}_s(\mathbb{R}^d)$ and $\mathcal{S}_s'(\mathbb{R}^d)$ respectively.

In several situations it is convenient to use a localized version of the Fourier transform, called the short-time Fourier transform, STFT for short. The short-time Fourier transform of $f \in \Sigma_s'(\mathbb{R}^d)$ with respect to the fixed *window function* $\phi \in \Sigma_s(\mathbb{R}^d)$ is defined by

$$(V_\phi f)(x, \xi) \equiv (2\pi)^{-\frac{d}{2}} (f, \phi(\,\cdot\, - x) e^{i\langle\,\cdot\,,\xi\rangle})_{L^2}, \quad x, \xi \in \mathbb{R}^d. \tag{2.5}$$

Here $(\,\cdot\,,\,\cdot\,)_{L^2}$ denotes the unique continuous extension of the inner product on $L^2(\mathbb{R}^d)$ restricted to $\Sigma_s(\mathbb{R}^d)$ into a continuous map from $\Sigma_s'(\mathbb{R}^d) \times \Sigma_s(\mathbb{R}^d)$ to $\mathbb{C}$.

We observe that using certain properties for tensor products of distributions (2.5) can be written as

$$(V_\phi f)(x, \xi) = \mathscr{F}(f \cdot \overline{\phi(\,\cdot\, - x)})(\xi), \quad x, \xi \in \mathbb{R}^d \tag{2.5$'$}$$

(cf. [17, 24]). If in addition $f \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$, then

$$(V_\phi f)(x, \xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(y) \overline{\phi(y - x)} e^{-i\langle y, \xi\rangle}\, dy, \quad x, \xi \in \mathbb{R}^d. \tag{2.5$''$}$$

Proposition 2.1. *The map*

$$(f, \phi) \mapsto V_\phi f : \Sigma_s(\mathbb{R}^d) \times \Sigma_s(\mathbb{R}^d) \to \Sigma_s(\mathbb{R}^{2d}) \tag{2.6}$$

is continuous, which extends uniquely to a continuous map

$$(f, \phi) \mapsto V_\phi f : \Sigma_s'(\mathbb{R}^d) \times \Sigma_s'(\mathbb{R}^d) \to \Sigma_s'(\mathbb{R}^{2d}), \tag{2.7}$$

which in turn restricts to an isometric map

$$(f, \phi) \mapsto V_\phi f : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d}). \tag{2.8}$$

If $\phi \in \Sigma_s(\mathbb{R}^d)$ and $f \in \Sigma_s'(\mathbb{R}^d)$, then (2.7) shows that $V_\phi f \in \Sigma_s'(\mathbb{R}^{2d})$. On the other hand, it is easy to see that the right-hand side of (2.5) defines a smooth function. Consequently beside (2.6) and (2.7), we also have the continuous map

$$(f,\phi) \mapsto V_\phi f : \Sigma_s'(\mathbb{R}^d) \times \Sigma_s(\mathbb{R}^d) \to \Sigma_s'(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d}).$$

By a straight-forward computation it follows that

$$|V_\phi f(x,\xi)| = |V_f \phi(-x,-\xi)|, \ \ (x,\xi) \in \mathbb{R}^{2d}, \ \ f \in \Sigma_1'(\mathbb{R}^d), \ \ \phi \in \Sigma_1(\mathbb{R}^d). \quad (2.9)$$

For the short-time Fourier transform, the Parseval identity is replaced by the so-called Moyal identity, also known as the *orthogonality relation* given by

$$(V_\phi f, V_\psi g)_{L^2(\mathbb{R}^{2d})} = (\psi,\phi)_{L^2(\mathbb{R}^d)} (f,g)_{L^2(\mathbb{R}^d)}, \quad (2.10)$$

when $f,g,\phi,\psi \in \Sigma_s(\mathbb{R}^d)$.

By Moyal's identity (2.10) it follows that if $\phi \in \Sigma_s(\mathbb{R}^d) \setminus 0$, then the identity operator on $\Sigma_s'(\mathbb{R}^d)$ is given by

$$\mathrm{Id} = \big(\|\phi\|_{L^2}^{-2}\big) \cdot V_\phi^* \circ V_\phi, \quad (2.11)$$

provided suitable mapping properties of the (L^2-)adjoint V_ϕ^* of V_ϕ can be established. Obviously, V_ϕ^* fullfils

$$(V_\phi^* F, g)_{L^2(\mathbb{R}^d)} = (F, V_\phi g)_{L^2(\mathbb{R}^{2d})} \quad (2.12)$$

when $F \in \Sigma_s(\mathbb{R}^{2d})$ and $g \in \Sigma_s(\mathbb{R}^d)$.

By expressing the scalar product and the short-time Fourier transform in terms of integrals in (2.12), it follows by straight-forward manipulations that the adjoint in (2.12) is given by

$$(V_\phi^* F)(x) = (2\pi)^{-\frac{d}{2}} \iint_{\mathbb{R}^{2d}} F(y,\eta)\phi(x-y) e^{\mathrm{i}\langle x,\eta\rangle}\, dy\, d\eta,$$

when $F \in \Sigma_s(\mathbb{R}^{2d})$ and $\phi \in \Sigma_s(\mathbb{R}^d)$. We may now use mapping properties like (2.7)–(2.8) to extend the definition of $V_\phi^* F$ when F and ϕ belong to various classes of function and distribution spaces. For example, by (2.6)–(2.8), it follows that the map

$$(F,g) \mapsto (F, V_\phi g)_{L^2(\mathbb{R}^{2d})}$$

defines a sesqui-linear form on

$$\Sigma_s(\mathbb{R}^{2d}) \times \Sigma_s'(\mathbb{R}^d), \quad \Sigma_s'(\mathbb{R}^{2d}) \times \Sigma_s(\mathbb{R}^d) \quad \text{and on} \quad L^2(\mathbb{R}^{2d}) \times L^2(\mathbb{R}^d).$$

This implies that if $\phi \in \Sigma_s(\mathbb{R}^d)$, then the mappings

$$V_\phi^* : \Sigma_s(\mathbb{R}^{2d}) \to \Sigma_s(\mathbb{R}^d), \qquad V_\phi^* : \Sigma_s'(\mathbb{R}^{2d}) \to \Sigma_s'(\mathbb{R}^d)$$
$$\text{and} \qquad V_\phi^* : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^d)$$

are continuous.

We will often use the (pointwise) estimates

$$|(V_{\phi_1} \circ V_{\phi_2}^*) F(x,\xi)| \le (|F| * |V_{\phi_1}\phi_2|)(x,\xi) \tag{2.13}$$

and

$$|V_{\phi_1} f(x,\xi)| \le \|\phi_2\|_{L^2}^{-2} (|V_{\phi_2} f| * |V_{\phi_1}\phi_2|)(x,\xi), \quad (x,\xi) \in \mathbb{R}^{2d}. \tag{2.14}$$

Clearly, (2.14) follows from (2.11) and (2.13). Here above F is a suitable distribution on $\mathbb{R}^{2d}$, f is a suitable distribution on $\mathbb{R}^d$, and ϕ_1, ϕ_2 are suitable functions on $\mathbb{R}^d$. We refer to [13, Proposition 11.3.2] for the proofs of (2.11) and (2.13) in the context of tempered distributions.

2.3. Mixed norm spaces of Lebesgue type

Let F be a (complex-valued) measurable function on $\mathbb{R}^{2d}$, $p,q \in (0,\infty]$, and let ω be a weight on $\mathbb{R}^{2d}$. Then we set

$$G_{F,\omega,p}(\xi) = \|F(\,\cdot\,,\xi)\omega(\,\cdot\,,\xi)\|_{L^p(\mathbb{R}^d)}, \quad H_{F,\omega,q}(x) = \|F(x,\,\cdot\,)\omega(x,\,\cdot\,)\|_{L^q(\mathbb{R}^d)},$$

and

$$\|F\|_{L^{p,q}_{(\omega)}(\mathbb{R}^{2d})} \equiv \|G_{F,\omega,p}\|_{L^q(\mathbb{R}^d)}, \quad \|F\|_{L^{p,q}_{*,(\omega)}(\mathbb{R}^{2d})} \equiv \|H_{F,\omega,q}\|_{L^p(\mathbb{R}^d)}.$$

Then the mixed-norm Lebesgue space $L^{p,q}_{(\omega)}(\mathbb{R}^{2d})$ ($L^{p,q}_{*,(\omega)}(\mathbb{R}^{2d})$) consists of all measurable functions F such that $\|F\|_{L^{p,q}_{(\omega)}} < \infty$ ($\|F\|_{L^{p,q}_{*,(\omega)}} < \infty$).

Discrete versions of mixed-norm Lebesgue spaces are given as follows. Let Ω_1, Ω_2 be discrete sets, ω be a positive function on $\Omega_1 \times \Omega_2$ and $\ell_0'(\Omega_1 \times \Omega_2)$ be the set of all formal (complex-valued) sequences $c = \{c(j,\iota)\}_{j \in \Omega_1, \iota \in \Omega_2}$. Then the discrete Lebesgue spaces, i.e. the Lebesgue sequence spaces

$$\ell^{p,q}_{(\omega)}(\Omega_1 \times \Omega_2) \quad \text{and} \quad \ell^{p,q}_{*,(\omega)}(\Omega_1 \times \Omega_2)$$

of mixed (quasi-)norm types consist of all $c \in \ell_0'(\Omega_1 \times \Omega_2)$ such that

$$\|c\|_{\ell^{p,q}_{(\omega)}(\Omega_1 \times \Omega_2)} < \infty \ \text{ respectively } \ \|c\|_{\ell^{p,q}_{*,(\omega)}(\Omega_1 \times \Omega_2)} < \infty,$$

where

$$\|c\|_{\ell^{p,q}_{(\omega)}(\Omega_1\times\Omega_2)} \equiv \|G_{c,\omega,p}\|_{\ell^q(\Omega_2)}, \quad \text{where} \quad G_{c,\omega,p}(\iota) = \|c(\,\cdot\,,\iota)\omega(\,\cdot\,,\iota)\|_{\ell^p(\Omega_1)}$$

and

$$\|c\|_{\ell^{p,q}_{*,(\omega)}(\Omega_1\times\Omega_2)} \equiv \|H_{c,\omega,q}\|_{\ell^p(\Omega_1)}, \quad \text{where} \quad H_{c,\omega,q}(j) = \|c(j,\,\cdot\,)\omega(j,\,\cdot\,)\|_{\ell^q(\Omega_2)}.$$

2.4. Convolutions and multiplications for discrete Lebesgue spaces

Next we discuss extended Hölder and Young inequalities for multiplications and convolutions on discrete Lebesgue spaces. The Hölder and Young conditions on the Lebesgue exponents are respectively given by

$$\frac{1}{q_0} \le \frac{1}{q_1} + \frac{1}{q_2}, \tag{2.15}$$

and

$$\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2} - \max\left(1, \frac{1}{p_1}, \frac{1}{p_2}\right). \tag{2.16}$$

Notice that, when $p_1, p_2 \in (0,1)$, then (2.16) becomes $p_0 \ge \max\{p_1, p_2\}$, while for $p_1, p_2 \ge 1$ it reduces to the common Young condition

$$1 + \frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2}.$$

By $\ell_0(\Omega)$ we denote the set if all sequences $c = \{c(j)\}_{j\in\Omega}$ with finitely many entries $c(j) \ne 0$.

Proposition 2.2. *Let $p_n, q_n \in (0,\infty]$, $n = 0,1,2$, be such that* (2.15) *and* (2.16) *hold, let $\Lambda \subseteq \mathbb{R}^d$ be a lattice and let ω_n be weights on Λ, $n = 0,1,2$. Then the following is true:*

(1) *if $\omega_0(j) \le \omega_1(j)\omega_2(j)$, $j \in \Lambda$, then the map $(a_1, a_2) \mapsto a_1 \cdot a_2$ from $\ell_0(\Lambda) \times \ell_0(\Lambda)$ to $\ell_0(\Lambda)$ extends uniquely to a continuous map from $\ell^{q_1}_{(\omega_1)}(\Lambda) \times \ell^{q_2}_{(\omega_2)}(\Lambda)$ to $\ell^{q_0}_{(\omega_0)}(\Lambda)$, and*

$$\|a_1 \cdot a_2\|_{\ell^{q_0}_{(\omega_0)}} \le \|a_1\|_{\ell^{q_1}_{(\omega_1)}} \|a_2\|_{\ell^{q_2}_{(\omega_2)}}, \qquad a_n \in \ell^{q_n}_{(\omega_n)}(\Lambda),\ n = 1,2;$$

(2) *if $\omega_0(j_1 + j_2) \le \omega_1(j_1)\omega_2(j_2)$, $j_1, j_2 \in \Lambda$, then the map $(a_1, a_2) \mapsto a_1 * a_2$ from $\ell_0(\Lambda) \times \ell_0(\Lambda)$ to $\ell_0(\Lambda)$ extends uniquely to a continuous map from $\ell^{p_1}_{(\omega_1)}(\Lambda) \times \ell^{p_2}_{(\omega_2)}(\Lambda)$ to $\ell^{p_0}_{(\omega_0)}(\Lambda)$, and*

$$\|a_1 * a_2\|_{\ell^{p_0}_{(\omega_0)}} \le \|a_1\|_{\ell^{p_1}_{(\omega_1)}} \|a_2\|_{\ell^{p_2}_{(\omega_2)}}, \qquad a_n \in \ell^{p_n}_{(\omega_n)}(\Lambda),\ n = 1,2. \tag{2.17}$$

The assertion (1) is the Hölder inequality for discrete Lebesgue spaces, and the assertion (2) is Young's inequality for Lebesgue spaces on lattices when $p_0, p_1, p_2 \in [1, \infty]$. We refer to Appendix A in [27] for the proof of Proposition 2.2.

3. *Modulation spaces and Wiener-Lebesgue amalgam spaces*

In this section we recall definitions and basic properties of modulation spaces and Wiener-Lebesgue amalgam spaces.

3.1. Modulation spaces

The (classical) modulation spaces, essentially introduced in [6] by Feichtinger are given in the following. (See e. g. [7] for definition of more general modulation spaces.)

Definition 3.1. Let $p, q \in (0, \infty]$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$ and $\phi \in \Sigma_1(\mathbb{R}^d) \setminus 0$.

1. The *modulation space* $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ consists of all $f \in \Sigma_1'(\mathbb{R}^d)$ such that

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \|V_\phi f\|_{L^{p,q}_{(\omega)}}$$

   is finite. The topology of $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ is defined by the (quasi-)norm $\|\cdot\|_{M^{p,q}_{(\omega)}}$.

2. The *modulation space (of Wiener amalgam type)* $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ consists of all $f \in \Sigma_1'(\mathbb{R}^d)$ such that

$$\|f\|_{W^{p,q}_{(\omega)}} \equiv \|V_\phi f\|_{L^{p,q}_{*,(\omega)}}$$

   is finite. The topology of $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ is defined by the (quasi-)norm $\|\cdot\|_{W^{p,q}_{(\omega)}}$.

In the next Remark p' respectively q' are the conjugate exponents of p respectively q. This means that

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{and} \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Remark 3.1. Modulation spaces possess several convenient properties. In fact, let $p, q \in (0, \infty]$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$ and $\phi \in \Sigma_1(\mathbb{R}^d) \setminus 0$. Then the following is true (see [6, 7, 9, 10, 12, 13] and their analyses for verifications):

- the definitions of $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ are independent of the choices of $\phi \in \Sigma_1(\mathbb{R}^d) \setminus 0$, and different choices give rise to equivalent quasi-norms;

- the spaces $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ are quasi-Banach spaces which increase with p and q, and decrease with ω. If in addition $p,q \ge 1$, then they are Banach spaces;

- if $p,q \ge 1$, then the $L^2(\mathbb{R}^d)$ scalar product, $(\,\cdot\,,\,\cdot\,)_{L^2(\mathbb{R}^d)}$, on $\Sigma_1(\mathbb{R}^d)\times\Sigma_1(\mathbb{R}^d)$ is uniquely extendable to dualities between $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ and $M^{p',q'}_{(1/\omega)}(\mathbb{R}^d)$, and between $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ and $W^{p',q'}_{(1/\omega)}(\mathbb{R}^d)$. If in addition $p,q<\infty$, then the dual spaces of $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ can be identified with $M^{p',q'}_{(1/\omega)}(\mathbb{R}^d)$ respectively $W^{p',q'}_{(1/\omega)}(\mathbb{R}^d)$, through the form $(\,\cdot\,,\,\cdot\,)_{L^2(\mathbb{R}^d)}$;

- if $\omega_0(x,\xi) = \omega(-\xi,x)$, then $\mathscr{F}$ on $\Sigma_1'(\mathbb{R}^d)$ restricts to a homeomorphism from $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ to $W^{q,p}_{(\omega_0)}(\mathbb{R}^d)$;

- when $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$ the following inclusions are continuous:
$$\Sigma_1(\mathbb{R}^d) \subseteq M^{p,q}_{(\omega)}(\mathbb{R}^d), W^{p,q}_{(\omega)}(\mathbb{R}^d) \subseteq \Sigma_1'(\mathbb{R}^d). \tag{3.1}$$
If in addition $p,q<\infty$, then these inclusions are dense.

- Let $s \ge 1$ be fixed and set
$$v_{r,t}(x,\xi) = \begin{cases} e^{r(|x|^{1/t}+|\xi|^{1/t})}, & t \in (0,\infty), \\ (1+|x|+|\xi|)^r, & t = \infty, \end{cases}$$
where $x,\xi \in \mathbb{R}^d$. Then
$$\begin{aligned} \Sigma_s(\mathbb{R}^d) &= \bigcap_{r>0} M^{p,q}_{(v_{r,s})}(\mathbb{R}^d) &&= \bigcap_{r>0} W^{p,q}_{(v_{r,s})}(\mathbb{R}^d), \\ \mathcal{S}_s(\mathbb{R}^d) &= \bigcup_{r>0} M^{p,q}_{(v_{r,s})}(\mathbb{R}^d) &&= \bigcup_{r>0} W^{p,q}_{(v_{r,s})}(\mathbb{R}^d), \\ \mathscr{S}(\mathbb{R}^d) &= \bigcap_{r>0} M^{p,q}_{(v_{r,\infty})}(\mathbb{R}^d) &&= \bigcap_{r>0} W^{p,q}_{(v_{r,\infty})}(\mathbb{R}^d), \\ \mathscr{S}'(\mathbb{R}^d) &= \bigcup_{r>0} M^{p,q}_{(1/v_{r,\infty})}(\mathbb{R}^d) &&= \bigcup_{r>0} W^{p,q}_{(1/v_{r,\infty})}(\mathbb{R}^d), \\ \mathcal{S}_s'(\mathbb{R}^d) &= \bigcap_{r>0} M^{p,q}_{(1/v_{r,s})}(\mathbb{R}^d) &&= \bigcap_{r>0} W^{p,q}_{(1/v_{r,s})}(\mathbb{R}^d), \end{aligned} \tag{3.2}$$
and
$$\Sigma_s'(\mathbb{R}^d) = \bigcup_{r>0} M^{p,q}_{(1/v_{r,s})}(\mathbb{R}^d) = \bigcup_{r>0} W^{p,q}_{(1/v_{r,s})}(\mathbb{R}^d). \tag{3.3}$$

The topologies of the spaces on the left-hand sides of (3.2)–(3.3) are obtained by replacing each intersection by projective limit with respect to $r > 0$ and each union with inductive limit with respect to $r > 0$.

Relations (3.2)–(3.3) are essentially special cases of [22, Theorem 3.9], see also [15, 20].

We finish this subsection with convolution and multiplication properties for modulation spaces. Similar results hold true for modulation spaces of Wiener amalgam type, cf. [21].

For multiplications of elements in modulation spaces the involved Lebesgue exponents should satisfy

$$\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q_0} \le \frac{1}{q_1} + \frac{1}{q_2} - \max\left(1, \frac{1}{p_0}, \frac{1}{q_1}, \frac{1}{q_2}\right), \tag{3.4}$$

and the condition on the weight functions is given by

$$\omega_0(x, \xi_1 + \xi_2) \le \omega_1(x, \xi_1)\omega_2(x, \xi_2), \qquad x, \xi_1, \xi_2 \in \mathbb{R}^d. \tag{3.5}$$

Theorem 3.1. *Let $p_n, q_n \in (0, \infty)$ and $\omega_n \in \mathscr{P}_E(\mathbb{R}^{2d})$, $n = 0, 1, 2$, be such that* (3.4) *and* (3.5) *hold. Then $(f_1, f_2) \mapsto f_1 \cdot f_2$ from $\Sigma_1(\mathbb{R}^d) \times \Sigma_1(\mathbb{R}^d)$ to $\Sigma_1(\mathbb{R}^d)$ is uniquely extendable to a continuous map from $M^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \times M^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d)$ to $M^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^d)$, and*

$$\|f_1 f_2\|_{M^{p_0,q_0}_{(\omega_0)}} \lesssim \|f_1\|_{M^{p_1,q_1}_{(\omega_1)}} \|f_2\|_{M^{p_2,q_2}_{(\omega_2)}}, \quad f_n \in M^{p_n,q_n}_{(\omega_n)}(\mathbb{R}^d), \ \ n = 1, 2.$$

For the corresponding results for convolutions the conditions on the involved Lebesgue exponents are given by (2.15) and (2.16), and the conditions on the weight functions are now given by

$$\omega_0(x_1 + x_2, \xi) \le \omega_1(x_1, \xi)\omega_2(x_2, \xi), \qquad x_1, x_2, \xi \in \mathbb{R}^d. \tag{3.6}$$

Theorem 3.2. *Let $p_n, q_n \in (0, \infty)$ and $\omega_n \in \mathscr{P}_E(\mathbb{R}^{2d})$, $n = 0, 1, 2$, be such that* (2.15)*,* (2.16) *and* (3.6) *hold. Then $(f_1, f_2) \mapsto f_1 * f_2$ from $\Sigma_1(\mathbb{R}^d) \times \Sigma_1(\mathbb{R}^d)$ to $\Sigma_1(\mathbb{R}^d)$ is uniquely extendable to a continuous map from $M^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \times M^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d)$ to $M^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^d)$, and*

$$\|f_1 * f_2\|_{M^{p_0,q_0}_{(\omega_0)}} \lesssim \|f_1\|_{M^{p_1,q_1}_{(\omega_1)}} \|f_2\|_{M^{p_2,q_2}_{(\omega_2)}}, \quad f_n \in M^{p_n,q_n}_{(\omega_n)}(\mathbb{R}^d), \ n = 1, 2.$$

3.2. Wiener-Lebesgue amalgam spaces

The construction of Wiener-Lebesgue (or just Wiener) amalgam spaces given below is a special case of the general definition of coorbit spaces, cf. [9, 10].

Let $\omega_0 \in \mathscr{P}_E(\mathbb{R}^d)$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$, $p, q, r \in (0, \infty]$, $Q_d = [0,1]^d$ be the unit cube. For measurable f on $\mathbb{R}^d$, set

$$\|f\|_{\mathsf{W}^r(\omega_0, \ell^p)} \equiv \|a_0\|_{\ell^p(\mathbb{Z}^d)} \tag{3.7}$$

when

$$a_0(j) \equiv \|f\, \omega_0\|_{L^r(j+Q_d)}, \qquad j \in \mathbb{Z}^d,$$

and for measurable F on $\mathbb{R}^{2d}$,

$$\|F\|_{\mathsf{W}^r(\omega, \ell^{p,q})} \equiv \|a\|_{\ell^{p,q}(\mathbb{Z}^{2d})} \quad \text{and} \quad \|F\|_{\mathsf{W}^r(\omega, \ell^{p,q}_*)} \equiv \|a\|_{\ell^{p,q}_*(\mathbb{Z}^{2d})} \tag{3.8}$$

when

$$a(j, \iota) \equiv \|F\, \omega\|_{L^r((j,\iota)+Q_{2d})}, \qquad j, \iota \in \mathbb{Z}^d.$$

The Wiener amalgam space

$$\mathsf{W}^r(\omega_0, \ell^p) = \mathsf{W}^r(\omega_0, \ell^p(\mathbb{Z}^d))$$

consists of all measurable $f \in L^r_{\mathrm{loc}}(\mathbb{R}^d)$ such that $\|f\|_{\mathsf{W}^r(\omega_0,\ell^p)}$ is finite. The Wiener amalgam spaces

$$\mathsf{W}^r(\omega, \ell^{p,q}) = \mathsf{W}^r(\omega, \ell^{p,q}(\mathbb{Z}^{2d})) \quad \text{and} \quad \mathsf{W}^r(\omega, \ell^{p,q}_*) = \mathsf{W}^r(\omega, \ell^{p,q}_*(\mathbb{Z}^{2d}))$$

consist of all measurable functions $F \in L^r_{\mathrm{loc}}(\mathbb{R}^{2d})$ such that $\|F\|_{\mathsf{W}^r(\omega,\ell^{p,q})}$ respectively $\|F\|_{\mathsf{W}^r(\omega,\ell^{p,q}_*)}$ are finite. We observe that $\mathsf{W}^r(\omega_0, \ell^p)$ is often denoted by $W(L^r, L^p_{(\omega_0)})$ or $W(L^r, \ell^p_{(\omega_0)})$ in the literature (see e.g., [9, 11, 12, 19]).

The choice of another compact set with nonempty interior instead of the unit cube yields the same space under an equivalent norm, see e.g., [16, Proposition 11.3.2]. For example, in the proof of Theorem 4.1 we will consider $\tilde{Q}_{2d} = [-1,1]^{2d}$ together with Q_{2d}.

The topologies are defined through their respectively quasi-norms in (3.7) and (3.8).

Obviously, $\mathsf{W}^r(\omega_0, \ell^p)$ and $\mathsf{W}^r(\omega, \ell^{p,q})$ increase with p, q, decrease with r, and

$$\mathsf{W}^\infty(\omega, \ell^{p,q}) \hookrightarrow L^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \cap \Sigma_1'(\mathbb{R}^{2d}) \hookrightarrow L^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \hookrightarrow \mathsf{W}^r(\omega, \ell^{p,q})$$

and

$$\|\cdot\|_{\mathsf{W}^r(\omega,\ell^{p,q})} \le \|\cdot\|_{L^{p,q}_{(\omega)}} \le \|\cdot\|_{\mathsf{W}^\infty(\omega,\ell^{p,q})}, \qquad r \le \min(1, p, q).$$

In addition, we have the following inclusions:

$$L^{p,q}_{(\omega)} \hookrightarrow \mathsf{W}^1(\omega, \ell^{p,q}), \quad p, q \ge 1 \quad \text{and} \quad \mathsf{W}^1(\omega, \ell^{p,q}) \subseteq L^{p,q}_{(\omega)} \cap L^1_{(\omega)}, \quad p, q \le 1.$$

For convenience, if $\omega \equiv 1$, we set

$$\mathsf{W}^r(\ell^{p,q}) = \mathsf{W}^r(1, \ell^{p,q}) \quad \text{and} \quad \mathsf{W}^r(\ell^{p,q}_*) = \mathsf{W}^r(1, \ell^{p,q}_*).$$

In an analogous fashion one can introduce norms of the form $\| \cdot \|_{\mathsf{W}^r(\omega, L^{p,q})}$ and $\| \cdot \|_{\mathsf{W}^r(\omega, L^{p,q}_*)}$, and define the Wiener amalgam spaces

$$\mathsf{W}^r(\omega, L^{p,q}) = \mathsf{W}^r(\omega, L^{p,q}(\mathbb{R}^{2d})) \quad \text{and} \quad \mathsf{W}^r(\omega, L^{p,q}_*) = \mathsf{W}^r(\omega, L^{p,q}_*(\mathbb{R}^{2d}))$$

as spaces of all measurable $F \in L^r_{\mathrm{loc}}(\mathbb{R}^{2d})$ such that $\|F\|_{\mathsf{W}^r(\omega, L^{p,q})}$ respectively $\|F\|_{\mathsf{W}^r(\omega, L^{p,q}_*)}$ are finite. We omit the details.

4. *Equivalence of norms in modulation and Wiener amalgam spaces*

As a preparation for the proof of our main result, we shall prove some auxiliary statements.

Lemma 4.1. *Let $p, q \in (0, \infty)$, $Q_{2d} = [0,1]^{2d}$ be the unit cube, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$, and let $F \in \mathsf{W}^1(\omega, \ell^{p,q})$. Then for every $(j, \iota) \in \mathbb{Z}^{2d}$,*

$$\left\| \iint_{(j,\iota)+Q_{2d}} F(\cdot + (z, \zeta))\, dz\, d\zeta \right\|_{L^{p,q}_{(\omega)}} \lesssim \|F\|_{\mathsf{W}^1(\omega, \ell^{p,q})}.$$

PROOF. Put $\alpha(F, j, \iota) \equiv \|F\|_{L^1((j,\iota)+Q)}$ with $Q = Q_{2d}$, $Q_d = [0,1]^d$, $F_\omega = F\omega$ and

$$\Omega = \{ m \in \mathbb{Z}^d \ : \ 0 \le m_n \le 2, \quad n = 1, \dots, d \}.$$

Then for $\xi \in \nu + Q_d$ with $\nu \in \mathbb{Z}^d$ we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left| \iint_{(j,\iota)+Q} F_\omega(x+z, \xi+\zeta)\, dz\, d\zeta \right|^p dx \\
&\quad = \sum_{n \in \mathbb{Z}^d} \int_{n+Q_d} \left| \iint_{(j,\iota)+Q} F_\omega((x+z, \xi+\zeta))\, dz\, d\zeta \right|^p dx \\
&\quad \le \sum_{n \in \mathbb{Z}^d} \int_{n+Q_d} \left| \sum_{k,\kappa \in \Omega} \alpha(F_\omega, j+n+k, \iota+\nu+\kappa) \right|^p dx \\
&\quad \le C^{\max(p,1)} \sum_{n \in \mathbb{Z}^d} \sum_{\kappa \in \Omega} |\alpha(F_\omega, n, \iota+\nu+\kappa)|^p \,,
\end{aligned}$$

where the constant $C > 1$ depends only on the dimension. Hence by the previous estimate,

$$
\begin{aligned}
&\left\| \iint_{(j,\iota)+Q} F_\omega(\cdot + (z,\zeta))\, dz\, d\zeta \right\|^q_{L^{p,q}_{(\omega)}} \\
&\quad = \sum_{\nu \in \mathbb{Z}^d} \int_{\nu + Q_d} \left(\int_{\mathbb{R}^d} \left| \iint_{(j,\iota)+Q} F_\omega((x+z,\xi+\zeta))\, dz\, d\zeta \right|^p dx \right)^{q/p} d\xi \\
&\quad \lesssim \sum_{\nu \in \mathbb{Z}^d} \left(\sum_{n \in \mathbb{Z}^d} |\alpha(F_\omega, n, \nu)|^p \right)^{q/p} = \|F\|^q_{\mathsf{W}^1(\omega,\ell^{p,q})}. \qquad \square
\end{aligned}
$$

In what follows we will consider conditions of the form

$$
\frac{1}{p_2} - \frac{1}{p_1} = \frac{1}{p} \quad \text{and} \quad \frac{1}{q_2} - \frac{1}{q_1} = \frac{1}{q}, \qquad p, q, p_n, q_n \in (0,\infty],\ n = 1,2. \tag{4.1}
$$

Lemma 4.2. *Let p, q, p_n and q_n be as in* (4.1)*, $n = 1,2$. If $F \in \mathsf{W}^1(\omega, \ell^{p,q})$ and $G \in \mathsf{W}^\infty(\omega_1, \ell^{p_1,q_1})$, then $FG \in \mathsf{W}^1(\omega_2, \ell^{p_2,q_2})$ where $\omega_2/\omega_1 \lesssim \omega$.*

PROOF. Let $Q = [0,1]^{2d}$ and $\alpha_r(H,j,\iota) \equiv \|H\|_{L^r((j,\iota)+Q)}$, when $0 < r \le \infty$, $j, \iota \in \mathbb{Z}^d$, and H is a measurable function on $\mathbb{R}^{2d}$. Then

$$
\begin{aligned}
\alpha_1(F\,G\,\omega_2, j, \iota) &= \int_{(j,\iota)+Q} |F(X)G(X)\omega_2(X)|\, dX \\
&\lesssim \|F\,\omega\|_{L^1((j,\iota)+Q)} \|G\,\omega_1\|_{L^\infty((j,\iota)+Q)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|FG\|_{\mathsf{W}^1(\omega_2,\ell^{p_2,q_2})} &= \|\alpha_1(F\,G\,\omega_2, \,\cdot\,)\|_{\ell^{p_2,q_2}} \\
&\lesssim \|\alpha_1(F\,\omega, \cdot)\alpha_\infty(G\,\omega_1, \,\cdot\,)\|_{\ell^{p_2,q_2}} \\
&\le \|\alpha_1(F\,\omega, \,\cdot\,)\|_{\ell^{p,q}} \cdot \|\alpha_\infty(G\,\omega_1, \,\cdot\,)\|_{\ell^{p_1,q_1}} \\
&= \|F\|_{\mathsf{W}^1(\omega,\ell^{p,q})} \cdot \|G\|_{\mathsf{W}^\infty(\omega_1,\ell^{p_1,q_1})}. \qquad \square
\end{aligned}
$$

The following result extends [25, Proposition 2′] by relaxing conditions on window functions. The arguments are different compared to [25]. For related results, when $r = \infty$, see [13] when $p, q \in [1,\infty]$, and [12, 23] when $p, q \in (0,\infty]$, and for $r \in (0,\infty]$, see [26].

Theorem 4.1. *Let $p,q,r,r_0 \in (0,\infty]$ be such that $r_0 \le \min(1,p,q)$, $\omega, v \in \mathscr{P}_E(\mathbb{R}^{2d})$ be such that ω is v-moderate, and let $\phi_1,\phi_2 \in M^{r_0}_{(v)}(\mathbb{R}^d)\setminus 0$. Then*

$$f \in M^{p,q}_{(\omega)}(\mathbb{R}^d) \quad \Leftrightarrow \quad V_{\phi_1} f \in L^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \quad \Leftrightarrow \quad V_{\phi_2} f \in \mathsf{W}^r(\omega, \ell^{p,q}),$$

and

$$\|f\|_{M^{p,q}_{(\omega)}} \asymp \|V_{\phi_1} f\|_{L^{p,q}_{(\omega)}} \asymp \|V_{\phi_2} f\|_{\mathsf{W}^r(\omega,\ell^{p,q})}.$$

For the proof of Theorem 4.1 we need the following special case where the window functions ϕ_1 and ϕ_2 are the standard Gaussian. We omit the proof because the result follows from (3.9) and (3.10) in [25].

Lemma 4.3. *Let $\phi_0(x) = \pi^{-\frac{d}{4}} e^{-\frac{1}{2}|x|^2}$, $x \in \mathbb{R}^d$, $p,q,r \in (0,\infty]$ and $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$. Then*

$$f \in M^{p,q}_{(\omega)}(\mathbb{R}^d) \quad \Leftrightarrow \quad V_{\phi_0} f \in L^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \quad \Leftrightarrow \quad V_{\phi_0} f \in \mathsf{W}^r(\omega, \ell^{p,q}),$$

and

$$\|f\|_{M^{p,q}_{(\omega)}} \asymp \|V_{\phi_0} f\|_{L^{p,q}_{(\omega)}} \asymp \|V_{\phi_0} f\|_{\mathsf{W}^r(\omega,\ell^{p,q})}.$$

Proof of Theorem 4.1. Let $\phi_0(x) = \pi^{-\frac{d}{4}} e^{-\frac{1}{2}|x|^2}$, $x \in \mathbb{R}^d$, be the standard Gaussian, $\phi \in M^{r_0}_{(v)}(\mathbb{R}^d)\setminus 0$ be arbitrary and let $f \in \Sigma_1'(\mathbb{R}^d)$ be fixed. We introduce the following notation

$$
\begin{aligned}
&F = |V_\phi f \cdot \omega|, \quad F_0 = |V_{\phi_0} f \cdot \omega|, \quad \Phi_1 = |V_{\phi_0}\phi \cdot v|, \qquad \Phi_2 = |V_\phi \phi_0 \cdot v|,\\
&Q = Q_{2d} = [0,1]^{2d}, \quad Q(j,\iota) = (j,\iota) + Q,\\
&\tilde Q = \tilde Q_{2d} = [-1,1]^{2d}, \quad \tilde Q(j,\iota) = (j,\iota) + \tilde Q,
\end{aligned}
$$

$$\alpha_{n,s}(j,\iota) = \|\Phi_n\|_{L^s(Q(j,\iota))}, \;\; \beta_s(j,\iota) = \|F\|_{L^s(Q(j,\iota))}, \;\; \tilde\beta_s(j,\iota) = \|F\|_{L^s(\tilde Q(j,\iota))},$$

$$\beta_{0,s}(j,\iota) = \|F_0\|_{L^s(Q(j,\iota))} \;\text{ and }\; \tilde\beta_{0,s}(j,\iota) = \|F_0\|_{L^s(\tilde Q(j,\iota))}, \;\; j,\iota \in \mathbb{Z}^d, \; s \in (0,\infty].$$

Lemma 4.3 gives

$$\|F_0\|_{L^{p,q}} = \|f\|_{M^{p,q}_{(\omega)}} \asymp \|F_0\|_{\mathsf{W}^r(\ell^{p,q})}, \tag{4.2}$$

and since v is even, by using (2.9) we get

$$\|\Phi_n\|_{L^{r_0}} \asymp \|\phi\|_{M^{r_0}_{(v)}} \asymp \|\Phi_n\|_{\mathsf{W}^r(\ell^{r_0})}, \quad n = 1,2. \tag{4.3}$$

We also have

$$
\begin{aligned}
&\|\alpha_{1,s}\|_{\ell^r} = \|\alpha_{2,s}\|_{\ell^r} = \|\Phi_n\|_{\mathsf{W}^s(\ell^r)} \asymp \|\phi\|_{M^r_{(v)}}, \quad n = 1,2,\\
&\|\beta_{0,s}\|_{\ell^{p,q}} = \|F_0\|_{\mathsf{W}^s(\ell^{p,q})},
\end{aligned} \tag{4.4}
$$

and

$$\|\beta_s\|_{\ell^{p,q}} = \|F\|_{\mathsf{W}^s(\ell^{p,q})} \asymp \|\tilde{\beta}_s\|_{\ell^{p,q}}, \quad s \in (0,\infty]$$

by Lemma 4.3, using the facts that v is an even function, and that the Gaussian ϕ_0 is included in the short-time Fourier transforms in Φ_n, $n = 1, 2$.

Let $\phi \in M^{r_0}_{(v)}(\mathbb{R}^d) \setminus 0$. We need to show

$$\|V_\phi f\|_{\mathsf{W}^r(\omega,\ell^{p,q})} \lesssim \|f\|_{M^{p,q}_{(\omega)}} \quad \text{and} \quad \|f\|_{M^{p,q}_{(\omega)}} \lesssim \|V_\phi f\|_{\mathsf{W}^r(\omega,\ell^{p,q})}.$$

We have $\|V_\phi f\|_{\mathsf{W}^r(\omega,\ell^{p,q})} \asymp \|F\|_{\mathsf{W}^r(\ell^{p,q})}$, and by Lemma 4.3 it follows that $\|f\|_{M^{p,q}_{(\omega)}} \asymp \|F_0\|_{\mathsf{W}^r(\ell^{p,q})}$. Hence the result follows if we prove

a) $\|F\|_{\mathsf{W}^r(\ell^{p,q})} \lesssim \|F_0\|_{\mathsf{W}^r(\ell^{p,q})}, \quad F_0 \in W^r(\ell^{p,q})$,

b) $\|F_0\|_{\mathsf{W}^r(\ell^{p,q})} \lesssim \|F\|_{\mathsf{W}^r(\ell^{p,q})}, \quad F \in \mathsf{W}^r(\ell^{p,q})$.

a) Assume that $F_0 \in \mathsf{W}^\infty(\ell^{p,q})$. Let us prove

$$F_0 \in \mathsf{W}^\infty(\ell^{p,q}) \quad \Longrightarrow F \in \mathsf{W}^\infty(\ell^{p,q}).$$

This implies a) since the Wiener amalgam spaces decrease with $r > 0$.

By the pointwise estimate $F \lesssim F_0 * \Phi_2$ we get

$$\begin{aligned}
\beta_\infty(j,\iota) = \|F\|_{L^\infty(Q(j,\iota))} &\lesssim \|F_0 * \Phi_2\|_{L^\infty(Q(j,\iota))} \qquad (4.5)\\
&\lesssim \sup_{(x,\xi)\in Q(j,\iota)} \left(\sum_{k,\kappa\in\mathbb{Z}^d} \alpha_{2,\infty}(k,\kappa) \iint_{Q(k,\kappa)} F_0(x-y,\xi-\eta)\,dy\,d\eta \right)\\
&\lesssim \sum_{k,\kappa\in\mathbb{Z}^d} \alpha_{2,\infty}(k,\kappa) \sup_{(x,\xi)\in Q(j,\iota)} \left(\iint_{Q(k,\kappa)} F_0(x-y,\xi-\eta)\,dy\,d\eta \right)\\
&\lesssim \sum_{k,\kappa\in\mathbb{Z}^d} \alpha_{2,\infty}(k,\kappa) \sup_{(x,\xi)\in Q(j,\iota)} \left(\sup_{(y,\eta)\in Q(k,\kappa)} |F_0(x-y,\xi-\eta)| \right)\\
&\lesssim \sum_{k,\kappa\in\mathbb{Z}^d} \alpha_{2,\infty}(k,\kappa) \sup_{(z,\zeta)\in\tilde{Q}(j-k,\iota-\kappa)} |F_0(z,\zeta)|\\
&\lesssim (\alpha_{2,\infty} * \tilde{\beta}_{0,\infty})(j,\iota).
\end{aligned}$$

Thus (by using (2.17)) we get

$$
\begin{aligned}
\|F\|_{\mathsf{W}^\infty(\ell^{p,q})} = \|\beta_\infty\|_{\ell^{p,q}} &\lesssim \|\alpha_{2,\infty} * \tilde{\beta}_{0,\infty}\|_{\ell^{p,q}} \\
&\lesssim \|\alpha_{2,\infty}\|_{\ell^{\min(1,p,q)}} \|\tilde{\beta}_{0,\infty}\|_{\ell^{p,q}} \\
&= \|\Phi_2\|_{\mathsf{W}^\infty(\ell^{\min(1,p,q)})} \|F_0\|_{\mathsf{W}^\infty(\ell^{p,q})} \\
&\lesssim \|\phi\|_{M^{r_0}_{(v)}} \|F_0\|_{\mathsf{W}^\infty(\ell^{p,q})}, \qquad (4.6)
\end{aligned}
$$

where we used (4.3) in the last estimate. Consequently, $F \in \mathsf{W}^\infty(\ell^{p,q})$, which gives $F \in \mathsf{W}^r(\ell^{p,q})$, $r > 0$, and hence a) follows.

b) Assume that $F \in \mathsf{W}^r(\ell^{p,q})$, $r > 0$. Since Wiener amalgam spaces decrease with $r > 0$, without loss of generality we assume that $r \le \min(1, p, q)$. We have to prove that $F_0 \in \mathsf{W}^r(\ell^{p,q})$ for $r \le \min(1, p, q)$.

By (4.4), $\|F_0\|_{\mathsf{W}^r(\ell^{p,q})} = \|\beta_{0,r}\|_{\ell^{p,q}}$. Hence we start with estimating $\beta_{0,r}(j,\iota)$. The pointwise inequality $F_0 \lesssim F * \Phi_1$ gives

$$
\begin{aligned}
\beta_{0,r}(j,\iota)^r &\lesssim \iint_{Q(j,\iota)} \left(\iint_{\mathbb{R}^{2d}} F(x-y,\xi-\eta)\, \Phi_1(y,\eta)\, dy\, d\eta \right)^r dx\, d\xi \\
&= \iint_{Q(j,\iota)} \left(\sum_{k,\kappa \in \mathbb{Z}^d} \iint_{Q(k,\kappa)} F(x-y,\xi-\eta)\, \Phi_1(y,\eta)\, dy\, d\eta \right)^r dx\, d\xi \\
&\lesssim \iint_{Q(j,\iota)} \left(\sum_{k,\kappa \in \mathbb{Z}^d} \alpha_{1,\infty}(k,\kappa) \iint_{Q(k,\kappa)} F(x-y,\xi-\eta)\, dy\, d\eta \right)^r dx d\xi \\
&\lesssim \sum_{k,\kappa \in \mathbb{Z}^d} \alpha_{1,\infty}(k,\kappa)^r \iint_{Q(j,\iota)} \left(\iint_{Q(k,\kappa)} F(x-y,\xi-\eta) dy d\eta \right)^r dx d\xi.
\end{aligned}
$$

By $F = F^r\, F^{1-r}$ we have

$$
\begin{aligned}
&\iint_{Q(j,\iota)} \left(\iint_{Q(k,\kappa)} F(x-y,\xi-\eta)\, dy\, d\eta \right)^r dx\, d\xi \\
&\quad \le \tilde{\beta}_\infty(j-k,\iota-\kappa)^{r-r^2} \iint_{Q(j,\iota)} \left(\iint_{Q(k,\kappa)} F(x-y,\xi-\eta)^r\, dy\, d\eta \right)^r dx\, d\xi \\
&\quad \le \tilde{\beta}_\infty(j-k,\iota-\kappa)^{r-r^2}\, \tilde{\beta}_r(j-k,\iota-\kappa)^{r^2}.
\end{aligned}
$$

We insert the previous inequality in the estimate of $\beta_{0,r}(j,\iota)^r$ and get

$$\beta_{0,r}(j,\iota)^r \lesssim \sum_{k,\kappa\in\mathbb{Z}^d} \alpha_{1,\infty}(k,\kappa)^r\, \tilde\beta_r(j-k,\iota-\kappa)^{r^2}\, \tilde\beta_\infty(j-k,\iota-\kappa)^{r-r^2}$$
$$= \alpha_{1,\infty}^r * \left(\tilde\beta_r^{r^2}\, \tilde\beta_\infty^{r-r^2}\right)(j,\iota). \tag{4.7}$$

Let $p_2 = p/r, q_2 = q/r$ and $s = 1/r$. Then $s' = 1/(1-r)$. Due to (4.7) we have

$$\begin{aligned}\|F_0\|_{\mathsf{W}^r(\ell^{p,q})} = \|\beta_{0,r}\|_{\ell^{p,q}} = \|\beta_{0,r}^r\|_{\ell^{p_2,q_2}}^{1/r} &\lesssim \left\|\alpha_{1,\infty}^r * \left(\tilde\beta_r^{r^2}\, \tilde\beta_\infty^{r-r^2}\right)\right\|_{\ell^{p_2,q_2}}^{1/r} \\ &\lesssim \|\alpha_{1,\infty}^r\|_{\ell^1}^{1/r} \left\|\tilde\beta_r^{r^2}\, \tilde\beta_\infty^{r-r^2}\right\|_{\ell^{p_2,q_2}}^{1/r} \\ &\lesssim \|\alpha_{1,\infty}\|_{\ell^r} \left(\left\|\tilde\beta_r^{r^2}\right\|_{\ell^{s\,p_2,s\,q_2}} \left\|\tilde\beta_\infty^{r-r^2}\right\|_{\ell^{s'\,p_2,s'\,q_2}}\right)^{1/r}.\end{aligned}$$

The choice of s, p_2 and q_2 provides by means of a straight forward calculation

$$\left\|\tilde\beta_r^{r^2}\right\|_{\ell^{s\,p_2,s\,q_2}} = \|\tilde\beta_r\|_{\ell^{p,q}}^{r^2} \qquad \text{and} \qquad \left\|\tilde\beta_\infty^{r-r^2}\right\|_{\ell^{s'\,p_2,s'\,q_2}} = \|\tilde\beta_\infty\|_{\ell^{p,q}}^{r-r^2}.$$

Totally, this gives

$$\begin{aligned}\|F_0\|_{\mathsf{W}^r(\ell^{p,q})} &\lesssim \|\alpha_{1,\infty}\|_{\ell^r} \|\tilde\beta_r\|_{\ell^{p,q}}^r \|\tilde\beta_\infty\|_{\ell^{p,q}}^{1-r} \\ &= \|\phi\|_{M^r_{(v)}} \|F\|_{\mathsf{W}^r(\ell^{p,q})}^r \|F\|_{\mathsf{W}^\infty(\ell^{p,q})}^{1-r}.\end{aligned} \tag{4.8}$$

Now from part a), i.e. estimate (4.6) with $r = r_0$ and Lemma 4.3, we obtain

$$\begin{aligned}\|F\|_{\mathsf{W}^\infty(\ell^{p,q})} &\lesssim \|\phi\|_{M^r_{(v)}} \|F_0\|_{\mathsf{W}^\infty(\ell^{p,q})} \\ &\lesssim \|\phi\|_{M^r_{(v)}} \|F_0\|_{\mathsf{W}^r(\ell^{p,q})}.\end{aligned}$$

By combining this estimate with (4.8) we get

$$\|F_0\|_{\mathsf{W}^r(\ell^{p,q})} \lesssim \|\phi\|_{M^r_{(v)}}^{2-r} \|F\|_{\mathsf{W}^r(\ell^{p,q})}^r \|F_0\|_{\mathsf{W}^r(\ell^{p,q})}^{1-r}.$$

Solving this inequality for $\|F_0\|_{\mathsf{W}^r(\ell^{p,q})}$ gives

$$\|F_0\|_{\mathsf{W}^r(\ell^{p,q})} \lesssim \|F\|_{\mathsf{W}^r(\ell^{p,q})},$$

which implies b). □

We note that the same result as in Theorem 4.1 holds true with $W^{p,q}_{(\omega)}$, $L^{p,q}_{*,(\omega)}$ and $\mathsf{W}^r(\omega,\ell^{p,q}_*)$ in place of $M^{p,q}_{(\omega)}$, $L^{p,q}_{(\omega)}$ and $\mathsf{W}^r(\omega,\ell^{p,q})$, respectively, at each occurrence.

Remark 4.1. Theorem 4.1 can be used, for example, to extend some known results on the continuity of short-time Fourier transform multipliers, also referred to as Toeplitz or localization operators. For instance, it can be shown that [2, Theorem 3] can be generalized to a wider range of Lebesgue parameters and a larger class of symbols. In particular, the quasi-Banach case can not addressed using the techniques from [2], as convolution is generally not well-defined in this context. In contrast, convolution estimates can be applied within our framework since

$$\mathsf{W}^1(\omega, \ell^{p,q}) \subseteq L^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \cap L^1_{(\omega)}(\mathbb{R}^{2d}), \qquad 0 < p, q \le 1, \;\; \omega \in \mathscr{P}_E(\mathbb{R}^{2d}).$$

These investigations are beyond the scope of the present paper and will be addressed elsewhere, cf. [28] and a forthcoming contribution by the authors.

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