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ABSTRACT NON-SCALAR VOLTERRA DIFFERENCE EQUATIONS OF SEVERAL VARIABLES

MARKO KOSTIĆ

Dedicated to the 100th anniversary of the birth of Academician Bogoljub Stanković

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A b s t r a c t. In this paper, we analyze some classes of the abstract non-scalar Volterra difference equations depending of several variables. We introduce the notion of a discrete multi-dimensional $(k, C, B, (A_i)_{1\leq i\leq n}, (\mathrm{v}_i)_{1\leq i\leq n})$-existence family and the notion of a discrete $(k, C, B, (A_i)_{1\leq i\leq m}, (\mathrm{v}_i)_{1\leq i\leq m}, \mathcal{I})$-existence family, providing also certain results about the existence and uniqueness of almost periodic type solutions to the abstract non-scalar Volterra difference equations of several variables.

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Key Words: Abstract non-scalar Volterra difference equations of several variables, multi-dimensional $(k, C, B, (A_i)_{1\leq i\leq n}, (\mathrm{v}_i)_{1\leq i\leq n})$-solution operator families, almost periodic type solutions.

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1. *Introduction and preliminaries*

Discrete fractional calculus and discrete fractional equations are rapidly growing fields of research of many authors. For more details about these subjects, we refer the reader to the research monographs [1, 2, 3, 7] and the doctoral dissertation of M. T. Holm [8]; cf. also the research monograph [6] by M. I. Gil for the abstract difference equations with integer-order derivatives. Concerning the Volterra difference equations and their applications, we can warmly recommend [4, 5, 10, 16, 17]; cf. also references cited therein.

The asymptotically almost periodic type solutions of the abstract multi–term discrete abstract Cauchy problem

$$Bu(v) = f(v) + \sum_{i=1}^{n} A_i\big(a_i *_0 u\big)(v + v_i), \quad v \in \mathbb{N}_0,$$

as well as the well-posedness and the existence and uniqueness of almost periodic type solutions of the following abstract multi-term Volterra difference equation

$$Bu(v) = A_1 \sum_{l=-\infty}^{v+v_1} a_1(v + v_1 - l)u(l) + \cdots + A_n \sum_{l=-\infty}^{v+v_n} a_n(v + v_n - l)u(l),\ v \in \mathbb{Z},$$

where $B,\ A_1, \ldots,\ A_n$ are closed linear operator on a complex Banach space X and $v_1, \ldots, v_n \in \mathbb{N}_0$, have recently been analyzed in [14]; for more details about almost periodic functions, almost automorphic functions and their applications, we refer the reader to the research monograph [11] and the list of references quoted therein.

The main aim of this research article is to report how a great number of results established in [14] can be extended to the multi-dimensional setting. The proofs of results clarified here are very similar to the proofs of the corresponding results established in [14] and therefore omitted.

1.1. Notation and terminology

In the sequel, we will always assume that $m,\ n \in \mathbb{N}$, $(X, \|\cdot\|)$ is a complex Banach space, $L(X)$ is the Banach space of all bounded linear operators on X and $C \in L(X)$. If A is a closed linear operator on X, then $[D(A)]$ denotes the Banach space $D(A)$ equipped with the graph norm. Set $\mathbb{N}_k := \{1, \ldots, k\}$ for $k \in \mathbb{N}$. If $\mathrm{j} = (j_1, \ldots, j_n) \in \mathbb{N}_0^n$ and $\mathrm{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, then we write $\mathrm{j} \leq \mathrm{k}$ if and only if $j_m \leq k_m$ for all $1 \leq m \leq n$. If the sequences $(a_{\mathrm{k}})_{\mathrm{k} \in \mathbb{N}_0^n}$ and $(b_{\mathrm{k}})_{\mathrm{k} \in \mathbb{N}_0^n}$ are given, then we define $(a *_0 b)(\cdot)$ by

$$(a *_0 b)(\mathrm{k}) := \sum_{j \in \mathbb{N}_0^n;\, j \leq \mathrm{k}} a_{\mathrm{k}-j} b_j,\ \mathrm{k} \in \mathbb{N}_0^n.$$

It can be simply proved that the convolution product $*_0$ is commutative and associative. If the sequences $(a_{\mathrm{k}})_{\mathrm{k}\in\mathbb{N}_0^n}$ and $(b_{\mathrm{k}})_{\mathrm{k}\in\mathbb{Z}^n}$ are given, then we define the Weyl convolution product $(a \circ b)(\cdot)$ by

$$(a \circ b)(\mathrm{v}) := \sum_{l\in\mathbb{Z}^n;l\leq \mathrm{v}} a(\mathrm{v}-l)b(l), \quad v \in \mathbb{Z}^n.$$

Under certain assumptions, the following equalities hold true:

$$\bigl(f *_0 g\bigr) \circ h = g \circ (f \circ h) = f \circ (g \circ h);$$

cf. [9, Theorem 3.12(ii)-(iii)] for more details given in the one-dimensional setting.

2. *Multi-dimensional* $(k, C, B, (A_i)_{1\leq i\leq n}, (\mathrm{v}_i)_{1\leq i\leq n})$*-solution operator families*

We have recently analyzed various classes of discrete (A, k, B)-regularized C-solution operator families for the abstract Volterra non-scalar difference equation

$$B(v)u(v) = f(v) + \sum_{j=0}^{v} A(v-j)u(j), \quad v \in \mathbb{N}_0,$$

where $B(k)$ is a closed linear operator acting in X ($k \in \mathbb{N}_0$) and $A : \mathbb{N}_0 \to L(Y, X)$; here, Y is any Banach space which is continuously embedded into X ([14]). We can similarly analyze the well-posedness of the abstract Volterra non-scalar difference equation

$$B(\mathrm{v})u(\mathrm{v}) = f(\mathrm{v}) + \sum_{j\in\mathbb{N}_0^n;j\leq \mathrm{v}} A(\mathrm{v}-j)u(j), \quad \mathrm{v} \in \mathbb{N}_0^n, \tag{2.1}$$

where $B(k)$ is a closed linear operator acting in X ($k \in \mathbb{N}_0^n$) and $A : \mathbb{N}_0^n \to L(Y, X)$. The notion of a discrete (weak) (A, k, B)-regularized C-resolvent family for (2.1) and the notion of a discrete (A, k, B)-regularized C-uniqueness family for (2.1) can be introduced in the same way as in the one-dimensional setting (cf. [14, Definition 2.1]). After that, we can simply transfer the statements of [14, Proposition 2.2, Proposition 2.3, Theorem 2.4] to the higher-dimensional setting. It is also worth noting that the notion introduced in [13, Definition 2.1] and [15, Definition 2.3] can be reconsidered in the multi-dimensional setting; all details can be left to the interested readers.

The main purpose of this section is to analyze some classes of the discrete $(k, C, B, (A_i)_{1\leq i\leq m}, (\mathrm{v}_i)_{1\leq i\leq m})$-solution operator families in the multi-dimensional setting as well as to provide certain applications of the introduced notion to the abstract non-scalar Volterra difference equations of several variables. The following notion plays an essential role in our study:

Definition 2.1. Suppose that B, $A_1, \ldots, A_m$ are closed linear operators on X, $C \in L(X)$, $\mathrm{v}_1, \ldots, \mathrm{v}_m \in \mathbb{N}_0^n$, $\mathcal{I} \subseteq \mathbb{N}_m$, $k : \mathbb{N}_0^n \to \mathbb{C}$ and $k \neq 0$. Then we say that the operator family $(S(\mathrm{v}))_{\mathrm{v}\in\mathbb{N}_0^n} \subseteq L(X)$ is a discrete:

(i) $(k, C, B, (A_i)_{1\leq i\leq m}, (\mathrm{v}_i)_{1\leq i\leq m})$-existence family if and only if the mapping $x \mapsto A_i\bigl(a_i *_0 S\bigr)(\mathrm{v}+\mathrm{v}_i)x$, $x \in X$ belongs to $L(X)$ for $\mathrm{v} \in \mathbb{N}_0^n$, $1 \leq i \leq m$ and

$$BS(\mathrm{v})x = k(\mathrm{v})Cx + \sum_{i=1}^{m} A_i\bigl(a_i *_0 S\bigr)(\mathrm{v}+\mathrm{v}_i)x, \quad \mathrm{v} \in \mathbb{N}_0^n,\ x \in X.$$

(ii) $(k, C, B, (A_i)_{1\leq i\leq m}, (\mathrm{v}_i)_{1\leq i\leq m}, \mathcal{I})$-existence family if and only if $(S(\mathrm{v}))_{\mathrm{v}\in\mathbb{N}_0^n}$ is $(k, C, B, (A_i)_{1\leq i\leq m}, (\mathrm{v}_i)_{1\leq i\leq m})$-existence family and $S(\mathrm{v})A_i \subseteq A_iS(\mathrm{v})$ for all $v \in \mathbb{N}_0^n$ and $i \in \mathbb{N}_m \setminus \mathcal{I}$.

If $\mathrm{v}_1 = \mathrm{v}_2 = \ldots = \mathrm{v}_m = 0$, then we omit the term "$(\mathrm{v}_i)_{1\leq i\leq m}$" from the notation. The proofs of the following results can be given as in the one-dimensional setting (cf. [14, Proposition 3.2, Theorem 3.3]):

Proposition 2.1. *Suppose that B, $A_1, \ldots, A_m$ are closed linear operators on X, $C \in L(X)$, $\mathrm{v}_1, \ldots, \mathrm{v}_m \in \mathbb{N}_0^n$, $k : \mathbb{N}_0^n \to \mathbb{C}$, $k \neq 0$, $1 \leq i \leq m$, $a_i(0) \neq 0$ and $(S(\mathrm{v}))_{\mathrm{v}\in\mathbb{N}_0^n} \subseteq L(X)$ is a discrete $(k, C, B, (A_i)_{1\leq i\leq m}, (\mathrm{v}_i)_{1\leq i\leq m})$-existence family. If $x \in X$ and $\mathrm{v}_i = 0$, then $S(\mathrm{v})x \in D(A_i)$ for all $\mathrm{v} \in \mathbb{N}_0^n$; the same holds provided that $S(\mathrm{j})x \in D(A_i)$ for all $\mathrm{j} \in \mathbb{N}_0^n \setminus (\mathrm{v}_i + \mathbb{N}_0^n)$.*

Theorem 2.1. *Suppose that B, $A_1, \ldots, A_m$ are closed linear operators on X, $C \in L(X)$ is injective, $k : \mathbb{N}_0^n \to \mathbb{C}$, $k(0) \neq 0$ and $a_i(0) \neq 0$ for $1 \leq i \leq m$.*

(i) *Suppose, further, that $(S(\mathrm{v}))_{\mathrm{v}\in\mathbb{N}_0^n} \subseteq L(X)$ is a discrete $(k, C, B, (A_i)_{1\leq i\leq m})$-existence family such that $S(0)Bx = BS(0)x$ and $S(0)A_ix = A_iS(0)x$ for all $x \in D(B) \cap D(A_1) \cap \ldots \cap D(A_m)$. Then $(B - \sum_{i=0}^{m} a_i(0)A_i)^{-1}C \in L(X)$, $S(0) = k(0)(B - \sum_{i=0}^{m} a_i(0)A_i)^{-1}C$,*

$$S(\mathrm{v})x = \left(B - \sum_{i=0}^{m} a_i(0)A_i\right)^{-1} \times \left[k(\mathrm{v})Cx + \sum_{i=1}^{m} A_i \sum_{j\in A_{\mathrm{v}}} a_i(\mathrm{v}-j)S(j)x\right], \quad \mathrm{v} \in \mathbb{N}_0^n \setminus \{0\},\ x \in X, \tag{2.2}$$

where

$$A_{\mathrm{v}} := \bigl\{j \in \mathbb{N}_0^n : j \leq \mathrm{v},\ j \neq \mathrm{v}\bigr\}, \quad \mathrm{v} \in \mathbb{N}_0^n \setminus \{0\},$$

and $A_iS(\mathrm{v}) \in L(X)$ for all $i \in \mathbb{N}_m$ and $\mathrm{v} \in \mathbb{N}_0^n$.

(ii) *Suppose that* $C \in L(X)$ *is injective,* $(B - \sum_{i=0}^{m} a_i(0)A_i)^{-1}C \in L(X)$ *and, for every* $l \in \mathbb{N}$ *and for every choice of integers* $a_j \in \mathbb{N}_m$ *for* $1 \leq j \leq l$*, we have*

$$\left[\prod_{j=1}^{l}\left(B - \sum_{i=0}^{m} a_i(0)A_i\right)^{-1} A_{a_j}\right] \cdot \left(B - \sum_{i=0}^{m} a_i(0)A_i\right)^{-1} C \in L(X).$$

Define $S(0) := k(0)(B - \sum_{i=0}^{m} a_i(0)A_i)^{-1}C$ *and* $S(\mathrm{v})$, $\mathrm{v} \in \mathbb{N}_0^n \setminus \{0\}$, *recursively by* (2.2). *Then* $(S(\mathrm{v}))_{\mathrm{v}\in\mathbb{N}_0^n} \subseteq L(X)$ *is well-defined,* $A_i S(\mathrm{v}) \in L(X)$ *for all* $i \in \mathbb{N}_m$*,* $\mathrm{v} \in \mathbb{N}_0^n$ *and* $(S(\mathrm{v}))_{\mathrm{v}\in\mathbb{N}_0^n}$ *is a unique discrete* $(k, C, B, (A_i)_{1\leq i\leq m})$*-existence family. Furthermore, if* $\mathcal{I} \subseteq \mathbb{N}_m$ *and*

$$\begin{aligned}
&CB \subseteq BC,\ CA_i \subseteq A_iC \text{ for all } i \in \mathbb{N}_m \setminus \mathcal{I},\\
&\big(\forall i \in \mathbb{N}_m \setminus \mathcal{I}\big)\ \big(\forall x \in D(A_i) \cap D(B)\big)\ Bx \in D(A_i),\\
&A_i x \in D(B) \text{ and } A_iBx = BA_ix,\\
&\big(\forall i \in \mathbb{N}_m \setminus \mathcal{I}\big)\big(\forall j \in \mathbb{N}_n\big)\ \big(\forall x \in D(A_i) \cap D(A_j)\big)\\
&A_jx \in D(A_i),\ A_ix \in D(A_j) \text{ and } A_iA_jx = A_jA_ix,
\end{aligned} \tag{2.3}$$

respectively, there exist a closed linear operator A *and the complex polynomials* $P_B(\,\cdot\,)$, $P_1(\,\cdot\,)$, …, $P_m(\,\cdot\,)$, *such that* $CA \subseteq AC$ *and* $B = P_B(A)$, $A_1 = P_1(A)$, …, $A_m = P_m(A)$, *then* $(S(\mathrm{v}))_{v\in\mathbb{N}_0^n}$ *is a discrete* $(k, C, B, (A_i)_{1\leq i\leq m}, \mathcal{I})$*-existence family, respectively* $(S(\mathrm{v}))_{v\in\mathbb{N}_0^n}$ *is a discrete* $(k, C, B, (A_i)_{1\leq i\leq m}, \emptyset)$*-existence family.*

(iii) *Suppose that* $C = \mathrm{I}$,

$$\left(B - \sum_{j=0}^{m} a_j(0)A_j\right)^{-1} \in L(X), \qquad \sum_{\mathrm{v}\in\mathbb{N}_0^n\setminus\{0\}} |a_i(\mathrm{v})| < +\infty$$

for $1 \leq i \leq m$, $\sum_{\mathrm{v}\in\mathbb{N}_0^n} |k(\mathrm{v})| < +\infty$, *and* (a) *or* (b) *holds, where:*

(a) $A_i \in L(X)$ *for* $1 \leq i \leq m$ *and*

$$1 > \sum_{i=1}^{m} \sum_{\mathrm{v}\in\mathbb{N}_0^n\setminus\{0\}} |a_i(\mathrm{v})| \cdot \left\| \left(B - \sum_{j=0}^{m} a_j(0)A_j\right)^{-1} A_i \right\|.$$

(b) *Suppose that* $C = \mathrm{I}$, (2.3) *holds or there exist a closed linear operator* A *and the complex polynomials* $P_B(\cdot), P_1(\cdot), \ldots,\ P_m(\cdot)$ *such that* $B = P_B(A)$, $A_1 = P_1(A), \ldots,\ A_m = P_m(A)$, *and*

$$1 > \sum_{i=1}^{m} \sum_{\mathrm{v}\in\mathbb{N}_0^n\setminus\{0\}} |a_i(\mathrm{v})| \cdot \left\| A_i \left(B - \sum_{j=0}^{m} a_j(0)A_j\right)^{-1} \right\|.$$

Then the requirements in (ii) *hold and we have*

$$\sum_{\mathrm{v}\in\mathbb{N}_0^n}\|S(\mathrm{v})\|<+\infty \quad \textit{and} \quad \sum_{\mathrm{v}\in\mathbb{N}_0^n}\Big\|A_i\big(a_i*_0 S\big)(\mathrm{v})\Big\|<+\infty \ \ (1\leq i\leq m), \tag{2.4}$$

provided that (a) *holds, resp. we have* (2.4) *and*

$$\sum_{\mathrm{v}\in\mathbb{N}_0^n}\|A_iS(\mathrm{v})\|<+\infty \ \ (1\leq i\leq m),$$

provided that (b) *holds.*

Unfortunately, the statement of [14, Theorem 3.5] cannot be so straightforwardly transferred to the higher-dimensional setting. The extension is straightforward only in the case that there exists a tuple $\mathrm{v}_i =: \mathrm{v}_{max}\in\mathbb{N}_0^n$, for some $i\in\mathbb{N}_m$, such that $\mathrm{v}_{i;j}\geq \mathrm{v}_{l;j}$ for all $l\in\mathbb{N}_m$ and $j\in\mathbb{N}_n$, with the meaning clear; in this case, we define $M\subseteq\mathbb{N}_m$ as a set of all indexes $i\in\mathbb{N}_m$ with the above property. Then the result of of [14, Theorem 3.5] can be simply extended to the higher-dimensional setting with almost the same notation used; for example, in part (i) of this result, we have to assume that the compatibility condition

$$BS(0)x=k(0)Cx+\sum_{i=1}^m A_i\Big[a_i(\mathrm{v}_i)S(0)x+\ldots+a_i(0)S(\mathrm{v}_i)x\Big], \quad x\in X$$

holds, so that the value of $S(\mathrm{v})$ will be uniquely determined for any $\mathrm{v}\in(\mathrm{v}_{max}+\mathbb{N}_0^n)\setminus\{\mathrm{v}_{max}\}$. This always happens if $m=1$; all other details can be left to the interested readers.

If there does not exist a tuple $\mathrm{v}_i\in\mathbb{N}_0^n$ with the above described property, then there is no easy way to generalize [14, Theorem 3.5] to the higher-dimensional setting; the main problem lies in the fact that the partial order relation $\sim\subseteq\mathbb{N}_0^n\times\mathbb{N}_0^n$, defined by

$$\mathrm{v}=\big(v_1,\ldots,v_n\big)\sim \mathrm{v}'=\big(v_1',\ldots,v_n'\big)\Leftrightarrow v_i\leq v_i',\ i\in\mathbb{N}_n,$$

is not a total order if $n\geq 2$.

We continue by providing some useful observations about the abstract multi-term Volterra difference equation:

$$Bu(\mathrm{v})=f(\mathrm{v})+\sum_{i=1}^m A_i\big(a_i*_0 u\big)(\mathrm{v}+\mathrm{v}_i), \quad \mathrm{v}\in\mathbb{N}_0^n, \tag{2.5}$$

where $\mathrm{v}_1,\ldots,\mathrm{v}_m\in\mathbb{N}_0^n$:

Remark 2.1. In [14, Subsection 3.1], we have analyzed the well-posedness, the existence and uniqueness of asymptotically almost periodic type solutions of (2.5). All established results about the well-posedness of problem (2.5) continue to hold in the multi-dimensional setting; concerning the existence and uniqueness of asymptotically almost periodic type solutions of (2.5) and similar problems, we would like to note that we must require some additional conditions on the solution operator family $(S(\mathrm{v}))_{\mathrm{v}\in\mathbb{N}_0^n}$, besides its uniform integrability, in order to see that the sequence $u(\mathrm{v}) := (g *_0 S)(\mathrm{v})$, $\mathrm{v} \in \mathbb{N}_0^n$ is $\mathbb{D}$-asymptotically almost periodic (in the sense that there exist an almost periodic sequence $H : \mathbb{Z}^n \to X$ and a continuous function $Q : \mathbb{N}_0^n \to X$ such that $u = H+Q$ on $\mathbb{N}_0^n$ and $\lim_{|\mathrm{v}|\to+\infty;\mathrm{v}\in\mathbb{D}} \|Q(\mathrm{v})\| = 0$, where $\mathbb{D}$ is a certain non-empty subset of $\mathbb{N}_0^n$), provided that the function $g(\cdot)$ is $\mathbb{D}$-asymptotically almost periodic. In the multi-dimensional setting, the main problem is the $\mathbb{D}$-asymptotical vanishing of the function

$$\mathrm{v} \mapsto \sum_{j\leq \mathrm{v};\neg(0\leq j)} S(\mathrm{v}-j)h(j), \quad \mathrm{v} \in \mathbb{N}_0^n$$

as $|\mathrm{v}| \to +\infty$, where $h(\cdot)$ is the almost periodic part of $g(\cdot)$.

The following results can be proved in the same way as in the corresponding parts of the proofs of [14, Theorem 4.1, Theorem 4.3, Theorem 4.5]:

Theorem 2.2. (i) *Suppose that* $\mathrm{v}_1 \in \mathbb{N}_0^n, \ldots, \mathrm{v}_m \in \mathbb{N}_0^n$, $(S(\mathrm{v}))_{\mathrm{v}\in\mathbb{N}_0^n} \subseteq L(X)$ *is a discrete* $(k, C, B, (A_i)_{1\leq i\leq m}, (\mathrm{v}_i)_{1\leq i\leq m})$*-existence family,* $\sum_{\mathrm{v}\in\mathbb{N}_0^n} \|S(\mathrm{v})\| < +\infty$ *and the following holds:*

(a) $f : \mathbb{Z}^n \to X$ *is a bounded sequence,* $k \in l^1(\mathbb{N}_0^n : Y)$ *and* $\sum_{v\in\mathbb{N}_0^n} |a_i(\mathrm{v})| < +\infty$ *for* $1 \leq i \leq m$*, or*

(b) $f \in l^1(\mathbb{Z}^n : X)$, $k : \mathbb{N}_0^n \to X$ *is a bounded sequence and* $a_i : \mathbb{Z}^n \to \mathbb{C}$ *is a bounded sequence for* $1 \leq i \leq m$*.*

Define

$$u(\mathrm{v}) := \sum_{l\in\mathbb{Z}^n;l\leq \mathrm{v}} S(\mathrm{v}-l)f(l), \quad \mathrm{v} \in \mathbb{Z}^n \tag{2.6}$$

and

$$\begin{aligned} g(\mathrm{v}) := A_1&\Bigg(\sum_{l\leq \mathrm{v}+\mathrm{v}_1} - \sum_{l\leq \mathrm{v}}\Bigg)\big(a_1 *_0 S\big)(\mathrm{v}+\mathrm{v}_1-l)f(l) + \cdots \\ &+ A_m\Bigg(\sum_{l\leq \mathrm{v}+\mathrm{v}_m} - \sum_{l\leq \mathrm{v}}\Bigg)\big(a_m *_0 S\big)(\mathrm{v}+\mathrm{v}_m-l)f(l), \quad v \in \mathbb{Z}^n. \end{aligned} \tag{2.7}$$

Then $u(\cdot)$ *is bounded if* (a) *holds,* $u \in l^1(\mathbb{Z}^n : X)$ *if* (b) *holds, and we have*

$$Bu(\mathrm{v}) = A_1 \sum_{l\in\mathbb{Z}^n; l\leq \mathrm{v}+\mathrm{v}_1} a_1(\mathrm{v}+\mathrm{v}_1-l)u(l)+\cdots + A_m \sum_{l\in\mathbb{Z}^n; l\leq \mathrm{v}+\mathrm{v}_m} a_1(\mathrm{v}+\mathrm{v}_m-l)u(l)+g(\mathrm{v}),\ v\in\mathbb{Z}^n.$$

(ii) *Suppose that* $\mathrm{v}_1 \in \mathbb{N}_0^n,\ \ldots,\ \mathrm{v}_m \in \mathbb{N}_0^n$, $\mathcal{I} \subseteq \mathbb{N}_m$, $(S(\mathrm{v}))_{\mathrm{v}\in\mathbb{N}_0^n} \subseteq L(X)$ *is a discrete* $(k, C, B, (A_i)_{1\leq i\leq m}, (\mathrm{v}_i)_{1\leq i\leq m}, \mathcal{I})$*-existence family,*

$$\sum_{\mathrm{v}\in\mathbb{N}_0^n} \|S(\mathrm{v})\| < +\infty, \quad \sum_{\mathrm{v}\in\mathbb{N}_0^n} \|A_i\big(a_i *_0 S\big)(\mathrm{v}+\mathrm{v}_i)\| < +\infty$$

for $i \in \mathcal{I}$ *and the following holds:*

(a) $f : \mathbb{Z}^n \to X$ *is a bounded sequence,* $k \in l^1(\mathbb{N}_0^n : X)$ *and* $\sum_{\mathrm{v}\in\mathbb{N}_0^n} |a_i(\mathrm{v})| < +\infty$ *for* $i \in \mathcal{I}$ *or*

(b) $f \in l^1(\mathbb{Z}^n : X)$, $k : \mathbb{N}_0^n \to X$ *is a bounded sequence and* $a_i : \mathbb{Z}^n \to \mathbb{C}\setminus\{0\}$ *is a bounded sequence for* $i \in \mathcal{I}$

as well as

(c) $A_i f : \mathbb{Z}^n \to X$ *is a bounded sequence,* $\sum_{\mathrm{v}\in\mathbb{N}_0^n} |a_i(\mathrm{v})| < +\infty$ *for* $i \in \mathbb{N}_m \setminus \mathcal{I}$ *and* $(S(\mathrm{v}))_{v\in\mathbb{N}_0} \subseteq L(X)$ *is a discrete*

$$(k, C, B, (A_i)_{1\leq i\leq m}, (\mathrm{v}_i)_{1\leq i\leq m}, \mathcal{I}) - \textit{existence family, or}$$

(d) $f \in l^1(\mathbb{Z}^n : X)$, $\sum_{\mathrm{v}\in\mathbb{N}_0^n} \|A_i S(\mathrm{v})\| < +\infty$ *for all* $i \in \mathbb{N}_m \setminus \mathcal{I}$ *and* $a_i : \mathbb{N}_0^n \to \mathbb{C}\setminus\{0\}$ *is a bounded sequence for* $i \in \mathbb{N}_m \setminus \mathcal{I}$*, or*

(e) $f \in l^1(\mathbb{Z}^n : [D(A_i)])$ *for all* $i \in \mathbb{N}_m \setminus \mathcal{I}$, $a_i : \mathbb{N}_0^n \to \mathbb{C}\setminus\{0\}$ *is a bounded sequence for* $i \in \mathbb{N}_m \setminus \mathcal{I}$ *and* $(S(\mathrm{v}))_{\mathrm{v}\in\mathbb{N}_0^n} \subseteq L(X)$ *is a discrete* $(k, C, B, (A_i)_{1\leq i\leq m}, (\mathrm{v}_i)_{1\leq i\leq m}, \mathcal{I})$*-existence family.*

Define $u(\cdot)$ *and* $g(\cdot)$ *in the same way as in part* (i)*. Then we have:*

$$Bu(\mathrm{v}) = \sum_{i\in\mathcal{I}} A_i \sum_{l\in\mathbb{Z}^n; l\leq \mathrm{v}+\mathrm{v}_i} a_i(\mathrm{v}+\mathrm{v}_i-l)u(l) + \sum_{i\in\mathbb{N}_m\setminus\mathcal{I}} \sum_{l\in\mathbb{Z}^n; l\leq \mathrm{v}+\mathrm{v}_i} a_i(\mathrm{v}+\mathrm{v}_i-l)A_i u(l) + (k\circ Cf)(\mathrm{v}) + g(\mathrm{v}),\ v\in\mathbb{Z}^n.$$

(iii) *Suppose that* $\omega > 0$, $\mathrm{v}_1 \in \mathbb{N}_0^n,\ \ldots,\ \mathrm{v}_m \in \mathbb{N}_0^n$, $\mathcal{I} \subseteq \mathbb{N}_m$, $(S(\mathrm{v}))_{\mathrm{v}\in\mathbb{N}_0^n} \subseteq L(X)$ *is a discrete* $(k, C, B, (A_i)_{1\leq i\leq m}, (\mathrm{v}_i)_{1\leq i\leq m}, \mathcal{I})$*-existence family,*
$\sum_{\mathrm{v}\in\mathbb{N}_0^n} \|e^{-\omega[v_1+\ldots+v_n]} S(\mathrm{v})\| < +\infty$,
$\sum_{\mathrm{v}\in\mathbb{N}_0^n} \|A_i\big(e^{-\omega[(\cdot_1-\mathrm{v}_{i;1})+\ldots+(\cdot_n-\mathrm{v}_{i;n})]} a_i *_0 [e^{-\omega[\cdot_1+\ldots+\cdot_n]} S]\big)(\mathrm{v}+\mathrm{v}_i)\| < +\infty$ *for* $i \in \mathcal{I}$ *and the following holds:*

(a) $e^{-\omega[\cdot_1+...+\cdot_n]}f : \mathbb{Z}^n \to X$ *is a bounded sequence,* $e^{-\omega[\cdot_1+...+\cdot_n]}k \in l^1(\mathbb{N}_0^n : X)$ *and* $\sum_{\mathrm{v}\in\mathbb{N}_0^n} |e^{-\omega[(v_1-\mathrm{v}_{i;1})+...+(v_n-\mathrm{v}_{i;n})]}a_i(\mathrm{v})| < +\infty$ *for* $i \in \mathcal{I}$ *or*

(b) $e^{-\omega[\cdot_1+...+\cdot_n]}f \in l^1(\mathbb{Z}^n : X)$, $e^{-\omega[\cdot_1+...+\cdot_n]}k : \mathbb{N}_0^n \to X$ *is a bounded sequence and* $e^{-\omega[(\cdot_1-\mathrm{v}_{i;1})+...+(\cdot_n-\mathrm{v}_{i;n})]}a_i : \mathbb{Z}^n \to \mathbb{C} \setminus \{0\}$ *is a bounded sequence for* $i \in \mathcal{I}$,

as well as

(c) $e^{-\omega[\cdot_1+...+\cdot_n]}A_i f : \mathbb{Z}^n \to X$ *is a bounded sequence,* $\sum_{\mathrm{v}\in\mathbb{N}_0^n} |e^{-\omega[(v_1-\mathrm{v}_{i;1})+...+(v_n-\mathrm{v}_{i;n})]}a_i(\mathrm{v})| < +\infty$ *for* $i \in \mathbb{N}_m \setminus \mathcal{I}$ *and* $(S(\mathrm{v}))_{v\in\mathbb{N}_0} \subseteq L(X)$ *is a discrete* $(k, C, B, (A_i)_{1\le i\le m}, (\mathrm{v}_i)_{1\le i\le m}, \mathcal{I})$*-existence family, or*

(d) $e^{-\omega[\cdot_1+...+\cdot_n]}f \in l^1(\mathbb{Z}^n : X)$, $\sum_{\mathrm{v}\in\mathbb{N}_0^n} \|e^{-\omega[v_1+...+v_n]}A_i S(\mathrm{v})\| < +\infty$ *for all* $i \in \mathbb{N}_m \setminus \mathcal{I}$ *and* $e^{-\omega[(\cdot_1-\mathrm{v}_{i;1})+...+(\cdot_n-\mathrm{v}_{i;n})]}a_i : \mathbb{N}_0^n \to \mathbb{C} \setminus \{0\}$ *is a bounded sequence for* $i \in \mathbb{N}_m \setminus \mathcal{I}$, *or*

(e) $e^{-\omega[\cdot_1+...+\cdot_n]}f \in l^1(\mathbb{Z}^n : [D(A_i)])\ \forall i \in \mathbb{N}_m \setminus \mathcal{I}$, $e^{-\omega[(\cdot_1-\mathrm{v}_{i;1})+...+(\cdot_n-\mathrm{v}_{i;n})]}a_i : \mathbb{N}_0^n \to \mathbb{C}\setminus\{0\}$ *is a bounded sequence for* $i \in \mathbb{N}_m \setminus \mathcal{I}$ *and* $(S(\mathrm{v}))_{\mathrm{v}\in\mathbb{N}_0} \subseteq L(X)$ *is a discrete* $(k, C, B, (A_i)_{1\le i\le m}, (\mathrm{v}_i)_{1\le i\le m}, \mathcal{I})$*-existence family.*

Define

$$u(\mathrm{v}) := e^{\omega[v_1+...+v_n]} \times \sum_{l\in\mathbb{Z}^n; l\le\mathrm{v}} \Big[e^{-\omega[(v_1-l_1)+\cdots+(v_n-l_n)]}S(\mathrm{v}-l)\Big]\Big[e^{-\omega[l_1+\cdots+l_n]}f(l)\Big], \tag{2.8}$$

for any $\mathrm{v} \in \mathbb{Z}^n$ *and* $g_\omega(\cdot)$ *in the same way as in part* (i)*, with the operator family* $S(\cdot)$, *the kernels* $a_i(\cdot)$ *and the function* $f(\cdot)$ *replaced therein with the operator family* $e^{-\omega[\cdot_1+...+\cdot_n]}S(\cdot)$, *the kernels* $e^{-\omega[(\cdot_1-\mathrm{v}_{i;1})+...+(\cdot_n-\mathrm{v}_{i;n})]}a_i(\cdot)$ *and the function* $e^{-\omega[\cdot_1+...+\cdot_n]}f(\cdot)$, *respectively* ($1 \le i \le m$)*. Then we have:*

$$\begin{aligned} Bu(\mathrm{v}) &= \sum_{i\in\mathcal{I}} A_i \sum_{l\in\mathbb{Z}^n; l\le\mathrm{v}+\mathrm{v}_i} a_i(\mathrm{v}+\mathrm{v}_i-l)u(l) \\ &+ \sum_{i\in\mathbb{N}_m\setminus\mathcal{I}} \sum_{l\in\mathbb{Z}^n; l\le\mathrm{v}+\mathrm{v}_i} a_i(\mathrm{v}+\mathrm{v}_i-l)A_i u(l) \\ &+ e^{-\omega[v_1+...+v_n]}(k\circ Cf)(\mathrm{v}) + g_\omega(\mathrm{v}), \end{aligned} \tag{2.9}$$

for any $\mathrm{v} \in \mathbb{Z}^n$.

Keeping in mind the representation formulae (2.6) and (2.8), we are in a position to consider the existence and uniqueness of almost periodic and almost automorphic type solutions to the abstract multi-term problems (2.7) and (2.9), respectively. For example, in the concrete siutation of Theorem 2.2(i), the almost periodicity of the inhomogeneity of $f(\cdot)$ implies the almost periodicity of the solution $u(\cdot)$; cf. the forthcoming research monograph [12] for the notion and more details about this problematic.

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Faculty of Technical Sciences
University of Novi Sad
Trg Dositeja Obradovića 6
Novi Sad 21125, Serbia
e-mail: marco.s@verat.net