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IRREGULARITY SOMBOR INDEX

IVAN GUTMAN, VEERABHADRAPPA R. KULLI, IZUDIN REDŽEPOVIĆ

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A b s t r a c t. The irregularity Sombor index ISO is a recently introduced measure for graph irregularity, defined as the sum over all pairs of adjacent vertices u, v of the term $\sqrt{|d_u^2 - d_v^2|}$, where d_u is the degree of the vertex u. Some basic mathematical properties of ISO are established.

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1. *Introduction*

In this paper we are concerned with connected simple graphs. Let G be such a graph. Its vertex and edge sets are $\mathbf{V}(G)$ and $\mathbf{E}(G)$, respectively. The degree d_u of the vertex $u \in \mathbf{V}(G)$ is the number of vertices adjacent to u. The edge connecting the vertices u and v is denoted by uv.

A graph whose all vertices have mutually equal vertices is said to be regular. Graphs that are not regular are said to be irregular. An important and long-time studied problem is to find a criterion for how irregular a graph is [3]. Usually, one speaks of *measures of irregularity* or *irregularity indices*.

If a real number $\mathcal{I} = \mathcal{I}(\mathcal{G})$, associated with the graph G, is a candidate for an irregularity index, then it must be

(a) $\mathcal{I}(\mathcal{G}) = 0$ if and only if G is a regular graph,

(b) $\mathcal{I}(\mathcal{G}) > 0$ if G is irregular.

In the current literature, a large number of irregularity measures have been considered; see the recent papers [1, 4, 5, 6, 8, 9, 16] and the references cited therein. Among them, the oldest and most thoroughly examined is the "Albertson index" [2, 7, 12]

$$Alb = Alb(G) = \sum_{uv \in \mathbf{E}(G)} |d_u - d_v| \,.$$

Short time ago, a vertex-degree-based graph invariant, called "Sombor index"

$$SO = SO(G) = \sum_{uv \in \mathbf{E}(G)} \sqrt{d_u^2 + d_v^2}$$

was put forward [10] which soon attracted much attention [11]. Within the study of the Sombor index, its variant

$$ISO = ISO(G) = \sum_{uv \in \mathbf{E}(G)} \sqrt{|d_u^2 - d_v^2|}$$

was also put forward [15]. Clearly, $ISO(G)$ may be viewed as an irregularity index. We call it *irregularity Sombor index*. Recently, it found applications in computer network theory [13, 14].

In this paper we determine some basic mathematical properties of the irregularity Sombor index.

2. *Relations between ISO and Albertson index*

Theorem 2.1. *If G is a connected graph of order n, then*

$$Alb(G) \le ISO(G) \le \alpha\, Alb(G), \tag{2.1}$$

where $\alpha = \sqrt{2n-3}$ if G is a general (connected) graph, and $\alpha = \sqrt{n}$ if G is a tree. Equality on the left-hand-side holds if and only if G is regular. Equality on the right-hand side holds if and only if G is either regular or is the 3-vertex path.

PROOF. Evidently, equality on both sides of (2.1) holds if G is a regular graph. Assume, therefore, that G is not regular, i.e., that for at least one of its edges, $d_u \neq d_v$. Then

$$\sqrt{|d_u^2 - d_v^2|} = \sqrt{(d_u + d_v)|d_u - d_v|} > \sqrt{|d_u - d_v||d_u - d_v|} = |d_u - d_v|$$

which implies $ISO(G) > Alb(G)$.

In order to verify the right-hand side of inequality (2.1), we need to determine α such that $\sqrt{|d_u^2 - d_v^2|} \le \alpha |d_u - d_v|$ holds for all edges of the underlying graph. Without loss of generality, assume that $d_u > d_v$. Thus, it must be

$$(d_u + d_v)(d_u - d_v) \le \alpha^2 (d_u - d_v)^2 \iff \alpha \ge \sqrt{\frac{d_u + d_v}{d_u - d_v}}$$

Since d_u, d_v are vertex degrees of a graph of order n (which differs from the complete graph K_n), the greatest possible value of $d_u + d_v$ is $(n-1)+(n-2) = 2n-3$, whereas the minimum possible value of $d_u - d_v$ is 1. Therefore, the maximal possible value of α is $\sqrt{2n-3}$.

The only graph whose all edges have property $d_u + d_v = 2n-3$ and $d_u - d_v = 1$ is the 3-vertex path. Therefore, the equality $ISO(G) \leq \sqrt{2n-3}\, Alb(G)$ holds only for the 3-vertex path (plus, of course, for all regular graphs).

For any edge uv of a tree, $d_u + d_v \leq n$. The only tree whose all edges have the property $d_u + d_v = n$ and $d_u - d_v = 1$ is (again) the 3-vertex path. Therefore, for trees, $\alpha = \sqrt{n}$ and $ISO(G) \leq \sqrt{n}\, Alb(G)$ holds only for the 3-vertex path (plus, of course, for the 2-vertex path, which is regular).

Theorem 2.1 suggests that the irregularity indices ISO and Alb should be linearly correlated. That this, indeed, is the case is illustrated in Fig. 1.

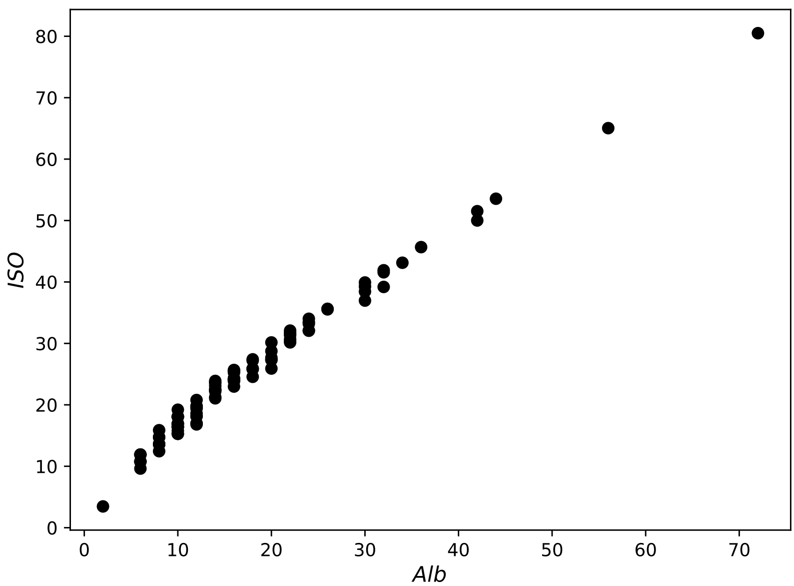

Figure 1: Correlation between ISO and Alb in the case of 10-vertex trees (The correlation coefficient is 0.9925)

Let $M_1(G)$ be the first Zagreb index, defined as

$$M_1(G) = \sum_{u \in \mathbf{V}(G)} d_u^2 = \sum_{uv \in \mathbf{E}(G)} \left(d_u + d_v\right).$$

Theorem 2.2. *If G is a connected graph of order n, then*

$$ISO(G) \leq \sqrt{M_1(G)\,Alb(G)}\,. \tag{2.2}$$

Equality holds if and only if G is either regular or a complete bipartite graph.

PROOF. Recall the Cauchy–Schwarz inequality,

$$\left(\sum_{i=1}^{n} a_i\,b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

in which equality holds if and only if $a_i = \lambda\,b_i$, $i = 1, 2, \ldots, n$, for some real number λ.

Applying this to ISO, we get

$$\begin{aligned} ISO(G)^2 &= \left(\sum_{uv\in\mathbf{E}(G)} \sqrt{|d_u^2 - d_v^2|}\right)^2 = \left(\sum_{uv\in\mathbf{E}(G)} \sqrt{d_u + d_v}\,\sqrt{|d_u - d_v|}\right)^2 \\ &\leq \left(\sum_{uv\in\mathbf{E}(G)} \big(d_u + d_v\big)\right)\left(\sum_{uv\in\mathbf{E}(G)} |d_u - d_v|\right) = M_1(G)\,Alb(G) \end{aligned}$$

and inequality (2.2) follows.

Equality in (2.2) will happen if the end-vertices of all edges uv have the property $d_u = x$, $d_v = y$ for some fixed values of x and y. If $x = y$, then G is a regular graph. If $x \neq y$, then G must be the complete bipartite graph $K_{x,y}$.

It is worth noting that the upper bound for ISO, Eq. (2.2), is reasonably well linearly correlated with ISO. A characteristic example is depicted in Fig. 2.

3. *Trees with extremal ISO-value*

In this section we are concerned with trees. The n-vertex path and star will be denoted by P_n and S_n. The trees with $n = 1, 2, 3$ are unique and those with $n = 1, 2$ are regular. No tree of order $n \geq 3$ is regular. If $n = 3$, then the unique tree is $P_3 \cong S_3$. There are exactly two trees with $n = 4$, which are just P_4 and S_4. Therefore, in what follows we assume that $n \geq 5$.

Theorem 3.1. *Let T_n be a tree of order $n \geq 5$, different from the path P_n and the star S_n. Then*

$$ISO(P_n) < ISO(T_n) < ISO(S_n)\,.$$

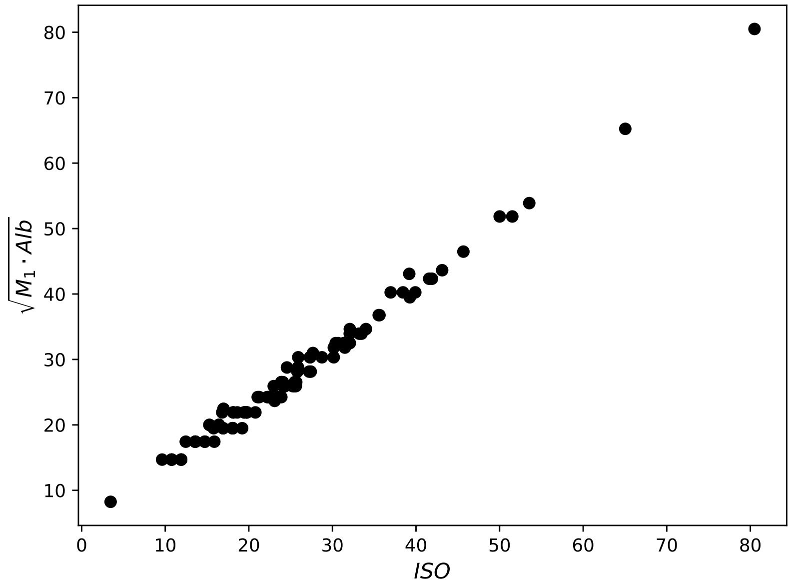

Figure 2: Correlation between ISO and and its upper bound, Eq. (2.2), in the case of 10-vertex trees (The correlation coefficient is 0.9945)

PROOF. An edge connecting a vertex of degree a with a vertex of degree b, will be referred to as an (a,b)-edge. If $a = 1$, then the respective edge is said to be pendent. The contribution of a pendent edge $(1,b)$ to ISO is $\sqrt{b^2-1}$. Evidently, this contribution will be minimal if $b = 2$.

In order that ISO be minimal, we seek for a tree in which
(a) all pendent edges are of $(1,2)$-type,
(b) there are as few as possible such pendent edges,
(c) all non-pendent edges are of (x,x)-type, preferably for $x = 2$,
provided such a tree does exist.

The path P_n, and only this tree, satisfies all the three above conditions. Therefore, among all n-vertex trees, P_n has the minimal ISO-value.

The maximal possible contribution of an (a,b)-edge to ISO is if $a+b=n$ and $|a-b| = n-2$, which happens for $a = 1$, $b = n-1$. All edges of the star S_n, and only of this tree, have this property. Therefore, among all n-vertex trees, S_n has the maximal ISO-value.

Remark 3.1. The tree with second-minimal ISO-value is obtained by attaching two pendent edges to a pendent vertex of P_{n-2}. The tree with second-maximal ISO-value is obtained by attaching a pendent edge to a pendent vertex of S_{n-1}.

The trees with third-minimal ISO-values are obtained by attaching a pendent edge to a vertex (any vertex) of P_{n-1} that is not pendent and not adjacent to a pendent vertex of P_{n-1}. The tree with third-maximal ISO-value is obtained by attaching two pendent edges to a pendent vertex of S_{n-2}.

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Faculty of Science
University of Kragujevac
P. O. Box 60
34000 Kragujevac
Serbia
e-mail: gutman@kg.ac.rs

Department of Mathematics
Gulbarga University
Kalaburgi 585 106
India
e-mail: vrkulli@gmail.com

State University of Novi Pazar
Novi Pazar
Serbia
e-mail: iredzepovic@np.ac.rs