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ON A BIVARIATE KIND OF q-BERNOULLI POLYNOMIALS

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A b s t r a c t. Two bivariate kinds of q-Bernoulli polynomials are introduced and several of their properties are stated and proved.

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1. *Introduction*

The Appell polynomials $A_n(x)$ defined by

$$f(t)\mathrm{e}^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \tag{1.1}$$

where f is a formal power series in t, have found remarkable applications in different branches of mathematics, theoretical physics and chemistry [1, 8, 14, 18]. A special case of Appell polynomials are Bernoulli polynomials $B_n(x)$, generated by $f(t) = t/(\mathrm{e}^t - 1)$ in (1.1). Also, Bernoulli numbers $B_n := B_n(0)$ are of considerable importance in number theory, combinatorics and numerical analysis. They are represented as

$$\frac{t}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \qquad (|t| < 2\pi),$$

or by the recurrence relation

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k = 0 \ \text{ for } n \geq 1 \ \text{ and } \ B_0 = 1.$$

Bernoulli numbers are directly related to several combinatorial numbers such as Stirling, Cauchy and harmonic numbers. For example, except B_1 we have

$$B_n = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m!}{m+1} S_2(n,m), \tag{1.2}$$

where

$$S_2(n,m) = \frac{1}{m!}\sum_{j=0}^{m}(-1)^j\binom{m}{j}(m-j)^n,$$

denote the second kind of Stirling numbers [5, 7] with $S_2(n,m) = 0$ for $n < m$.

They have found various extensions such as poly-Bernoulli numbers, which are somehow connected to multiple zeta values. Al-Salam [2] introduced the first q-extension of Bernoulli numbers and polynomials and gave many of their properties. The q-extension of Bernoulli numbers and polynomials has now found many applications in combinatoric, statistics and various branches of applied mathematics.

Recently in [9], the authors introduced a new kind of bivariate Bernoulli polynomials and studied their main properties. As a valuable application of these extended polynomials, they introduced an extension of the well-known Euler-Maclaurin quadrature formula. In this paper, we introduce a q-extension of the aforesaid bivariate Bernoulli polynomials and establish their properties. Several connection and inversion formulas are stated and proved. In the following section, some preliminaries and definitions are given and in Section 3, a bivariate kind of q-Bernoulli polynomias is introduced and some of its basic properties are stated and proved.

2. *Preliminaries and definitions*

For any complex number a, the basic number and the q-factorial are defined, respectively, by

$$[a]_q = \frac{1-q^a}{1-q}, \quad q \neq 1 \tag{2.1}$$

$$[n]_q! = [n]_q[n-1]_q \cdots [1]_q = \prod_{k=1}^{n}[k]_q, \quad n \in \mathbb{N}, \quad [0]_q! = 1, \tag{2.2}$$

and the q-Pochhammer is defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \ \ n \in \mathbb{N}. \tag{2.3}$$

The limit $\lim_{n\to\infty}(a;q)_n$ is denoted by $(a;q)_\infty$, provided that $|q|<1$. Then,

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \quad n\in\mathbb{N}_0, \quad |q|<1, \tag{2.4}$$

and for any complex number α,

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, \quad |q|<1, \tag{2.5}$$

where the principal value of q^α is taken.

The so-called q-power basis (see e.g. [13]) is defined by

$$(x\ominus y)_q^n = \begin{cases} (x-y)(x-yq)\cdots(x-yq^{n-1}) & n=1,2,\ldots, \\ 1 & n=0. \end{cases}$$

It should be noted that

$$(x\ominus y)_q^n = x^n\left(\frac{y}{x},q\right)_n, \quad x\neq 0.$$

The q-binomial coefficient is defined for positive integers n, k, as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q. \tag{2.6}$$

The basic hypergeometric or q-hypergeometric series ${}_r\phi_s$ is defined as

$${}_r\phi_s\left(\begin{array}{c} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{array}\middle| q;z\right) := \sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(b_1,\cdots,b_s;q)_n}\left((-1)^n q^{\binom{k}{2}}\right)^{1+s-r}\frac{z^n}{(q;q)_n},$$

where $(a_1,\cdots,a_r)_n := (a_1;q)_n\cdots(a_r;q)_n$.

The following so-called q-binomial theorem [12, p. 16] can be written as

$${}_1\phi_0\left(\begin{array}{c} a \\ - \end{array}\middle| q;z\right) = \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad 0<|q|<1, \ \ |z|<1. \tag{2.7}$$

The q-derivative operator is defined by [10, 12, 13]

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}, \quad x\neq 0,$$

satisfying the important product rule

$$D_q(f(x)g(x)) = f(x)D_q g(x) + g(qx)D_q f(x). \tag{2.8}$$

In this sense, note that when we deal with functions $f(x_1, x_2, \ldots, x_n)$ of more than one variable, we denote $D_q f$ by $D_{q,x_i} f$ to make clear that the derivative is taken with respect to the variable x_i.

The q-integral operator is defined by [10, 12]

$$\int_0^z f(z)d_qt = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k).$$

This definition can be established based on a simple geometric series.

The usual exponential function may have two different natural q-extensions, denoted by $e_q(z)$ and $E_q(z)$, which are defined, respectively, by

$$e_q(z) := {}_1\phi_0\left(\begin{array}{c} 0 \\ - \end{array}\middle|\, q; (1-q)z\right) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!}, \quad 0 < |q| < 1, \ \ |z| < 1, \tag{2.9}$$

and

$$E_q(z) := {}_0\phi_0\left(\begin{array}{c} - \\ - \end{array}\middle|\, q, -(1-q)z\right) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_q!} z^n, \quad 0 < |q| < 1. \tag{2.10}$$

It is worth noting that $e_q(z)$ and $E_q(z)$ are linked by the well known relation

$$e_q(z)E_q(-z) = 1. \tag{2.11}$$

In [16], Schork has studied Ward's "Calculus of Sequences" and introduced a q-addition $x \oplus_q y$ by

$$(x \oplus_q y)^n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k y^{n-k},$$

and although this q-addition was already known to Jackson, it was generalized later on by Ward and Al-Salam. For more informations about different q-additions, see e.g., [6]. Similarly the q-subtraction can be defined in the same way by [11]

$$(x \ominus_q y)^n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k (-y^{n-k}) = (x \oplus_q (-y))^n.$$

By noting (2.9), the following relation holds [6, 11]

$$(\forall x, y \in \mathbb{R}) \qquad e_q(x)e_q(y) = e_q(x \oplus_q y). \tag{2.12}$$

2.1. q-Appell sets, q-Bernoulli polynomials and some related properties

Let $\{P_n(x)\}_{n=0}^{\infty}$ be a polynomial set, where the polynomial $P_n(x)$ is of exact degree n. $\{P_n(x)\}_{n=0}^{\infty}$ is a q-Appell set if

$$D_q P_{n+1}(x) = [n+1]_q P_n(x).$$

Such sets were first introduced by Sharma and Chak [17] and they called them q-harmonic.

The following characterization theorem holds in this regard.

Theorem 2.1 (see [17]). *Let $\{P_n(x)\}$ ba a polynomial set. The following assertions are equivalent:*

1. $\{P_n(x)\}$ *is a q-Appell polynomial set.*
2. *There exists a sequence $(a_k)_{k\geq 0}$ independent of n; $a_0 = 1$, such that*

$$P_n(x) = \sum_{k=0}^{n} a_k \frac{[n]_q!}{[n-k]_q!} x^{n-k}.$$

3. $\{P_n(x)\}$ *is generated by*

$$A(t) e_q(xt) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{[n]_q!},$$

where

$$A(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{[k]_q!}, \quad a_0 = 1.$$

The q-Bernoulli polynomials are essentially defined by the generating function [2]

$$\frac{t e_q(xt)}{e_q(t)-1} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{[n]_q},$$

in which

$$B_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q} x^{n-k},$$

where the $B_{k,q} = B_{k,q}(1)$ stands for the k-th q-Bernoulli numbers (see also a determinant approach to q-Bessel polynomials in [15]). It is not difficult to see that since

$$D_q B_{n,q}(x) = [n]_q B_{n-1,q}(x),$$

q-Bernoulli polynomials belong to q-Appell set.

3. *A bivariate kind of q-Bernoulli polynomials*

Let $x, y \in \mathbb{R}$. It is well-known that the Taylor expansion of the two functions $e^{xt}\cos yt$ and $e^{xt}\sin yt$ are as follows [9]

$$e^{xt}\cos yt = \sum_{k=0}^{\infty} C_k(x,y)\frac{t^k}{k!}, \tag{3.1}$$

and

$$e^{xt}\sin yt = \sum_{k=0}^{\infty} S_k(x,y)\frac{t^k}{k!}, \tag{3.2}$$

where

$$C_k(x,y) = \sum_{j=0}^{[\frac{k}{2}]}(-1)^j\binom{k}{2j}x^{k-2j}y^{2j}, \tag{3.3}$$

and

$$S_k(x,y) = \sum_{j=0}^{[\frac{k-1}{2}]}(-1)^j\binom{k}{2j+1}x^{k-2j-1}y^{2j+1}. \tag{3.4}$$

Here we introduce a q-extension of the two above polynomials $C_k(x,y)$ and $S_k(x,y)$ as follows:

Theorem 3.1. *Let $x, y \in \mathbb{R}$. Then the generating functions*

$$e_q(xt)\cos_q yt = \sum_{k=0}^{\infty} C_{k,q}(x,y)\frac{t^k}{[k]_q!} \tag{3.5}$$

and

$$e_q(xt)\sin_q yt = \sum_{k=0}^{\infty} S_{k,q}(x,y)\frac{t^k}{[k]_q!} \tag{3.6}$$

hold, such that

$$C_{k,q}(x,y) = \sum_{j=0}^{[\frac{k}{2}]}(-1)^j\begin{bmatrix} k \\ 2j \end{bmatrix}_q x^{k-2j}y^{2j} \tag{3.7}$$

and

$$S_{k,q}(x,y) = \sum_{j=0}^{[\frac{k-1}{2}]}(-1)^j\begin{bmatrix} k \\ 2j+1 \end{bmatrix}_q x^{k-2j-1}y^{2j+1}. \tag{3.8}$$

PROOF. Since

$$\cos_q(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{[2n]_q!} = \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2[n]_q!}(iz)^n,$$

we have

$$\begin{aligned} e_q(xt)\cos_q(yt) &= \left(\sum_{n=0}^{\infty} \frac{(xt)^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty} \frac{1+(-1)^n}{2[n]_q!}(iyt)^n\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q \frac{1+(-1)^k}{2}(iy)^k x^{n-k}\right)\frac{t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{[\frac{k}{2}]}(-1)^j \left[\begin{array}{c} n \\ k \end{array}\right]_q x^{k-2j}y^{2j}\right)\frac{t^n}{[n]_q!}, \end{aligned}$$

which proves (3.5). The proof of (3.6) is similar. The following series manipulation holds true

$$\left(\sum_{k=0}^{\infty} a_k \frac{t^k}{[k]_q!}\right)\left(\sum_{k=0}^{\infty} b_k \frac{t^k}{[k]_q!}\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\left[\begin{array}{c} k \\ j \end{array}\right]_q a_j b_{k-j}\right)\frac{t^k}{[k]_q!}. \tag{3.9}$$

Proposition 3.1. *The following derivative rules are valid*

$$D_{q,x}C_{k,q}(x,y) = [k]_q C_{k-1,q}(x,y), \tag{3.10}$$

$$D_{q,y}C_{k,q}(x,y) = -[k]_q S_{k-1,q}(x,y), \tag{3.11}$$

$$D_{q,x}S_{k,q}(x,y) = [k]_q S_{k-1,q}(x,y), \tag{3.12}$$

$$D_{q,y}S_{k,q}(x,y) = [k]_q C_{k-1,q}(x,y). \tag{3.13}$$

PROOF. Relation (3.5) yields

$$\begin{aligned} \sum_{n=1}^{\infty} D_{q,x}C_{n,q}(x,y)\frac{t^n}{[n]_q!} &= te_q(xt)\cos_q yt = \sum_{n=0}^{\infty} C_{n,q}(x,y)\frac{t^{n+1}}{[n]_q!} \\ &= \sum_{n=1}^{\infty} C_{n-1,q}(x,y)\frac{t^n}{[n-1]_q!} \\ &= \sum_{n=0}^{\infty}[n]_q C_{n-1,q}(x,y)\frac{t^n}{[n]_q!}, \end{aligned}$$

proving (3.10). The other equalities (3.11), (3.12) and (3.13) can be similarly proved.

Proposition 3.2. *The following identities hold*

$$\begin{aligned} C_{k,q}(x,y) &= \sum_{k=0}^{n} \begin{bmatrix} k \\ j \end{bmatrix}_q C_{j,q}(x,0)x^k, \\ S_{k,q}(x,y) &= \sum_{k=0}^{n} \begin{bmatrix} k \\ j \end{bmatrix}_q S_{j,q}(x,0)x^k, \end{aligned}$$

and are straightforward to prove.

Proposition 3.3. *The following power representations hold*

$$y^{2n} = \sum_{k=0}^{2n} (-1)^{n-k} q^{\binom{k}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q C_{2n-k,q}(x,y)x^k, \tag{3.14}$$

and

$$y^{2n+1} = \sum_{k=0}^{2n+1} (-1)^{n-k} q^{\binom{k}{2}} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q S_{2n+1-k,q}(x,y)x^k. \tag{3.15}$$

PROOF. Multiplying both sides of (3.5) by $E_q(-xt)$ and using (2.11), it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} (-1)^n y^{2n} \frac{t^{2n}}{[n]_q!} &= \left(\sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(-x)^n t^n}{[n]_q!} \right) \left(\sum_{n=0}^{\infty} C_{n,q}(x,y) \frac{t^n}{[n]_q!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q C_{n-k,q}(x,y)x^k \right) \frac{t^n}{[n]_q!}, \end{aligned}$$

which proves (3.14). The proof of (3.15) is similar.

We can now introduce two kinds of bivariate q-Bernoulli polynomials as

$$\frac{te_q(xt)}{e_q(t)-1} \cos_q(yt) = \sum_{n=0}^{\infty} B_{n,q}^{(c)}(x,y) \frac{t^n}{[n]_q!}, \tag{3.16}$$

and

$$\frac{te_q(xt)}{e_q(t)-1} \sin_q(yt) = \sum_{n=0}^{\infty} B_{n,q}^{(s)}(x,y) \frac{t^n}{[n]_q!}, \tag{3.17}$$

and give some basic properties of them in the sequel.

Proposition 3.4. *$B^{(c)}_{n,q}(x,y)$ and $B^{(s)}_{n,q}(x,y)$ can be represented in terms of q-Bernoulli numbers as follows*

$$B^{(c)}_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q B_k C_{n-k,q}(x,y), \tag{3.18}$$

and

$$B^{(s)}_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q B_k S_{n-k,q}(x,y), \tag{3.19}$$

PROOF. Using the relation (3.9), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B^{(c)}_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \frac{t}{e_q(t)-1}e_q(xt)\cos_q(yt)\\ &= \left(\sum_{k=0}^{\infty} B_k\frac{t^k}{[k]_q!}\right)\left(\sum_{k=0}^{\infty} C_{k,q}(x,y)\frac{t^k}{[k]_q!}\right)\\ &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q B_{j,q}C_{k-j,q}(x,y)\right)\frac{t^k}{[k]_q!},\end{aligned}$$

which proves (3.18). The proof of (3.19) is similar.

Similiarly, we can prove that

$$B^{(c)}_{n,q}(x,y)=\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}_q B_{n-2k}(x)y^{2k}, \tag{3.20}$$

and

$$B^{(s)}_{n,q}(x,y)=\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\begin{bmatrix} n \\ 2k+1 \end{bmatrix}_q B_{n-2k-1,q}(x)y^{2k+1}. \tag{3.21}$$

Equality (3.20) follows since

$$\sum_{n=0}^{\infty} B^{(c)}_{n,q}(x,y)\frac{t^n}{[n]_q}=\frac{te_q(xt)}{e_q(t)-1}\cos_q(yt),$$

i.e.,

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,q}^{(c)}(x,y)\frac{t^n}{[n]_q} &= \left(\sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty}\frac{1+(-1)^n}{2[n]_q!}(iyt)^n\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\frac{1+(-1)^k}{2}(iy)^k B_{n-k,q}(x)\right)\frac{t^n}{[n]_q!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\begin{bmatrix} n\\ k\end{bmatrix}_q B_{n-2k}(x)y^{2k}\right)\frac{t^n}{[n]_q!}.\end{aligned}$$

The proof of (3.21) is similar.

Proposition 3.5. *The polynomials $C_{n,q}(x,y)$ and $S_{n,q}(x,y)$ can be represented in terms of the polynomials $B_{n,q}^{(c)}(x,y)$ and $B_{n,q}^{(s)}(x,y)$ as follows*

$$C_{n,q}(x,y)=\sum_{k=0}^{n}\frac{1}{[k+1]_q}\begin{bmatrix} n\\ k\end{bmatrix}_q B_{n-k}^{(c)}(x,y), \tag{3.22}$$

and

$$S_{n,q}(x,y)=\sum_{k=0}^{n}\frac{1}{[k+1]_q}\begin{bmatrix} n\\ k\end{bmatrix}_q B_{n-k,q}^{(s)}(x,y). \tag{3.23}$$

PROOF. From (3.16), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,q}^{(c)}(x,y)\frac{t^n}{[n]_q!} &= \frac{t}{e_q(t)-1}e_q(xt)\cos_q(yt)\\ &= \frac{t}{e_q(t)-1}\sum_{n=0}^{\infty} C_{n,q}(x,y)\frac{t^n}{[n]_q!}.\end{aligned}$$

Hence, it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} C_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \frac{e_q(t)-1}{t}\sum_{n=0}^{\infty} B_{n,q}^{(c)}(x,y)\frac{t^n}{[n]_q!}\\ &= \left(\sum_{n=0}^{\infty}\frac{1}{[n+1]_q}\frac{t^n}{[n]_q!}\right)\left(\sum_{n=0}^{\infty} B_{n,q}^{(c)}(x,y)\frac{t^n}{[n]_q!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{[k+1]_q}\begin{bmatrix} n\\ k\end{bmatrix}_q B_{n-k}^{(c)}(x,y)\right)\frac{t^n}{[n]_q!}\end{aligned}$$

and (3.22) follows. The proof of (3.23) is similar.

Proposition 3.6. *For every $n \in \mathbb{N}$, the following identities hold*

$$B^{(c)}_{n,q}((1 \oplus_q x), y) - B^{(c)}_{n,q}(x, y) = [n]_q C_{n-1,q}(x, y), \tag{3.24}$$

$$B^{(s)}_{n,q}((1 \oplus_q x), y) - B^{(s)}_{n,q}(x, y) = [n]_q S_{n-1,q}(x, y). \tag{3.25}$$

PROOF. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} B^{(c)}_{n,q}((1 \oplus_q x), y)\frac{t^n}{[n]_q!} &= \frac{te_q((1 \oplus_q x)t}{e_q(t)-1}\cos_q(yt) \\
&= \frac{te_q(xt)[e_q(t)-1+1]}{e_q(t)-1}\cos_q(yt) \\
&= te_q(xt)\cos_q(yt) + \frac{te_q(xt)}{e_q(t)-1}\cos_q(yt) \\
&= \sum_{n=0}^{\infty} C_{n,q}(x, y)\frac{t^{n+1}}{[n]_q!} + \sum_{n=0}^{\infty} B^{(c)}_{n,q}(x, y)\frac{t^n}{[n]_q!},
\end{aligned}$$

which proves (3.24). Eq. (3.25) is proved similarly.

Corollary 3.1. *The following relations hold*

$$B^{(c)}_{2n+1,q}(1, y) - B^{(c)}_{2n+1}(0, y) = [2n+1]_q(-1)^n y^{2n},$$

$$B^{(s)}_{2n,q}(1, y) - B^{(s)}_{2n}(0, y) = [2n]_q(-1)^{n+1} y^{2n-1}.$$

PROOF. If we replace n by $2n+1$ in (3.24), and x by 0, we obtain

$$B^{(c)}_{2n+1,q}(1, y) - B^{(c)}_{2n+1,q}(0, y) = [n]_q C_{2n,q}(0, y).$$

The first relation is proved since from (3.7) we have $C_{2n,q}(0, y) = (-1)^n y^{2n}$. The second relation is proved similarly.

Proposition 3.7. *For every $n \in \mathbb{N}$, the following identities hold*

$$B^{(c)}_{n,q}((x \oplus_q z), y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B^{(c)}_{k,q}(x, y) z^{n-k}, \tag{3.26}$$

and

$$B^{(s)}_{n,q}((x \oplus_q z), y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B^{(s)}_{k,q}(x, y) z^{n-k}. \tag{3.27}$$

PROOF. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,q}^{(c)}((x \oplus_q z), y)\frac{t^n}{[n]_q!} &= \frac{te_q((x\oplus_q z)t)}{e_q(t)-1}\cos_q(yt)\\
&= \frac{te_q(xt)}{e_q(t)-1}\cos_q(t)\times e_q(zt)\\
&= \sum_{n=0}^{\infty} B_{n,q}^{(c)}(x,y)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{(tz)^n}{[n]_q!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_q B_{k,q}^{(c)}(x,y)z^{n-k}\right)\frac{t^n}{[n]_q!},
\end{aligned}$$

which proves (3.26). The proof of (3.27) is similar.

Proposition 3.8. *The following equations can be concluded*

$$\sum_{k=0}^{n}\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q B_{k,q}^{(c)}(x,y) = [n+1]_q C_{n,q}(x,y), \tag{3.28}$$

$$\sum_{k=0}^{n}\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q B_{k,q}^{(s)}(x,y) = [n+1]_q S_{n,q}(x,y). \tag{3.29}$$

PROOF. From (3.26), we have

$$B_{n+1,q}^{(c)}((x\oplus_q 1),y) - B_{n+1}^{(c)}(x,y) = \sum_{k=0}^{n}\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q B_{k,q}^{(c)}(x,y).$$

Hence, by using (3.24), relation (3.28) is derived. The proof of (3.29) is concluded in a similar way.

Corollary 3.2. *Relations* (3.28) *and* (3.29) *imply that*

$$\sum_{k=0}^{n}\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q B_{n,k}^{(c)}(0,y) = \begin{cases} (-1)^m[2m+1]_q y^{2m} & \textit{if} \quad n=2m \quad \textit{is odd},\\ 0 & \textit{if} \quad n=2m+1 \quad \textit{is even}, \end{cases}$$

and

$$\sum_{k=0}^{n}\left[\begin{array}{c} n+1 \\ k \end{array}\right]_q B_{n,k}^{(s)}(0,y) = \begin{cases} 0 & \textit{if} \quad n=2m \quad \textit{is odd},\\ (-1)^m[2m+2]_q y^{2m+1} & \textit{if} \quad n=2m+1 \quad \textit{is even}. \end{cases}$$

Corollary 3.3. *For every $n \in \mathbb{N}$, the following partial q-differential equations hold*

$$\begin{aligned} D_{q,x} B^{(c)}_{n,q}(x,y) &= [n]_q B^{(c)}_{n-1,q}(x,y), \\ D_{q,y} B^{(c)}_{n,q}(x,y) &= -[n]_q B^{(c)}_{n-1,q}(x,y), \\ D_{q,x} B^{(s)}_{n,q}(x,y) &= [n]_q B^{(s)}_{n-1,q}(x,y), \end{aligned}$$

and

$$D_{q,y} B^{(c)}_{n,q}(x,y) = [n]_q B^{(s)}_{n-1,q}(x,y).$$

Corollary 3.4. *The following equalities are valid*

$$\begin{aligned} \int_0^1 B^{(c)}_{2n,q}(x,y) d_q x &= (-1)^n q^{2n}, \\ \int_0^1 B^{(s)}_{2n+1,q}(x,y) d_q x &= (-1)^n q^{2n+1}, \end{aligned}$$

which are proved by combining Proposition 3.3 *and Corollary* 3.1 *using the definition of the q-integral.*

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