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METRICAL BESICOVITCH ALMOST AUTOMORPHY AND APPLICATIONS

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A b s t r a c t. In this paper, we reconsider and slightly generalize various classes of Besicovitch almost automorphic functions from ([*Selected Topics in Almost Periodicity, W. de Gruyter, Berlin,* 2022] *and* [*J. Nonl. Evol. Equ. Appl.* **3** (2023), 35-52]). *More precisely, we consider here various classes of metrically Besicovitch almost automorphic functions of the form* $F : \mathbb{R}^n \times X \to Y$ *and metrically Besicovitch almost automorphic sequences of the form* $F : \mathbb{Z}^n \times X \to Y$*, where* X *and* Y *are complex Banach spaces. The main structural characterizations for the introduced classes of metrically Besicovitch almost automorphic functions and sequences are established. In addition to the above, we provide some applications of our results to the abstract Volterra integro-differential equations.*

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1. *Introduction and preliminaries*

The notion of almost automorphy, which generalizes the well-known notion of almost periodicity, was introduced by S. Bochner in 1955 ([4]). Suppose that

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$F : \mathbb{R}^n \to X$ is a continuous function, where $(X, \|\cdot\|)$ is a complex Banach space. Then $F(\cdot)$ is said to be almost automorphic if and only if for every sequence $(\mathbf{b}_k)$ in $\mathbb{R}^n$ there exist a subsequence $(\mathbf{a}_k)$ of $(\mathbf{b}_k)$ and a mapping $G : \mathbb{R}^n \to X$ such that

$$\lim_{k\to\infty} F\big(\mathbf{t} + \mathbf{a}_k\big) = G(\mathbf{t}) \text{ and } \lim_{k\to\infty} G\big(\mathbf{t} - a_k\big) = F(\mathbf{t}), \tag{1.1}$$

pointwisely for $\mathbf{t} \in \mathbb{R}^n$. In this case, the range of $F(\cdot)$ is relatively compact in X and the limit function $G(\cdot)$ is bounded on $\mathbb{R}^n$ but not necessarily continuous on $\mathbb{R}^n$. Furthermore, if the convergence of limits appearing in (1.1) is uniform on compact subsets of $\mathbb{R}^n$, then $F(\cdot)$ is said to be compactly almost automorphic. For more details about almost periodic functions, almost automorphic functions and their applications, the reader may consult the monographs [3, 6, 8, 9, 11, 20, 21].

On the other hand, various classes of (multi-dimensional) Bohr ρ-almost periodic sequences and their Weyl, Besicovitch and Doss generalizations have recently been considered in [17]. In [19], we have recently analyzed the multi-dimensional almost automorphic sequences of the form $F : \mathbb{Z}^n \times X \to Y$, where $(Y, \|\cdot\|_Y)$ is a complex Banach space, and provided several applications to the abstract Volterra difference equations depending on several variables.

The main aim of this research study is to introduce and analyze several new classes of metrically Besicovitch almost automorphic functions of the form $F : \mathbb{R}^n \times X \to Y$ and metrically Besicovitch almost automorphic sequences of the form $F : \mathbb{Z}^n \times X \to Y$, where X and Y are complex Banach spaces. We provide certain applications to the abstract Volterra integro-differential equations and the abstract Volterra difference equations, continuing thus the research studies raised in [1, 13, 15, 16, 17, 18, 19].

The organization of this paper can be briefly described as follows. We first explain the notion and terminology used throughout the paper as well as the most important function spaces we are dealing with. In Section 2., we introduce various classes of metrically Besicovitch almost automorphic functions and metrically Besicovitch almost automorphic sequences. In Section 3., we primarily consider the convolution invariance of metrical Besicovitch almost type automorphy and explain how we can provide certain applications to the abstract Volterra integro-differential equations and the abstract Volterra difference equations. The final section of paper is reserved for conclusions and final remarks about the notion of metrical Besicovitch almost automorphy.

Notation and terminology. We will always assume henceforth that $(X, \|\cdot\|)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ are complex Banach spaces, $\mathcal{B}$ is a non-empty collection of non-empty subsets of X and R is a non-empty collection of sequences in $\mathbb{R}^n$ $[\mathbb{Z}^n]$. Furthermore, we will always assume henceforth that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. By $L(X, Y)$ we denote the Banach space of all bounded linear operators from X into Y; $L(X,X) \equiv L(X)$. Define $\mathbb{N}_0 := \{0,1,...,m,...\}$; if A and B are non-empty sets, then we set $\mathrm{B}^{\mathrm{A}} := \{f | f : \mathrm{A} \to \mathrm{B}\}$. If $A \subseteq \mathbb{R}^n$, then its convex hull is denoted by $CH(A)$.

Suppose now that $0 < p < 1$ and Ω is any Lebesgue measurable subset of $\mathbb{R}^n$ with positive Lebesgue measure. Then $L^p(\Omega : X)$ consists of all Lebesgue measurable functions $f : \Omega \to X$ such that $\int_\Omega \|f(\mathbf{u})\|^p \, d\mathbf{u} < +\infty$; the metric on $L^p(\Omega : X)$ is given by $d(f,g) := \int_\Omega \|f(\mathbf{u}) - g(\mathbf{u})\|^p \, d\mathbf{u}$ for all $f,\ g \in L^p(\Omega : X)$. Let us recall that $(L^p(\Omega : X), d)$ is a complete quasi-normed metric space; the notion and properties of the metric space $(L^p(\Omega : X), d)$ are well-known if $p \geq 1$. Concerning the Lebesgue spaces with variable exponents $L^{p(x)}$, we will use the same notion and notation as in the monograph [11] and the research article [16]; for further information in this direction, we refer the reader to the important research monograph [7] by L. Diening et al. If a set $I \subseteq \mathbb{R}^n$ is Lebesgue measurable and $\nu : I \to (0,\infty)$ is a Lebesgue measurable function, then we consider the Banach space

$$L_\nu^{p(\mathbf{t})}(I : Y) := \big\{ u : I \to Y \ ; \ u(\cdot) \text{ is measurable and } ||u||_{p(\mathbf{t})} < \infty \big\},$$

where $p \in \mathcal{P}(I)$, the collection of all measurable functions from I into $[1, +\infty]$, and

$$\|u\|_{p(\mathbf{t})} := \big\| u(\mathbf{t})\nu(\mathbf{t}) \big\|_{L^{p(\mathbf{t})}(I:Y)}.$$

We similarly define the space $L_\nu^p(I : Y)$ with $p > 0$.

2. *Metrically Besicovitch almost automorphic functions and metrically Besicovitch almost automorphic sequences*

The main aim of this section is to introduce and analyze various classes of metrically Besicovitch almost automorphic functions and metrically Besicovitch almost automorphic sequences as well as to slightly generalize the notion introduced recently in [15]. Unless stated otherwise, we will always assume henceforth that $\Omega := [-1,1]^n \subseteq \mathbb{R}^n$ $[\Omega := [-1,1]^n \cap \mathbb{Z}^n \subseteq \mathbb{Z}^n]$, $\mathbb{F} : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$ $[\mathbb{F} : (0,\infty) \times \mathbb{Z}^n \to (0,\infty)]$ as well as that, for every $l > 0$, (P_l, d_l) is a pseudometric space, where $P_l \subseteq Y^{l\Omega}$ $[P_l \subseteq Y^{l\Omega \cap \mathbb{Z}^n}]$ is closed under the addition and subtraction of functions, containing the zero-function. Define $\|f\|_l := d_l(f, 0)$ for all $f \in P_l$ $(l > 0)$. We will always assume henceforth that R is a collection of sequences in $\mathbb{R}^n$ $[\mathbb{Z}^n]$; for simplicity and better understanding, we will not consider here the corresponding classes of functions with the collections R_X of sequences in $\mathbb{R}^n \times X$ $[\mathbb{Z}^n \times X]$.

The following notion generalizes the notion introduced recently in [15, Definition 2.1] (for simplicity, we will consider the case $\phi(x) \equiv x$ here, only):

Definition 2.1. Suppose that $F : \mathbb{R}^n \times X \to Y$ [$F : \mathbb{Z}^n \times X \to Y$] satisfies that for each $x \in X$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ [$\mathbf{t} \in \mathbb{Z}^n$] we have $F(\mathbf{t} + \cdot; x) \in P_l$. Let for every $l > 0$, $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_m} = (b_{k_m}^1, b_{k_m}^2, \ldots, b_{k_m}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to P_l$ [$F^* : \mathbb{Z}^n \times X \to P_l$] such that for each $x \in B$, $l > 0$ and $\mathbf{t} \in \mathbb{R}^n$ [$\mathbf{t} \in \mathbb{Z}^n$] we have:

(i)

$$\lim_{m\to+\infty} \limsup_{l\to+\infty} \mathbb{F}(l, \mathbf{t}) \Bigl\| F(\mathbf{t} + \cdot + (b_{k_m}^1, \ldots, b_{k_m}^n); x) - \bigl[F^*(\mathbf{t}; x)\bigr](\cdot) \Bigr\|_l = 0 \quad (2.1)$$

and

$$\lim_{m\to+\infty} \limsup_{l\to+\infty} \mathbb{F}(l, \mathbf{t}) \Bigl\| \bigl[F^*(\mathbf{t} - (b_{k_m}^1, \ldots, b_{k_m}^n); x)\bigr](\cdot) - F(\mathbf{t} + \cdot; x) \Bigr\|_l = 0, \quad (2.2)$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ [$\mathbf{t} \in \mathbb{Z}^n$], or

(ii)

$$\lim_{m\to+\infty} \liminf_{l\to+\infty} \mathbb{F}(l, \mathbf{t}) \Bigl\| F(\mathbf{t} + \cdot + (b_{k_m}^1, \ldots, b_{k_m}^n); x) - \bigl[F^*(\mathbf{t}; x)\bigr](\cdot) \Bigr\|_l = 0 \quad (2.3)$$

and

$$\lim_{m\to+\infty} \liminf_{l\to+\infty} \mathbb{F}(l, \mathbf{t}) \Bigl\| \bigl[F^*(\mathbf{t} - (b_{k_m}^1, \ldots, b_{k_m}^n); x)\bigr](\cdot) - F(\mathbf{t} + \cdot; x) \Bigr\|_l = 0, \quad (2.4)$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ [$\mathbf{t} \in \mathbb{Z}^n$],

(iii)

$$\lim_{l\to+\infty} \limsup_{m\to+\infty} \mathbb{F}(l, \mathbf{t}) \Bigl\| F(\mathbf{t} + \cdot + (b_{k_m}^1, \ldots, b_{k_m}^n); x) - \bigl[F^*(\mathbf{t}; x)\bigr](\cdot) \Bigr\|_l = 0 \quad (2.5)$$

and

$$\lim_{l\to+\infty} \limsup_{m\to+\infty} \mathbb{F}(l, \mathbf{t}) \Bigl\| \bigl[F^*(\mathbf{t} - (b_{k_m}^1, \ldots, b_{k_m}^n); x)\bigr](\cdot) - F(\mathbf{t} + \cdot; x) \Bigr\|_l = 0, \quad (2.6)$$

pointwise for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ [$\mathbf{t} \in \mathbb{Z}^n$], or

(iv)

$$\lim_{l\to+\infty} \liminf_{m\to+\infty} \mathbb{F}(l, \mathbf{t}) \Bigl\| F(\mathbf{t} + \cdot + (b_{k_m}^1, \ldots, b_{k_m}^n); x) - \bigl[F^*(\mathbf{t}; x)\bigr](\cdot) \Bigr\|_l = 0 \quad (2.7)$$

and

$$\lim_{l\to+\infty}\liminf_{m\to+\infty}\mathbb{F}(l,\mathbf{t})\Big\|\big[F^{*}(\mathbf{t}-(b^1_{k_m},\ldots,b^n_{k_m});x)\big](\cdot)-F(\mathbf{t}+\cdot;x)\Big\|_l=0, \quad (2.8)$$

pointwise for all $x\in B$ and $\mathbf{t}\in\mathbb{R}^n$ [$\mathbf{t}\in\mathbb{Z}^n$].

In the case that (i), resp., (ii) holds, resp. [(iii), resp., (iv) holds], then we say that $F(\cdot;\cdot)$ is Besicovitch-$(\mathbb{F},\mathcal{P},\mathrm{R},\mathcal{B})$-multi-almost automorphic, resp., weakly Besicovitch-$(\mathbb{F},\mathcal{P},\mathrm{R},\mathcal{B})$-multi-almost automorphic [Besicovitch-$(\mathbb{F},\mathcal{P},\mathrm{R},\mathcal{B})$-multi-almost automorphic of type 1, resp., weakly Besicovitch-$(\mathbb{F},\mathcal{P},\mathrm{R},\mathcal{B})$-multi-almost automorphic of type 1].

We will also consider the notion in which the limit function $F^{*}(\cdot;\cdot)$ from Definition 2.1 is bounded by the function $\omega:\mathbb{R}^n\to(0,\infty)$ [$\omega:\mathbb{Z}^n\to(0,\infty)$] in the sense that there exists $M>0$ such that, for every $x\in B$, $l>0$ and $\mathbf{u}\in l\Omega$, we have $\|[F^{*}(\mathbf{t};x)](\mathbf{u})\|_Y\leq M\omega(|\mathbf{t}|)$, $\mathbf{t}\in\mathbb{R}^n$ [$\|[F^{*}(\mathbf{t};x)](\mathbf{u})\|_Y\leq M\omega(|\mathbf{t}|)$, $\mathbf{t}\in\mathbb{Z}^n$]. If this is the case, then we say that the function $F(\cdot;\cdot)$ is Besicovitch-$(\mathbb{F},\mathcal{P},\mathrm{R},\mathcal{B},\omega)$-multi-almost automorphic, e.g.; furthermore, if $\omega(\cdot)\equiv 1$, then we write "b" in place of "ω". For simplicity, we will not consider here the notion in which the first limits in the above equations are repalced by $\limsup_{m\to}\cdot$, $\liminf_{m\to}\cdot$ and the corresponding limits with the variable $l>0$. As in all our previous investigations, we omit the term "R" if R is the collection of all sequences in $\mathbb{R}^n$ [$\mathbb{Z}^n$] and we omit the term "$\mathcal{B}$" if $X=\{0\}$.

If $1\leq p<\infty$, then the notion of a (weakly) Besicovitch-p-almost automorphic function (of type 1) $F:\mathbb{R}^n\to Y$ is obtained with $\Omega=[-1,1]^n$, R being the collection of all sequnces in $\mathbb{R}^n$, $\mathbb{F}(l,\mathbf{t})\equiv l^{-n/p}$ and $P_l=L^p([-l,l]^n:Y)$; the notion of a (weakly) Besicovitch-p-almost automorphic sequence (of type 1) $F:\mathbb{Z}^n\to Y$ is new and can be obtained with $\Omega=[-1,1]^n$, R being the collection of all sequences in $\mathbb{Z}^n$, $\mathbb{F}(l,\mathbf{t})\equiv l^{-n/p}$ and $P_l=L^p([-l,l]^n\cap\mathbb{Z}^n:Y)$, where

$$\|f\|_l\equiv\Bigg[\sum_{j\in[-l,l]^n\cap\mathbb{Z}^n}\|f(j)\|_Y^p\Bigg]^{1/p},\ \text{ if } p\geq 1, \quad (2.9)$$

$$\text{resp.,}\ \|f\|_l\equiv\sum_{j\in[-l,l]^n\cap\mathbb{Z}^n}\|f(j)\|_Y^p,\ \text{ if } p\in(0,1). \quad (2.10)$$

We similarly define the notion of (weak) Besicovitch-(p,R)-almost automorphy (of type 1) [(weak) Besicovitch-(p,R,b)-almost automorphy (of type 1)], where R is a general collection of sequences obeying our requirements. The important metrical generalizations of (2.9)-(2.10) are given by

$$\|f\|_l \equiv \Biggl[\sum_{j\in[-l,l]^n\cap\mathbb{Z}^n}\|f(j)\|_Y^p\nu^p(j)\Biggr]^{1/p}, \text{ if } p\geq 1,$$

$$\text{resp., } \|f\|_l \equiv \sum_{j\in[-l,l]^n\cap\mathbb{Z}^n}\|f(j)\|_Y^p\nu^p(j), \text{ if } p\in(0,1),$$

where $\nu : \mathbb{Z}^n \rightarrow (0,\infty)$ is an arbitrary sequence.

Now we will introduce the corresponding notion with $0 < p < 1$:

Definition 2.2. Suppose that $0 < p < 1$ and $F : \mathbb{R}^n \rightarrow Y$ $[F : \mathbb{Z}^n \rightarrow Y]$. Then we say that $F(\cdot)$ is (weakly) Besicovitch-p-almost automorphic function (of type 1) [(weakly) Besicovitch-(p, b)-almost automorphic sequence (of type 1)] if and only if $F(\cdot)$ is (weakly) Besicovitch-$(\mathbb{F}, \mathcal{P}, \mathrm{R})$-multi-almost automorphic (of type 1) [(weakly) Besicovitch-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, b)$-multi-almost automorphic (of type 1)], where $\Omega = [-1, 1]^n$, R is the collection of all sequences in $\mathbb{R}^n$ $[\mathbb{Z}^n]$, $\mathbb{F}(l, \mathbf{t}) \equiv l^{-n}$ and $P_l = L^p([-l, l]^n : Y)$ $[P_l = l^p([-l, l]^n \cap \mathbb{Z}^n : Y)]$. If R is a general collection of sequences obeying our requirements, then we also say that $F(\cdot)$ is (weakly) Besicovitch-(p, R)-almost automorphic (of type 1) [(weakly) Besicovitch-(p, R, b)-almost automorphic (of type 1)].

We can further generalize the notion introduced in the above definitions following our approach from [15, Definition 2.6]. The notion in which $P_l = L^p([-l, l]^n : Y)$ $[P_l = l^p([-l, l]^n \cap \mathbb{Z}^n : Y)]$ and $\mathbb{F}(l, \mathbf{t}) \equiv l^{-n/p}$, if $1 \leq p < +\infty$, resp., $\mathbb{F}(l, \mathbf{t}) \equiv l^{-n}$, if $0 < p < 1$, is the most important, when we have the following result; the proof is almost the same as the corresponding proof for the Weyl classes of metrically almost automorphic functions and therefore omitted ([1]):

Proposition 2.1. *Let P_l and $\mathbb{F}(l, \cdot)$ be as above $(l > 0)$. Then the following holds:*

- (i) *Suppose that $0 < p \leq q < +\infty$. Then any (weakly) Besicovitch-q-almost automorphic function $F : \mathbb{R}^n \rightarrow Y$ $[F : \mathbb{Z}^n \rightarrow Y]$ (of type 1) is (weakly) Besicovitch-p-almost automorphic (of type 1); furthermore, the same holds for the corresponding classes of (weakly) Besicovitch-(q, b)-almost automorphic functions (of type 1) and (weakly) Besicovitch-(p, b)-almost automorphic functions (of type 1).*

- (ii) *Suppose that $0 < p \leq q < +\infty$. Then $F : \mathbb{R}^n \rightarrow Y$ $[F : \mathbb{Z}^n \rightarrow Y]$ is essentially bounded, (weakly) Besicovitch-(q, b)-almost automorphic (of type 1) if and only if $F(\cdot)$ is essentially bounded, (weakly) Besicovitch-(p, b)-almost automorphic (of type 1).*

We continue by observing that the statement of [15, Proposition 2.5] can be simply formulated for a class of metrically Besicovich almost automorphic functions of type 1, with a general exponent $p > 0$; for example, if $p > 0$, $\sigma > 0$, $\mathbb{F}(l) \equiv l^{-\sigma}$, $f \in L^p_{loc}(\mathbb{R} : X)$ and there exist a strictly increasing sequence (l_k) of positive real numbers tending to plus infinity, a sequence (b_k) of real numbers and a positive real number $\epsilon_0 > 0$ such that, for every $k \in \mathbb{N}$ and for every subsequence of (b_{k_m}) of (b_k), we have

$$\lim_{m \to +\infty} l_k^{-\sigma} \int_{b_{k_m}-l_k}^{b_{k_m}+l_k} \|f(x)\|^p \nu^p(x)\, dx = +\infty,$$

where $\nu : \mathbb{R} \to (0, \infty)$ is a Lebesgue measurable function, then $f(\cdot)$ cannnot be Besicovitch-$(\mathbb{F}, \mathcal{P})$-multi-almost automorphic, where $P_l = L^p_\nu([-l, l] : X)$ for all $l > 0$. A similar statement can be formulated for the sequences.

Furthermore, the statement of [15, Proposition 2.8], which concerns the pointwise multiplication of Besicovitch almost automorphic type functions, can be slightly generalized with the usage of the Banach spaces $P_l = L^p_\nu([-l, l]^n : Y)$ for all $l > 0$, provided that a Lebesgue measurable function $\nu : \mathbb{R}^n \to Y$ satisfies that there exists a bounded function $\varphi : \mathbb{R}^n \to (0, \infty)$ such that $\nu(x + y) \leq \nu(x)\varphi(y)$ for all $x,\ y \in \mathbb{R}^n$. This can be simply employed for the construction of multi-dimensional Besicovitch almost periodic type functions in general metric; cf. [15, Example 2.9].

Concerning the composition principle for a class of Besicovich almost automorphic functions of type 1, established in [15, Theorem 2.10], we would like to note that the same proof shows the validity of this result for all exponents $p,\ q > 0$ such that $p = \alpha q$. Moreover, we can similarly deduce the following result in which we consider the metrical Besicovitch-p-almost automorphy of the multi-dimensional Nemytskii operator $W : \mathbb{R}^n \times X \to Z$, given by

$$W(\mathbf{t}; x) := G(\mathbf{t}; F(\mathbf{t}; x)), \quad \mathbf{t} \in \mathbb{R}^n,\ x \in X, \tag{2.11}$$

where $F : \mathbb{R}^n \times X \to Y$ and $G : \mathbb{R}^n \times Y \to Z$ (for the notion of uniform $(\mathrm{R}, \mathcal{B}')$-almost automorphy, we refer the reader to [5]; the result of [15, Theorem 2.10(ii)] can be generalized in a similar fashion):

Theorem 2.1. *Suppose that* $0 < p,\ q < +\infty$, $\alpha > 0$, $p = \alpha q$, $\mathbb{F}(l, t) \equiv l^{-n/p}$, $F(\cdot; \cdot)$ *is Besicovitch-*$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \mathcal{B})$*-multi-almost automorphic of type* 1, *where* $P_l = L^p_\nu([-l, l]^n : Y)$ *for all* $l > 0$ *and some* p*-locally integrable function* $\nu : \mathbb{R}^n \to (0, \infty)$, *and for every* $B \in \mathcal{B}$ *and* $(\mathbf{b}_k) \in \mathrm{R}$, *the subsequence* $(\mathbf{b}_{k_m})$ *of* $(\mathbf{b}_k)$ *and the function* $F^* : \mathbb{R}^n \times X \to Y$ *from the corresponding definition satisfy* $F^*(\mathbf{t}; x) \in \overline{\bigcup_{\mathbf{s} \in \mathbb{R}^n} F(\mathbf{s}; x)}$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$. *Define* $B' := \overline{\bigcup_{\mathbf{t} \in \mathbb{R}^n} F(\mathbf{t}; B)}$ *for each set* $B \in \mathcal{B}$, *and* $\mathcal{B}' := \{B' : B \in \mathcal{B}\}$. *Assume, additionally, that, for every sequence from* R, *any its subsequence also belongs to* R, $G : \mathbb{R}^n \times Y \to Z$ *is* *uniformly* $(\mathrm{R}, \mathcal{B}')$*-almost automorphic and there exists a finite real constant* $a > 0$ *such that*

$$\Big\|G(\mathbf{t}; y) - G(\mathbf{t}; y')\Big\|_Z \leq a\big\|y - y'\big\|_Y^{\alpha}, \quad \mathbf{t} \in \mathbb{R}^n,\ y,\ y' \in Y.$$

Then the function $W(\cdot;\cdot)$*, given by* (2.11)*, is Besicovitch-*$(\mathbb{F}^{p/q}, \mathcal{P}_q, \mathrm{R}, \mathcal{B})$*-multi-almost automorphic of type* 1*, where* $P_l = L^q_{\nu^{p/q}}([-l, l]^n : Y)$ *for all* $l > 0$*.*

We can apply such results in the analysis of the existence and uniqueness of Besicovitch-$(\mathbb{F}^{p/q}, \mathcal{P}_q, \mathrm{R}, \mathcal{B})$-multi-almost automorphic solutions for various classes of the abstract (fractional) semilinear Cauchy problems; see the third application in [15, Section 3] and the third application in [14, Section 4].

Concerning the extensions of metrically Besicovitch almost automorphic sequences, we will only state the following result; the proof is very similar to the proof of [1, Theorem 3.2] and therefore omitted:

Theorem 2.2. *Suppose that* $p > 0$ *and* $F : \mathbb{Z}^n \to Y$ *is bounded, (weakly) Besicovitch-*(p, R)*-multi-almost automorphic sequence (of type* 1*), where* R *is any collection of sequences in* $\mathbb{Z}^n$ *such that the assumption* $(b_k) \in \mathrm{R}$ *implies that any subsequence of* (b_k) *also belongs to* R*. Let* R' *be the collection of all sequences* (a_k) *in* $\mathbb{R}^n$ *satisfying that there exists a sequence* $(b_k) \in \mathrm{R}$ *such that* $\sup_{k \in \mathbb{N}} |a_k - b_k| < +\infty$*. Then there exists a bounded, uniformly continuous, (weakly) Besicovitch-*(p, R)*-multi-almost automorphic function* $\tilde{F} : \mathbb{R}^n \to Y$ *(of type* 1*) such that*

$$R(\tilde{F}(\cdot)) \subseteq CH(\overline{R(F)}), \quad \|\tilde{F}\|_\infty = \|F\|_\infty \ \text{and} \ \ \tilde{F}(k) = F(k) \ \text{for all} \ \ k \in \mathbb{Z}^n.$$

Before proceeeding to the next section, we would like to note that we have recently analyzed, in [1], several new classes of vectorial Weyl almost automorphic type functions. We will skip all details concerning the corresponding classes of vectorial Besicovitch almost automorphic type functions.

3. *Some applications*

In this section, we will first analyze the convolution invariance of metrical Besicovitch almost type automorphy. Let the operator family $(R(t))_{t>0} \subseteq L(X, Y)$ satisfy that there exist finite real constants $M > 0$, $\beta \in (0, 1]$ and $\gamma > \beta$ such that

$$\big\|R(t)\big\|_{L(X,Y)} \leq M\frac{t^{\beta-1}}{1 + t^{\gamma}}, \quad t > 0. \tag{3.1}$$

If this is the case, then we are in a position to state and prove the following slight generalization of [15, Proposition 3.2]:

Proposition 3.1. *Suppose that* (3.1) *holds,* $a > 0$, $\alpha > 0$, $1 \le p < +\infty$, $p \ge 1$, $p(\beta-1)/(p-1) > -1$ *if* $p > 1$, *and* $\beta = 1$ *if* $p = 1$. *Suppose, further, that* $b \in [0, \gamma - \beta)$, $w(t) := (1+|t|)^b$, $t \in \mathbb{R}$ *and the function* $f : \mathbb{R} \to X$ *is Besicovitch-*$(t^{-a}, \mathcal{P}, \mathrm{R}, w)$*-almost automorphic, where* $P_l = L^p_\nu([-l,l] : X)$ *for all* $l > 0$ *and some Lebesgue measurable function* $\nu : \mathbb{R} \to (0, \infty)$ *satisfying that*

$$\limsup_{l \to +\infty} \left[l^{-ap} \int_{-l}^{l} \nu^p(s)\, ds \right] < +\infty \tag{3.2}$$

and there exists a Lebesgue measurable function $\varphi : \mathbb{R} \to (0, \infty)$ *such that* $\nu(x) \le \nu(y)\varphi(x-y)$ *for all* $x,\ y \in \mathbb{R}$ *and there exists* $\zeta \in ((1/p) + b, (1/p) + \gamma - \beta)$ *with*

$$\int_{-\infty}^{+\infty} \frac{\varphi^p(s)}{(1+|s|^\zeta)^p}\, ds < +\infty. \tag{3.3}$$

If there exists a finite real constant $M' > 0$ *such that* $\|f(t)\| \le M' w(t)$, $t \in \mathbb{R}$, *then the function* $F(\cdot)$, *given by*

$$t \mapsto F(t) := \int_{-\infty}^{t} R(t-s) f(s)\, ds, \quad t \in \mathbb{R}, \tag{3.4}$$

is continuous and Besicovitch-$(t^{-a}, \mathcal{P}_Y, \mathrm{R}, w)$*-almost automorphic, where* $P_{l;Y} = L^p_\nu([-l,l] : Y)$ *for all* $l > 0$, *and there exists a finite real constant* $M'' > 0$ *such that* $\|F(t)\|_Y \le M'' w(t)$, $t \in \mathbb{R}$.

PROOF. We will provide the main details of the proof, only. Arguing as in the proof of [14, Proposition 4.2], we can show that the function $F(\cdot)$ is well-defined, continuous and there exists a finite real constant $M'' > 0$ such that $\|F(t)\|_Y \le M'' w(t)$, $t \in \mathbb{R}$. Let a sequence $(b_k) \in \mathrm{R}$ be given and let $l > 0$. Then we know that there exist a subsequence (b_{k_m}) of (b_k), a function $f^* : \mathbb{R} \to X$ and a finite real constant $M > 0$ such that $\|[f^*(t)](s)\| \le M w(t)$, $t \in \mathbb{R}$, $s \in [-l, l]$,

$$\lim_{m \to +\infty} \limsup_{l \to +\infty} \left[l^{-ap} \int_{-l}^{l} \left\| f\left(t + s + b_{k_m}\right) - [f^*(t)](s) \right\|^p \nu^p(s)\, ds \right] = 0$$

and

$$\lim_{m \to +\infty} \limsup_{l \to +\infty} \left[l^{-ap} \int_{-l}^{l} \left\| \left[f^*\left(t - b_{k_m}\right)\right](s) - f(t+s) \right\|^p \nu^p(s)\, ds \right] = 0.$$

Define $F^* : \mathbb{R} \to Y$ by $[F^*(t)](u) := \int_{-\infty}^{t} R(t-s)[f^*(s)](u)\, ds$, $t \in \mathbb{R}$, $u \in [-l, l]$. Take now any real number $\zeta \in ((1/p) + b, (1/p) + \gamma - \beta)$ such that (3.3) holds. Then we have:

$$\frac{1}{2l^{ap}}\int\limits_{-l}^{l}\bigl\|F(s+b_{k_m}+t)-\bigl[F^*(t)\bigr](s)\bigr\|^p\nu^p(s)\,ds$$

$$\leq\frac{M_1}{2l^{ap}}\int\limits_{-l}^{l}\int\limits_{-\infty}^{0}\frac{1}{(1+|z|^{\alpha\zeta})^p}\bigl\|F(s+b_{k_m}+t+z)-F^*(s+t+z)\bigr\|^p\nu^p(s)\,dz\,ds$$

$$=\frac{M_1}{2l^{ap}}\int\limits_{-l}^{l}\int\limits_{z-s}^{l}\frac{1}{(1+|z-s|^{\alpha\zeta})^p}\bigl\|F(b_{k_m}+t+z)-F^*(t+z)\bigr\|^p\nu^p(s)\,ds\,dz$$

$$+\frac{M_1}{2l^{ap}}\int\limits_{-\infty}^{-l}\int\limits_{-l}^{l}\frac{1}{(1+|z-s|^{\alpha\zeta})^p}\bigl\|F(b_{k_m}+t+z)-F^*(t+z)\bigr\|^p\nu^p(s)\,ds\,dz$$

$$\leq\frac{M_1}{l^{ap}}\int_{-l}^{l}\bigl\|F(b_{k_m}+t+z)-F^*(t+z)\bigr\|^p\nu^p(z)\cdot\int_{-\infty}^{+\infty}\frac{\varphi^p(s-z)\,ds}{(1+|s-z|^{\zeta})^p}\,dz$$

$$+\frac{M_1}{2l^{ap}}\int\limits_{-\infty}^{-3l}\int\limits_{-l}^{l}\frac{\nu^p(z)\varphi^p(s-z)}{(1+|z-s|^{\alpha\zeta})^p}\bigl\|F(b_{k_m}+t+z)-F^*(t+z)\bigr\|^p\,ds\,dz$$

$$+\frac{M_1}{2l^{ap}}\int\limits_{-3l}^{3l}\int\limits_{-l}^{l}\frac{\nu^p(z)\varphi^p(s-z)}{(1+|z-s|^{\alpha\zeta})^p}\bigl\|F(b_{k_m}+t+z)-F^*(t+z)\bigr\|^p\,ds\,dz$$

$$\leq\frac{M_1}{l^{ap}}\int\limits_{-l}^{l}\bigl\|F(b_{k_m}+t+z)-F^*(t+z)\bigr\|^p\nu^p(z)\cdot\int_{-\infty}^{+\infty}\frac{\varphi^p(s-z)\,ds}{(1+|s-z|^{\zeta})^p}\,dz$$

$$+\frac{M_1}{l^{ap}}\int\limits_{-3l}^{3l}\bigl\|F(b_{k_m}+t+z)-F^*(t+z)\bigr\|^p\nu^p(z)\cdot\int_{-\infty}^{+\infty}\frac{\varphi^p(s-z)\,ds}{(1+|s-z|^{\zeta})^p}\,dz$$

$$+\frac{cM_1}{2l^{ap}}\int\limits_{-\infty}^{-3l}\frac{\nu^p(z)}{(1+|z/2|^{\alpha\zeta})^p}\bigl\|F(b_{k_m}+t+z)-F^*(t+z)\bigr\|^p\,dz\left(\int\limits_{-l}^{l}\nu^p(s)\,ds\right),$$

for any $t\in\mathbb{R}$; here we have applied the Hölder inequality as well as the Fubini theorem and an elementary change of variables in the double integral. The first limit equation follows by applying (3.2); the second limit equation can be shown analogously.

It is clear that Proposition 3.1 can be applied in the analysis of the existence and uniqueness of Besicovitch-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \omega)$-almost automorphic type solutions for a substantially large class of the abstract Volterra integro-differential inclusions without initial conditions; see [10] for more details about applications of this type. We can similarly analyze the invariance of Besicovitch-$(\mathbb{F}, \mathcal{P}, \mathrm{R}, \omega)$-multi-almost automorphy under the actions of the usual convolution product

$$F \mapsto (h * F)(\mathbf{t}) \equiv \int_{\mathbb{R}^n} h(\mathbf{t} - \mathbf{s}) F(\mathbf{s})\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n$$

and provide certain applications to the heat equation in $\mathbb{R}^n$; cf. also [15, Theorem 3.3].

Concerning certain applications to the abstract Volterra difference equations, we will only note that we can consider the existence and uniqueness of metrically Besicovitch almost automorphic type solutions of the first-order difference equation

$$u(k+1) = Au(k) + f(k), \quad k \in \mathbb{Z}, \tag{3.5}$$

where $A \in L(X)$ and $(f(k))_{k \in \mathbb{Z}}$ satisfies certain assumptions. Speaking-matter-of-factly, we must impose the additional condition on the sequence $(f(k))_{k \in \mathbb{Z}}$ by requiring that the limits in Definition 2.1 are uniform with respect to the point $t \in \mathbb{Z}$; this is almost inevitable since the discrete version of the dominated convergence theorem is no longer applicable in our new framework (see, e.g., [1, Definition 4.3, Theorem 4.4] and the applications made in the pioneering paper [2]). Keeping in mind this extra condition, we can similarly analyze the abstract difference equations

$$u(k+1) = Au(k) + f(k), \quad k \geq 0; \ \ u(0) = u_0,$$

and

$$u(k,m) = A(k,m)u(k-1, m-1) + f(k,m), \quad k, \ m \in \mathbb{N},$$

the later subjected with the initial conditions

$$u(k,0) = u_{k,0}; \quad u(0,m) = u_{0,m}, \quad k, \ m \in \mathbb{N}_0.$$

See [1] and [19] for more details in this direction.

4. *Conclusions and final remarks*

In this paper, we have slightly generalized various notions of Besicovitch almost automorphy ([11], [15]). We have considered various classes of metrically Besicovitch almost automorphic functions of the form $F : \mathbb{R}^n \times X \to Y$ and metrically Besicovitch almost automorphic sequences of the form $F : \mathbb{Z}^n \times X \to Y$, where X and Y are complex Banach spaces. We have provided some applications, as well.

We close the paper with the observation that all open problems raised in [1], [15] and [16] remain open after this research study.

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