Bulletin T. CLV de l'Académie serbe des sciences et des arts − 2022
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 47

ON THE DIFFERENCE BETWEEN ATOM-BOND SUM-CONNECTIVITY AND SUM-CONNECTIVITY INDICES

KALKERE JAYANNA GOWTHAM, IVAN GUTMAN

(Presented and accepted at the 8th Meeting, held on November 25, 2022)

A b s t r a c t. The atom-bond sum-connectivity index ABS is a recently introduced vertex-degree-based graph invariant, defined as the sum over all pairs of adjacent vertices u, v of the term $\sqrt{(d_u + d_v - 2)/(d_u + d_v)}$, where d_u is the degree of the vertex u. A much older such invariant, the sum-connectivity index SC, is defined via $\sqrt{(1/(d_u + d_v)}$. We study the difference between ABS and SC and establish various bounds for it, in terms several vertex-degree-based graph invariants.

AMS Mathematics Subject Classification (2020): 05C07, 05C09.

Key Words: degree (of vertex), vertex-degree-based graph invariant, sum-connectivity index, atom-bond, sum-connectivity index.

1. *Introduction*

In this paper we consider simple graphs (i.e., graphs without directed, weighted, and multiple edges, and without self-loops). Let G be such a graph, and assume that it is connected. Its vertex set is $\mathbf{V}(G)$ and its edge set is $\mathbf{E}(G)$. The order and size of G are $|\mathbf{V}(G)| = n$ and $|\mathbf{E}(G)| = m$, respectively. The degree d_u of the vertex $u \in \mathbf{V}(G)$ is the number of vertices adjacent to u. The maximum and minimum vertex degree of the graph G are denoted by Δ and δ, respectively. The edge connecting the vertices u and v will be denoted by uv.

In the present-days literature, a large number of vertex-degree-based graph invariants (usually referred to as"topological indices") are being considered [12, 23, 24, 16]. Among them we mention here the first Zagreb index [6, 13, 14]

$$M_1 = M_1(G) = \sum_{u \in \mathbf{V}(G)} (d_u)^2 = \sum_{uv \in \mathbf{E}(G)} (d_u + d_v) \tag{1.1}$$

the connectivity index (Randić index) [17, 20, 21] and its sum-connectivity variant [7, 19, 27]

$$R = R(G) = \sum_{uv \in \mathbf{E}(G)} \frac{1}{\sqrt{d_u\, d_v}} \quad , \quad SC = SC(G) = \sum_{uv \in \mathbf{E}(G)} \frac{1}{\sqrt{d_u + d_v}} \tag{1.2}$$

the atom-bond connectivity index [1, 9] and its sum-connectivity counterpart [3, 4, 11]

$$ABC = ABC(G) = \sum_{uv \in \mathbf{E}(G)} \sqrt{\frac{d_u + d_v - 2}{d_u\, d_v}} \tag{1.3}$$

$$ABS = ABS(G) = \sum_{uv \in \mathbf{E}(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \tag{1.4}$$

and the harmonic index [5, 10, 25]

$$H = H(G) = \sum_{uv \in \mathbf{E}(G)} \frac{2}{d_u + d_v} \,. \tag{1.5}$$

All graph invariants listed in Eqs. (1.1)–(1.5) have similar algebraic forms. One may, therefore, expect that there exist relations between them. Indeed, many such relations have already been established, see e.g. [2, 8, 11, 15, 22, 26, 28]. The ABS-index was introduced quite recently [3], and its properties have not yet been fully determined. In this paper we examine the relations between ABS and sum-connectivity index (SC). In particular, we establish lower and upper bounds for the difference $ABS - SC$ in terms of first Zagreb and harmonic indices.

In what follows, we denote $ABS(G) - SC(G)$ by $\Xi(G)$, i.e.,

$$\Xi = \Xi(G) = \sum_{uv \in \mathbf{E}(G)} \frac{\sqrt{d_u + d_v - 2} - 1}{\sqrt{d_u + d_v}} \,.$$

2. *Elementary relation*

Let P_n be the n-vertex path.

Remark 2.1. *Let G be a connected graph, different from P_1, P_2, and P_3. Then*

$$ABS(G) > SC(G) \quad \text{and} \quad \Xi(G) > 0\,.$$

PROOF. Because P_1 has no edges, it is $ABS(P_1) = SC(P_1) = 0$ and thus $\Xi(P_1) = 0$. The two vertices of P_2 have degrees 1, implying $d_u + d_v - 2 = 0$ and $ABS(P_2) = 0$. Therefore, $ABS(P_2) < SC(P_2)$ and $\Xi(P_2) < 0$. Both edges of P_3 connect vertices of degree 1 and 2, implying $d_u + d_v - 2 = 1$. Therefore, $ABS(P_3) = SC(P_3)$ and $\Xi(P_3) = 0$.

All other graphs possess at least one edge for which $d_u + d_v - 2 > 1$.

3. *Main results*

Theorem 3.1. *If G is a graph with m edges, then*

$$\Xi(G) \leq \sqrt{\frac{m}{2\delta}\big(M_1(G) - m\big)}\,.$$

PROOF. By the Cauchy–Schwarz inequality, we acquire

$$\begin{aligned}
\Xi(G) &= \sum_{uv\in \mathbf{E}(G)} \frac{\sqrt{d_u+d_v-2}-1}{\sqrt{d_u+d_v}} \\
&\leq \sqrt{\sum_{uv\in \mathbf{E}(G)} (\sqrt{d_u+d_v-2}-1)^2 \sum_{uv\in \mathbf{E}(G)} \frac{1}{d_u+d_v}} \\
&< \sqrt{\sum_{uv\in \mathbf{E}(G)} (d_u+d_v-2+1-2\sqrt{d_u+d_v-2}) \sum_{uv\in \mathbf{E}(G)} \frac{1}{d_u+d_v}} \\
&< \sqrt{\sum_{uv\in \mathbf{E}(G)} (d_u+d_v-1) \sum_{uv\in \mathbf{E}(G)} \frac{1}{d_u+d_v}} \leq \sqrt{\frac{m}{2\delta}\big(M_1(G)-m\big)}\,.
\end{aligned}$$

Theorem 3.2. *If G be a graph with m edges, then*

$$(\sqrt{2\delta-2}-1)^2\left(\frac{H(G)}{2} + \frac{m(m-1)}{2\Delta}\right) \leq \Xi(G)$$

$$\leq (\sqrt{2\Delta-2}-1)^2\left(\frac{H(G)}{2}+\frac{m(m-1)}{2\delta}\right).$$

The equality holds if and only G is regular.

PROOF. From the definition of $\Xi(G)$ we have,

$$\begin{aligned}
\Xi(G)^2 &= \sum_{uv\in\mathbf{E}(G)}\frac{(\sqrt{d_u+d_v-2}-1)^2}{d_u+d_v}\\
&\quad +2\sum_{uv\neq rs\in\mathbf{E}(G)}\left(\frac{\sqrt{d_u+d_v-2}-1}{\sqrt{d_u+d_v}}\right)\left(\frac{\sqrt{d_r+d_s-2}-1}{\sqrt{d_r+d_s}}\right)\\
&\leq \sum_{uv\in\mathbf{E}(G)}\frac{(\sqrt{2\Delta-2}-1)^2}{d_u+d_v}\\
&\quad +2\sum_{uv\neq rs\in\mathbf{E}(G)}\left(\frac{\sqrt{2\Delta-2}-1}{\sqrt{2\delta}}\right)\left(\frac{\sqrt{2\Delta-2}-1}{\sqrt{2\delta}}\right)\\
&= (\sqrt{2\Delta-2}-1)^2\left(\frac{H(G)}{2}+\frac{m(m-1)}{2\delta}\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Xi(G)^2 &= \sum_{uv\in\mathbf{E}(G)}\frac{(\sqrt{d_u+d_v-2}-1)^2}{d_u+d_v}\\
&\quad +2\sum_{uv\neq rs\in\mathbf{E}(G)}\left(\frac{(\sqrt{d_u+d_v-2}-1)}{\sqrt{d_u+d_v}}\right)\left(\frac{(\sqrt{d_r+d_s-2}-1)}{\sqrt{d_r+d_s}}\right)\\
&\geq \sum_{uv\in\mathbf{E}(G)}\frac{(\sqrt{2\delta-2}-1)^2}{d_u+d_v}\\
&\quad +2\sum_{uv\neq rs\in\mathbf{E}(G)}\left(\frac{(\sqrt{2\delta-2}-1)}{\sqrt{2\Delta}}\right)\left(\frac{(\sqrt{2\delta-2}-1)}{\sqrt{2\Delta}}\right)\\
&= (\sqrt{2\delta-2}-1)^2\left(\frac{H(G)}{2}+\frac{m(m-1)}{2\Delta}\right).
\end{aligned}$$

The equalities are true if and only if G is a regular graph.

Theorem 3.3. *Let G be a graph with $n \geq 3$ vertices and m edges. Then,*

$$\frac{\sqrt{2\delta-2}-1}{\sqrt{2\Delta}} < \Xi(G) < m\sqrt{2\Delta-2}\,.$$

PROOF. Lower bound:

$$\Xi(G) = \sum_{uv\in \mathbf{E}(G)} \frac{\sqrt{d_u+d_v-2}-1}{\sqrt{d_u+d_v}} > \frac{\sum\limits_{uv\in \mathbf{E}(G)} \sqrt{d_u+d_v-2}-1}{\sum\limits_{uv\in \mathbf{E}(G)} \sqrt{d_u+d_v}}.$$

Clearly, $2\delta \leq d_u+d_v \leq 2\Delta$ and therefore,

$$\frac{\sum\limits_{uv\in \mathbf{E}(G)} \sqrt{d_u+d_v-2}-1}{\sum\limits_{uv\in \mathbf{E}(G)} \sqrt{d_u+d_v}} > \frac{\sqrt{2\delta-2}-1}{\sqrt{2\Delta}}\,.$$

Upper bound:

$$\sum_{uv\in \mathbf{E}(G)} \frac{\sqrt{d_u+d_v-2}-1}{\sqrt{d_u+d_v}} < \sum_{uv\in \mathbf{E}(G)} \sqrt{d_u+d_v-2},$$

and by the Cauchy–Schwarz inequality,

$$\begin{aligned}\sum_{uv\in \mathbf{E}(G)} \sqrt{d_u+d_v-2} \;&<\; \sqrt{\sum_{uv\in \mathbf{E}(G)} 1}\sqrt{\sum_{uv\in \mathbf{E}(G)} (d_u+d_v-2)}\\ &\leq\; m\sqrt{2\Delta-2}\,.\end{aligned}$$

By rearranging, we arrive at the required result.

Theorem 3.4. *Let G be a graph with m edges and p pendent vertices. Then,*

$$\begin{aligned}&\frac{\sqrt{2\delta-2}-1}{\sqrt{2\Delta}}(m-p) + \frac{\sqrt{\delta-1}-1}{\sqrt{1+\Delta}}p\\ &\quad\leq \Xi(G) \leq \frac{\sqrt{2\Delta-2}-1}{\sqrt{2\delta}}(m-p) + \frac{\sqrt{\Delta-1}-1}{\sqrt{1+\delta}}p\,.\end{aligned}$$

PROOF.

$$\Xi(G) \;=\; \sum_{\substack{uv\in \mathbf{E}(G)\\ d_u,d_v\neq 1}} \frac{\sqrt{d_u+d_v-2}-1}{\sqrt{d_u+d_v}} + \sum_{\substack{uv\in \mathbf{E}(G)\\ d_u=1}} \frac{\sqrt{d_v-1}-1}{\sqrt{1+d_v}}$$

$$\leq \sum_{\substack{uv\in\mathbf{E}(G)\\ d_u,d_v\neq 1}} \frac{\sqrt{2\Delta-2}-1}{\sqrt{2\delta}} + \sum_{\substack{uv\in\mathbf{E}(G)\\ d_u=1}} \frac{\sqrt{\Delta-1}-1}{\sqrt{1+\delta}}$$

$$= \frac{\sqrt{2\Delta-2}-1}{\sqrt{2\delta}}(m-p) + \frac{\sqrt{\Delta-1}-1}{\sqrt{1+\delta}}p$$

Similarly,

$$\Xi(G) = \sum_{\substack{uv\in\mathbf{E}(G)\\ d_u,d_v\neq 1}} \frac{\sqrt{d_u+d_v-2}-1}{\sqrt{d_u+d_v}} + \sum_{\substack{uv\in\mathbf{E}(G)\\ d_u=1}} \frac{\sqrt{d_v-1}-1}{\sqrt{1+d_v}}$$

$$\geq \sum_{\substack{uv\in\mathbf{E}(G)\\ d_u,d_v\neq 1}} \frac{\sqrt{2\delta-2}-1}{\sqrt{2\Delta}} + \sum_{\substack{uv\in\mathbf{E}(G)\\ d_u=1}} \frac{\sqrt{\delta-1}-1}{\sqrt{1+\Delta}}$$

$$= \frac{\sqrt{2\delta-2}-1}{\sqrt{2\Delta}}(m-p) + \frac{\sqrt{\delta-1}-1}{\sqrt{1+\Delta}}p$$

Equality holds if and only if $d_u + d_v = 2\Delta = 2\delta$.

Theorem 3.5. *Let G be a graph with m edges. Then,*

$$\Xi(G) \leq m\left(\frac{\sqrt{2\Delta-2}-1}{\sqrt{2\delta}}\right).$$

The equality holds if and only if G is regular.

PROOF. By the definition of Ξ and form the Cauchy–Scharwz inequality, we obtain

$$\Xi(G)^2 = \left(\sum_{uv\in\mathbf{E}(G)} \frac{\sqrt{d_u+d_v-2}-1}{\sqrt{d_u+d_v}}\right)^2$$

$$\leq m\sum_{uv\in\mathbf{E}(G)}\left(\frac{\sqrt{2\Delta-2}-1}{\sqrt{2\delta}}\right)^2 = m^2\left(\frac{\sqrt{2\Delta-2}-1}{\sqrt{2\delta}}\right)^2.$$

By simplifying, we get the required result. Equality holds if G is regular.

Lemma 3.1 (Pólya–Szegő inequality). *Let a_i and b_i be positive real numbers for $i = 1, 2, \ldots, m$, such that $a \le a_i \le A$ and $b \le b_i \le B$. Then*

$$\sum_{i=1}^{m} b_i^2 \sum_{i=1}^{m} a_i^2 \le \frac{1}{4}\left(\sqrt{\frac{AB}{ab}} + \sqrt{\frac{ab}{AB}}\right)^2 \left(\sum_{i=1}^{m} a_i b_i\right)^2.$$

Equality holds if and only if

$$\rho = \frac{A/a}{A/a + B/b} \quad \text{and} \quad \sigma = \frac{B/a}{A/a + B/b}$$

are integers, $a_1 = a_2 = \cdots = a_\rho = a$, $a_{\rho+1} = a_{\rho+2} = \cdots = a_m = A$ and $b_1 = b_2 = \cdots = b_\sigma = B$, $b_{\sigma+1} = b_{\sigma+2} = \cdots = b_m = b$.

Theorem 3.6. *Let G be a graph with m edges and $\delta, \Delta > 1$. Then*

$$\Xi(G) \ge \frac{m\left(\frac{\sqrt{2\delta - 2} - 1}{\sqrt{2\Delta}}\right)}{\frac{1}{2}\left(\sqrt{\frac{\sqrt{\Delta(2\Delta - 2)} - \sqrt{\Delta}}{\sqrt{\delta(2\delta - 2)} - \sqrt{\delta}}} + \sqrt{\frac{\sqrt{\delta(2\delta - 2)} - \sqrt{\delta}}{\sqrt{\Delta(2\Delta - 2)} - \sqrt{\Delta}}}\right)}.$$

The equality holds if and only if G is regular graph.

PROOF. Setting

$$a_i = \frac{\sqrt{d_u + d_v - 2} - 1}{\sqrt{d_u + d_v}}, \quad b_i = 1, \quad A = \frac{\sqrt{2\Delta - 2} - 1}{\sqrt{2\delta}}, \quad a = \frac{\sqrt{2\delta - 2} - 1}{\sqrt{2\Delta}},$$

and $B = b = 1$ in Lemma 3.1, we get

$$\sum_{uv \in \mathbf{E}(G)} \left(\frac{\sqrt{d_u + d_v - 2} - 1}{\sqrt{d_u + d_v}}\right)^2 \sum_{uv \in \mathbf{E}(G)} 1$$
$$\le \frac{1}{4}\left(\sqrt{\frac{\sqrt{\Delta(2\Delta - 2)} - \sqrt{\Delta}}{\sqrt{\delta(2\delta - 2)} - \sqrt{\delta}}} + \sqrt{\frac{\sqrt{\delta(2\delta - 2)} - \sqrt{\delta}}{\sqrt{\Delta(2\Delta - 2)} - \sqrt{\Delta}}}\right)^2 \Xi(G)^2.$$

Clearly, $2\delta \le d_u + d_v \le 2\Delta$ and so,

$$m^2 \left(\frac{\sqrt{2\delta - 2} - 1}{\sqrt{2\Delta}}\right)^2$$
$$\le \frac{1}{4}\left(\sqrt{\frac{\sqrt{\Delta(2\Delta - 2)} - \sqrt{\Delta}}{\sqrt{\delta(2\delta - 2)} - \sqrt{\delta}}} + \sqrt{\frac{\sqrt{\delta(2\delta - 2)} - \sqrt{\delta}}{\sqrt{\Delta(2\Delta - 2)} - \sqrt{\Delta}}}\right)^2 \Xi(G)^2.$$

By simplifying we get the required result. Equality holds if G is regular graph.

Lemma 3.2. [18] *Let $a = (a_i)$ and $b = (b_i)$ be sequences of non-negative real numbers and positive real numbers, respectively, for $i = 1, 2, \ldots, n$. For any real number r with $r \geq 1$ or $r \leq 0$,*

$$\sum_{i=1}^{n} a_i b_i^r \geq \sum_{i=1}^{n} a_i \left(\frac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} a_i} \right)^r .$$

Theorem 3.7. *Let G be a connected graph with m edges. Then,*

$$\Xi(G) \leq \sqrt{m(\sqrt{2\Delta - 2} - 1)\left(\frac{m\sqrt{2\Delta - 2}}{2\delta} - \frac{H(G)}{2} \right)}.$$

PROOF. Clearly,

$$\sum_{uv \in \mathbf{E}(G)} \frac{\sqrt{d_u + d_v - 2}}{d_u + d_v} - \sum_{uv \in \mathbf{E}(G)} \frac{1}{d_u + d_v} \leq \frac{m\sqrt{2\Delta - 2}}{2\delta} - \frac{H(G)}{2}$$

$$\sum_{uv \in \mathbf{E}(G)} \frac{\sqrt{d_u + d_v - 2} - 1}{(\sqrt{d_u + d_v})^2} \leq \frac{m\sqrt{2\Delta - 2}}{2\delta} - \frac{H(G)}{2} \tag{3.1}$$

Setting $r = 2$, $a_i = \sqrt{d_u + d_v - 2} - 1$, and $b_i = 1/\sqrt{d_u + d_v}$ in Lemma 3.2 we get,

$$\sum_{uv \in \mathbf{E}(G)} \frac{\sqrt{d_u + d_v - 2} - 1}{(\sqrt{d_u + d_v})^2}$$

$$\geq \sum_{uv \in \mathbf{E}(G)} (\sqrt{d_u + d_v - 2} - 1) \left(\frac{\sum_{uv \in \mathbf{E}(G)} \frac{\sqrt{d_u + d_v - 2} - 1}{\sqrt{d_u + d_v}}}{\sum_{uv \in \mathbf{E}(G)} \sqrt{d_u + d_v - 2} - 1} \right)^2 .$$

Therefore

$$\sum_{uv \in \mathbf{E}(G)} \frac{\sqrt{d_u + d_v - 2} - 1}{(\sqrt{d_u + d_v})^2} \geq \left(\frac{\Xi(G)^2}{m(\sqrt{2\Delta - 2} - 1)} \right) \tag{3.2}$$

from equation (3.1) and (3.2) we arrive at the result.

REFERENCES

[1] A. Ali, K. C. Das, D. Dimitrov, B. Furtula, *Atom-bond connectivity index of graphs: a review over extremal results and bounds*, Discr. Math. Lett. **5** (2021) 68–93.

[2] A. Ali, Z. Du, *On the difference between atom-bond connectivity index and Randić index of binary and chemical trees*, Int. J. Quantum Chem. **117** (2017) #e25446.

[3] A. Ali, B. Furtula, I. Redžepović, I. Gutman, *Atom-bond sum-connectivity index*, J. Math. Chem. **60** (2022) 2081–2093.

[4] A. Ali, I. Gutman, I. Redžepović, *Atom-bond sum-connectivity index of unicyclic graphs and some applications*, El. J. Math. (**5** (2023) 1–7.

[5] A. Ali, L. Zhong, I. Gutman, *Harmonic index and its generalization: extremal results and bounds*, MATCH Commun. Math. Comput. Chem. **81** (2019) 249–311.

[6] B. Borovićanin, K. C. Das, B. Furtula, I. Gutman, *Bounds for Zagreb indices*, MATCH Commun. Math. Comput. Chem. **78** (2017) 17–100.

[7] K. C. Das, S. Das, B. Zhou, *Sum-connectivity index of a graph*, Front. Math. China **11**(1) (2016) 47–54.

[8] K. C. Das, N. Trinajstić, *Comparison between the first geometric-arithmetic index and atom-bond connectivity index*, Chem. Phys. Lett. **497** (2010) 149–151.

[9] E. Estrada, L. Torres, L. Rodríguez, I. Gutman, *An atom-bond connectivity index: modelling the enthalpy of formation of alkanes*, Indian J. Chem. **A 37** (1998) 849–855.

[10] S. Fajtlowicz, *On conjectures of Grafti–II*, Congr. Numer. **60** (1987) 187–197.

[11] K. J. Gowtham, I. Gutman, *Relating atom-bond sum-connectivity with other topological indices* Discr. Math. Lett., in press.

[12] I. Gutman, *Degree-based topological indices*, Croat. Chem. Acta **86** (2013) 351–361.

[13] I. Gutman, K. C. Das, *The first Zagreb index 30 years after*, MATCH Commun. Math. Comput. Chem. **50** (2004) 83–92.

[14] I. Gutman, N. Trinajstić, *Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons*, Chem. Phys. Lett. **17** (1972) 535–538.

[15] R. Hasni, M. N. Husin, F. Movahedi, R. Gobithaasan, M. H. Akhbari, *On the difference of atom-bond connectivity index and Randić index with some topological indices*, Iran. J. Math. Chem. **13** (2022) 19–32.

[16] V. R. Kulli, Graph indices, in: M. Pal, S. Samanta, A. Pal (Eds.), *Handbook of Research of Advanced Applications of Graph Theory in Modern Society*, Global, Hershey, 2020, pp. 66–91.

[17] X. Li, Y. Shi, *A survey on the Randić index*, MATCH Commun. Math. Comput. Chem. **59** (2008) 127–156.

[18] B. Liu, M. M. Matejić, E. I. Milovanović, I. Z. Milovanović, *Some new inequalities for the forgotten topological index and coindex of graphs*, MATCH Commun. Math. Comput. Chem. **84** (2020) 719–738.

[19] B. Lučić, S. Nikolić, N. Trinajstić N, B. Zhou, S. I. Turk, *Sum-connectivity index*, in: I. Gutman, B. Furtula (Eds.), *Novel Molecular Structure Descriptors: Theory and Applications I*, Univ. Kragujevac, Kragujevac, 2010, pp. 101–136.

[20] M. Randić, *On characterization of molecular branching*, J. Am. Chem. Soc. **97** (1975) 6609–6615.

[21] M. Randić, *The connectivity index 25 years after*, J. Mol. Graph. Model. **20** (2001) 19–35.

[22] Z. Raza, A. A. Bhatti, A. Ali, *More comparison between the first geometric-arithmetic index and atom-bond connectivity index*, Miskolc Math. Notes **17** (2016) 561–570.

[23] R. Todeschini, V. Consonni, *Handbook of Molecular Descriptors*, Wiley–VCH, Weinheim, 2000.

[24] R. Todeschini, V. Consonni, *Molecular Descriptors for Chemoinformatics*, Wiley–VCH, Weinheim, 2009.

[25] L. Zhong, *The harmonic index for graphs*, Appl. Math. Lett. **25** (2012) 561–566.

[26] L. Zhong, Q. Cui, *On a relation between the atom-bond connectivity and the first geometric-arithmetic indices*, Discr. Appl. Math. **185** (2015) 249–253.

[27] B. Zhou, N. Trinajstić, *On a novel connectivity index*, J. Math. Chem. **46** (2009) 1252–1270.

[28] B. Zhou, N. Trinajstić, *Relations between the product- and sum-connectivity indices*, Croat. Chem. Acta **83** (2012) 363–365.

Department of Mathematics
University College of Science
Tumkur University
Tumakuru 572 103
Karnataka, India
e-mail: gowtham_k_j@yahoo.com

Faculty of Science
University of Kragujevac
P. O. Box 60
34000 Kragujevac
Serbia
e-mail: gutman@kg.ac.rs