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RECIPROCAL CONVOLUTION SUMS OF BERNOULLI AND EULER POLYNOMIALS

WENCHANG CHU

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A b s t r a c t. By inroducing two weight factors, we examine convolution sums about Bernoulli and Euler polynomials. Several reciprocal formulae are established, including polynomial extensions of Miki's well–known identity on Bernoulli numbers.

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1. *Introduction and outline*

In mathematics, physics and computer sciences, Bernoulli and Euler polynomials play an important role (see for example [17, §6.5 & §7.6]). They are defined by the exponential generating functions (cf. [10, §1.14] and [25, §2.5])

$$\frac{T\mathrm{e}^{Tx}}{\mathrm{e}^{T}-1}=\sum_{n\geq 0}B_n(x)\frac{T^n}{n!}\qquad\text{and}\qquad\frac{2\mathrm{e}^{Tx}}{\mathrm{e}^{T}+1}=\sum_{n\geq 0}E_n(x)\frac{T^n}{n!}.$$

The corresponding Bernoulli and Euler numbers are respectively given by

$$\frac{T}{\mathrm{e}^{T}-1}=\sum_{n\geq 0}B_n\frac{T^n}{n!}\qquad\text{and}\qquad\frac{2\mathrm{e}^{T}}{\mathrm{e}^{2T}+1}=\sum_{n\geq 0}E_n\frac{T^n}{n!}.$$

Both polynomials can be written as binomial sums

$$B_n(x) \;=\; \sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k} \quad \text{and} \quad E_n(x) \;=\; \sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\Big(x-\frac{1}{2}\Big)^{n-k},$$

and admit the following binomial relations

$$B_n(x+y) \;=\; \sum_{k=0}^{n}\binom{n}{k}B_k(x)y^{n-k} \quad \text{and} \quad E_n(x+y) \;=\; \sum_{k=0}^{n}\binom{n}{k}E_k(x)y^{n-k}.$$

They satisfy differential equations

$$B_n'(x) \;=\; nB_{n-1}(x) \quad \text{and} \quad E_n'(x) \;=\; nE_{n-1}(x)$$

and also reciprocal relations

$$B_n(1-x) \;=\; (-1)^nB_n(x) \quad \text{and} \quad E_n(1-x) \;=\; (-1)^nE_n(x)$$

as well as difference equations

$$B_n(x)-(-1)^nB_n(-x)=-nx^{n-1} \quad \text{and} \quad E_n(x)+(-1)^nE_n(-x)=2x^n.$$

In addition, Euler polynomials can be expressed in terms of Bernoulli polynomials

$$E_n(x)=\frac{2}{n+1}\Big\{B_{n+1}(x)-2^{n+1}B_{n+1}(x/2)\Big\}.$$

There exist numerous formulae about Bernoulli and Euler numbers/polynomials scattered in the literature (see [18, §50 & §51] and [1]–[3], [6]–[9], [12]). In particular in 1978, Miki [22] discovered a surprising identity on binomial and ordinary convolutions of Bernoulli numbers

$$\sum_{k=2}^{\ell-2}\frac{B_kB_{\ell-k}}{k(\ell-k)}-\sum_{k=2}^{\ell-2}\binom{\ell}{k}\frac{B_kB_{\ell-k}}{k(\ell-k)}=\frac{2H_\ell}{\ell}B_\ell,$$

where H_n stands for th harmonic numbers defined by

$$H_0=0 \quad \text{and} \quad H_n=\sum_{k=1}^{n}\frac{1}{k} \quad \text{for} \quad n\in\mathbb{N}.$$

By making use of MATHEMATICA, Matiyasevich [21] found another similar formula

$$\sum_{k=0}^{\ell}B_kB_{\ell-k}-2\sum_{k=2}^{\ell}\binom{\ell+1}{k+1}\frac{B_kB_{\ell-k}}{k+2}=(\ell+1)B_\ell.$$

These elegant identities spurred several interesting further works. Different proofs and extensions can be found in [11, 13, 14, 15, 16, 19, 20]. Especially, Pan and sun [23, 24] made generalizations of these identities to Bernoulli and Euler polynomials.

By employing the generating function approach, the author [4] made a systematic investigation on Miki–like identities about Bernoulli and Euler numbers/polynomials. The main inspiration came from a useful reciprocity theorem on polynomials (cf. Chu and Magli [5]). By introducing two weight factors, we shall examine convolutions on the four principal summation formulae obtained in [4] (in next four respective sections). Several reciprocal relations for general convolution sums will be established with a few remarkable sample ones being highlighted.

In order to carry out the related computation on convolutions, it is necessary to record some basic facts about the two binomial weight factors. They are briefly reviewed below in the remaining part of this section.

1.1. Binomial sums with weight factor $\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}$

First, it is not difficult to evaluate the two binomial sums below:

$$\sum_{m+n=\ell}\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}\binom{m}{k}=\binom{k+\tau}{\tau}\binom{\ell+\tau+\sigma+1}{\ell-k}, \tag{1.1}$$

$$\sum_{m+n=\ell}\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}\binom{n}{k}=\binom{k+\sigma}{\sigma}\binom{\ell+\tau+\sigma+1}{\ell-k}. \tag{1.2}$$

For $\tau,\sigma,\gamma\in\mathbb{N}_0$, by introducing the binomial sum

$$\Lambda_\gamma\big[\begin{smallmatrix}\tau,\sigma\\ k,\ell\end{smallmatrix}\big]=\sum_{m=\max\{0,k\}}^{\ell}\binom{-\gamma}{m-k}\frac{(\tau+m)!}{\tau!}\frac{(\sigma+\ell-m)!}{\sigma!}, \tag{1.3}$$

which satisfies the reciprocal relation $\Lambda_1\big[\begin{smallmatrix}\tau,\sigma\\ 0,\ell\end{smallmatrix}\big]=(-1)^\ell\Lambda_1\big[\begin{smallmatrix}\sigma,\tau\\ 0,\ell\end{smallmatrix}\big]$, we can express further four binomial sums

$$\sum_{m+n=\ell}m!n!\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}\binom{\delta-\gamma}{m-k}=\Lambda_{\gamma-\delta}\big[\begin{smallmatrix}\tau,\sigma\\ k,\ell\end{smallmatrix}\big], \tag{1.4}$$

$$\sum_{m+n=\ell}m!n!\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}\binom{\delta-\gamma}{n-k}=\Lambda_{\gamma-\delta}\big[\begin{smallmatrix}\sigma,\tau\\ k,\ell\end{smallmatrix}\big]; \tag{1.5}$$

$$\sum_{m+n=\ell} m!n!\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}\binom{-k}{m} = \Lambda_k\left[\begin{smallmatrix}\tau,\sigma\\0,\ell\end{smallmatrix}\right], \tag{1.6}$$

$$\sum_{m+n=\ell} m!n!\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}\binom{-k}{n} = \Lambda_k\left[\begin{smallmatrix}\sigma,\tau\\0,\ell\end{smallmatrix}\right]. \tag{1.7}$$

For the sake of brevity, denote the binomial difference by

$$\lambda_\ell[\tau,\sigma] = \begin{cases} \dfrac{1}{\tau}\dbinom{\ell+\tau+\sigma}{\ell} - \dfrac{1}{\tau}\dbinom{\ell+\sigma}{\ell}, & \tau \neq 0; \\[2ex] \big\{H_{\ell+\sigma} - H_\sigma\big\} \times \dbinom{\ell+\sigma}{\ell}, & \tau = 0. \end{cases} \tag{1.8}$$

Then we have two summation formulae:

$$\sum_{m+n=\ell} \frac{1}{m+1}\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma} = \lambda_{\ell+1}[\tau,\sigma], \tag{1.9}$$

$$\sum_{m+n=\ell} \frac{1}{n+1}\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma} = \lambda_{\ell+1}[\sigma,\tau]; \tag{1.10}$$

which contain the following limiting cases

$$\sum_{m+n=\ell} \frac{1}{m+1}\binom{n+\sigma}{\sigma} = \lambda_{\ell+1}[0,\sigma], \tag{1.11}$$

$$\sum_{m+n=\ell} \frac{1}{n+1}\binom{m+\tau}{\tau} = \lambda_{\ell+1}[0,\tau]. \tag{1.12}$$

In particular, $\Lambda_\gamma\left[\begin{smallmatrix}\tau,\sigma\\k,\ell\end{smallmatrix}\right]$ admits the following special values, that will be useful afterwards in the remaining sections of this paper:

$$\Lambda_0\left[\begin{smallmatrix}\tau,\sigma\\k,\ell\end{smallmatrix}\right] = \begin{cases} 0, & k < 0; \\ \dfrac{(\tau+k)!(\sigma+\ell-k)!}{\tau!\sigma!}, & k \geq 0; \end{cases} \tag{1.13}$$

$$\Lambda_{-1}\left[\begin{smallmatrix}\tau,\sigma\\k,\ell\end{smallmatrix}\right] = \begin{cases} 0, & k < -1; \\ \dfrac{(\sigma+\ell)!}{\sigma!}, & k = -1; \\ \dfrac{(\tau+\ell)!}{\tau!}, & k = \ell; \\ \dfrac{(\tau+k)!(\sigma+\ell-k-1)!(\tau+\sigma+\ell+1)}{\tau!\sigma!}, & 0 \le k < \ell; \end{cases} \tag{1.14}$$

$$\Lambda_{1}\left[\begin{smallmatrix}\tau,\sigma\\k,\ell\end{smallmatrix}\right] = \begin{cases} \dfrac{(-1)^k\tau!(\sigma+\ell+1)!+(-1)^{\ell-k}\sigma!(\tau+\ell+1)!}{\tau!(\tau+\sigma+\ell+2)}, & k < 0; \\ \dfrac{(\tau+k)!(\sigma+\ell-k+1)!+(-1)^{\ell-k}\sigma!(\tau+\ell+1)!}{\tau!\sigma!(\tau+\sigma+\ell+2)}, & k \ge 0. \end{cases} \tag{1.15}$$

1.2. Binomial sums with weight factor $\binom{m+n}{m}p^m q^n$

For two indeterminates p, q, it is routine to verify the binomial identities

$$\sum_{m+n=\ell}\binom{m+n}{m}\binom{m}{k}p^m q^n = \binom{\ell}{k}p^k(p+q)^{\ell-k}, \tag{1.16}$$

$$\sum_{m+n=\ell}\binom{m+n}{m}\binom{n}{k}p^m q^n = \binom{\ell}{k}q^k(p+q)^{\ell-k}; \tag{1.17}$$

and

$$\sum_{m+n=\ell}\binom{m+n}{m}\frac{p^m q^n}{m+1} = \frac{(p+q)^{\ell+1}-q^{\ell+1}}{p(\ell+1)}, \tag{1.18}$$

$$\sum_{m+n=\ell}\binom{m+n}{m}\frac{p^m q^n}{n+1} = \frac{(p+q)^{\ell+1}-p^{\ell+1}}{q(\ell+1)}. \tag{1.19}$$

Then by defining the partial binomial sum

$$\Theta_\gamma\left[\begin{smallmatrix}p,q\\k,\ell\end{smallmatrix}\right] = \sum_{m=\max\{0,k\}}^{\ell}\binom{-\gamma}{m-k}p^m q^{\ell-m}, \tag{1.20}$$

which satisfies the reciprocal relation $\Theta_1\left[\begin{smallmatrix}p,q\\0,\ell\end{smallmatrix}\right] = (-1)^\ell\Theta_1\left[\begin{smallmatrix}q,p\\0,\ell\end{smallmatrix}\right]$, we can readily express the four binomial sums

$$\sum_{m+n=\ell} m!n!\binom{m+n}{m}\binom{\delta-\gamma}{m-k}p^m q^n = \ell!\Theta_{\gamma-\delta}\left[{p,q \atop k,\ell}\right], \tag{1.21}$$

$$\sum_{m+n=\ell} m!n!\binom{m+n}{m}\binom{\delta-\gamma}{n-k}p^m q^n = \ell!\Theta_{\gamma-\delta}\left[{q,p \atop k,\ell}\right]; \tag{1.22}$$

$$\sum_{m+n=\ell} m!n!\binom{m+n}{m}\binom{-k}{m}p^m q^n = \ell!\Theta_{k}\left[{p,q \atop 0,\ell}\right], \tag{1.23}$$

$$\sum_{m+n=\ell} m!n!\binom{m+n}{m}\binom{-k}{n}p^m q^n = \ell!\Theta_{k}\left[{q,p \atop 0,\ell}\right]. \tag{1.24}$$

Taking into account the binomial relation

$$\theta_\ell[p,q] = \sum_{n=0}^{\ell-1}\binom{\ell}{n+1}p^n q^{\ell-n-1} = \frac{(p+q)^\ell - q^\ell}{p}, \tag{1.25}$$

we can also determine the following particular values:

$$\Theta_0\left[{p,q \atop k,\ell}\right] = \begin{cases} 0, & k<0; \\ p^k q^{\ell-k}, & k\geq 0; \end{cases} \tag{1.26}$$

$$\Theta_{-1}\left[{p,q \atop k,\ell}\right] = \begin{cases} 0, & k<-1; \\ q^\ell, & k=-1; \\ p^\ell, & k=\ell; \\ p^k(p+q)p^{\ell-k-1}, & 0\leq k<\ell; \end{cases} \tag{1.27}$$

$$\Theta_1\left[{p,q \atop k,\ell}\right] = \begin{cases} \dfrac{(-1)^{\ell-k}p^{\ell+1}+(-1)^k q^{\ell+1}}{p+q}, & k<0; \\[2ex] \dfrac{(-1)^{\ell-k}p^{\ell+1}+p^k q^{\ell-k+1}}{p+q}, & k\geq 0. \end{cases} \tag{1.28}$$

2. *The first class of reciprocal convolutions*

By making use of the binomial summation formulae presented in the introduction, we shall compute the weighted convolution for the reciprocal equality given in the lemma below and show further reciprocal convolution formulae on Bernoulli polynomials.

Lemma 2.1 (Chu [4, Theorem 9]). *We have*

$$\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m-k+1}(x-y)B_{n+k+\gamma}(y)}{(m-k+1)(n+k+1)_\gamma} \tag{2.1a}$$

$$-\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{B_{n-k+1}(x-y)B_{m+k+\gamma}(x)}{(n-k+1)(m+k+1)_\gamma} \tag{2.1b}$$

$$=m!n!\sum_{k=0}^{m+1}\binom{1-\gamma}{k}\frac{B_{m-k+1}(x)B_{n+k+\gamma}(y)}{(m-k+1)!(n+k+\gamma)!} \tag{2.1c}$$

$$+m!n!\sum_{k=1}^{\gamma}(-1)^{m+n+k}\binom{-k}{m}\frac{B_{\gamma-k}(x)B_{m+n+k+1}(x-y)}{(\gamma-k)!(m+n+k+1)!} \tag{2.1d}$$

$$-\frac{B_{m+n+\gamma+1}(x)}{(n+1)(m+n+2)_\gamma}-\frac{B_{m+n+\gamma+1}(y)}{(m+1)(m+n+2)_\gamma} \tag{2.1e}$$

$$=m!n!\sum_{k=0}^{n+1}\binom{1-\gamma}{k}\frac{B_{n-k+1}(y)B_{m+k+\gamma}(x)}{(n-k+1)!(m+k+\gamma)!} \tag{2.1f}$$

$$-m!n!\sum_{k=1}^{\gamma}\binom{-k}{n}\frac{B_{\gamma-k}(y)B_{m+n+k+1}(x-y)}{(\gamma-k)!(m+n+k+1)!} \tag{2.1g}$$

$$-\frac{B_{m+n+\gamma+1}(y)}{(m+1)(m+n+2)_\gamma}-\frac{B_{m+n+\gamma+1}(x)}{(n+1)(m+n+2)_\gamma}. \tag{2.1h}$$

2.1. Convolution sums with weight factor $\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}$

By multiplying across the equation in Lemma 2.1 by the factor $\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}$, we are going to compute the convolution with respect to $m+n=\ell$. For the sake of brevity, denote by $\Phi(x)$ the expression labeled by "(x)". Then we can proceed with the following computations.

- First, replacing k by $m-k$ in (2.1a), then interchanging the order of summation, and finally evaluating the inner sum by the binomial identity (1.1), we can manipulate the double sum as follows:

$$\begin{aligned}\Phi(2.1\text{a}) &= \sum_{m+n=\ell}\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}\sum_{k=0}^{m}\binom{m}{k}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma}\\
&=\sum_{k=0}^{\ell}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma}\sum_{m+n=\ell}\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}\binom{m}{k}\\
&=\sum_{k=0}^{\ell}\binom{k+\tau}{\tau}\binom{\ell+\tau+\sigma+1}{\ell-k}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma}.\end{aligned}$$

- For (2.1b), we can make analogous computation by making use of (1.2), so that $\Phi(2.1\text{b})$ becomes

$$\begin{aligned}&\sum_{m+n=\ell}\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}\\
&=\sum_{k=0}^{\ell}(-1)^{k+1}\binom{n}{k}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}\sum_{m+n=\ell}\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}\binom{n}{k}\\
&=\sum_{k=0}^{\ell}(-1)^{k+1}\binom{k+\sigma}{\sigma}\binom{\ell+\tau+\sigma+1}{\ell-k}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}.\end{aligned}$$

- The convolution for (2.1c) can be treated, by applying (1.4) as follows:

$$\begin{aligned}\Phi(2.1\text{c}) &= \sum_{m+n=\ell}\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}m!n!\sum_{k=-1}^{m}\binom{1-\gamma}{m-k}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\\
&=\sum_{k=-1}^{\ell}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\sum_{m+n=\ell}m!n!\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}\binom{1-\gamma}{m-k}\\
&=\sum_{k=-1}^{\ell}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\Lambda_{\gamma-1}\left[\begin{smallmatrix}\tau,\sigma\\ k,\ell\end{smallmatrix}\right].\end{aligned}$$

- Instead, the convolution for (2.1d) can be done by employing (1.6), so that $\Phi(2.1\text{d})$

becomes

$$\begin{aligned}
&\sum_{m+n=\ell}\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}m!n!\sum_{k=1}^{\gamma}(-1)^{\ell-k}\binom{-k}{m}\frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\\
&=\sum_{k=1}^{\gamma}(-1)^{\ell-k}\frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\sum_{m+n=\ell}m!n!\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}\binom{-k}{m}\\
&=\sum_{k=1}^{\gamma}(-1)^{\ell-k}\frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\Lambda_k\big[{\tau,\sigma \atop 0,\ell}\big].
\end{aligned}$$

- The convolution for (2.1e) is easier to calculate by appealing to (1.9) and (1.10):

$$\begin{aligned}
\Phi\text{(2.1e)} &= -\sum_{m+n=\ell}\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}\left\{\frac{B_{\ell+\gamma+1}(x)}{(n+1)(\ell+2)_\gamma}+\frac{B_{\ell+\gamma+1}(y)}{(m+1)(\ell+2)_\gamma}\right\}\\
&= -\frac{B_{\ell+\gamma+1}(x)}{(\ell+2)_\gamma}\lambda_{\ell+1}[\sigma,\tau]-\frac{B_{\ell+\gamma+1}(y)}{(\ell+2)_\gamma}\lambda_{\ell+1}[\tau,\sigma].
\end{aligned}$$

- The convolution for (2.1f) can be done as (2.1c) by invoking (1.5) as follows:

$$\begin{aligned}
\Phi\text{(2.1f)} &= \sum_{m+n=\ell}\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}m!n!\sum_{k=-1}^{n}\binom{1-\gamma}{n-k}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\\
&= \sum_{k=-1}^{\ell}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\sum_{m+n=\ell}m!n!\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}\binom{1-\gamma}{n-k}\\
&= \sum_{k=-1}^{\ell}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\Lambda_{\gamma-1}\big[{\sigma,\tau \atop k,\ell}\big].
\end{aligned}$$

- The convolution for (2.1g) can be done as (2.1d) by invoking (1.7) as follows:

$$\begin{aligned}
\Phi\text{(2.1g)} &= -\sum_{m+n=\ell}\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}m!n!\sum_{k=1}^{\gamma}\binom{-k}{n}\frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\\
&= -\sum_{k=1}^{\gamma}\frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\sum_{m+n=\ell}m!n!\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}\binom{-k}{n}\\
&= -\sum_{k=1}^{\gamma}\frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\Lambda_k\big[{\sigma,\tau \atop 0,\ell}\big].
\end{aligned}$$

- Finally for (2.1h), the convolution is the same as that for (2.1e):

$$\Phi\text{(2.1h)} = -\frac{B_{\ell+\gamma+1}(y)}{(\ell+2)_\gamma}\lambda_{\ell+1}[\tau,\sigma]-\frac{B_{\ell+\gamma+1}(x)}{(\ell+2)_\gamma}\lambda_{\ell+1}[\sigma,\tau].$$

Now by equating the resulting expression

$$\Phi(2.1\text{a}) + \Phi(2.1\text{b}) = \Phi(2.1\text{f}) + \Phi(2.1\text{g}) + \Phi(2.1\text{h})$$

we find a general reciprocal relation about convolution sums of Bernoulli polynomials as in the following theorem.

Theorem 2.1 (Reciprocal formula)**.** *We have*

$$\begin{aligned}
&\sum_{k=0}^{\ell}\binom{\tau+k}{\tau}\binom{\ell+\tau+\sigma+1}{\ell-k}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma}\\
&\quad-\sum_{k=0}^{\ell}(-1)^k\binom{\sigma+k}{\sigma}\binom{\ell+\tau+\sigma+1}{\ell-k}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}\\
&=\sum_{k=-1}^{\ell}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\Lambda_{\gamma-1}\left[\begin{smallmatrix}\sigma,\tau\\ k,\ell\end{smallmatrix}\right]-\frac{B_{\ell+\gamma+1}(x)}{(\ell+2)_\gamma}\lambda_{\ell+1}[\sigma,\tau]\\
&\quad-\sum_{k=1}^{\gamma}\frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\Lambda_{k}\left[\begin{smallmatrix}\sigma,\tau\\ 0,\ell\end{smallmatrix}\right]-\frac{B_{\ell+\gamma+1}(y)}{(\ell+2)_\gamma}\lambda_{\ell+1}[\tau,\sigma].
\end{aligned}$$

We remark that if considering the equality

$$\Phi(2.1\text{a}) + \Phi(2.1\text{b}) = \Phi(2.1\text{c}) + \Phi(2.1\text{d}) + \Phi(2.1\text{e}),$$

then we would find another reciprocity formula, equivalent to the last one.

Theorem 2.2 (Reciprocal formula)**.** *We have*

$$\begin{aligned}
&\sum_{k=0}^{\ell}\binom{\tau+k}{\tau}\binom{\ell+\tau+\sigma+1}{\ell-k}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma}\\
&\quad-\sum_{k=0}^{\ell}(-1)^k\binom{\sigma+k}{\sigma}\binom{\ell+\tau+\sigma+1}{\ell-k}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}\\
&=\sum_{k=-1}^{\ell}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\Lambda_{\gamma-1}\left[\begin{smallmatrix}\tau,\sigma\\ k,\ell\end{smallmatrix}\right]-\frac{B_{\ell+\gamma+1}(x)}{(\ell+2)_\gamma}\lambda_{\ell+1}[\sigma,\tau]\\
&\quad+\sum_{k=1}^{\gamma}(-1)^{\ell-k}\frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\Lambda_{k}\left[\begin{smallmatrix}\tau,\sigma\\ 0,\ell\end{smallmatrix}\right]-\frac{B_{\ell+\gamma+1}(y)}{(\ell+2)_\gamma}\lambda_{\ell+1}[\tau,\sigma].
\end{aligned}$$

Both Theorems 2.1 and 2.2 contain many interesting special cases. As a show case, we limit to record the following two common special cases. First, letting $\gamma = 0$ and then making replacements $k \to k-1$ and $\ell \to \ell-1$, we obtain, after some routine simplifications, the following elegant formula.

Proposition 2.1 (Theorems 2.1 or 2.2: $\boldsymbol{\gamma = 0}$)**.** *We have*

$$\begin{aligned}(\tau+\sigma+\ell)\sum_{k=0}^{\ell}&\binom{\tau+k-1}{k}\binom{\sigma+\ell-k-1}{\ell-k}B_k(x)B_{\ell-k}(y)\\&=\sigma\sum_{k=0}^{\ell}\binom{\tau+k-1}{k}\binom{\tau+\sigma+\ell}{\ell-k}B_k(x-y)B_{\ell-k}(y)\\&\quad+\tau\sum_{k=0}^{\ell}(-1)^k\binom{\sigma+k-1}{k}\binom{\tau+\sigma+\ell}{\ell-k}B_k(x-y)B_{\ell-k}(x).\end{aligned}$$

We point out that the special case $\sigma = 1$ of this formula has previously been obtained by Sun and Pan [24, Eq. 1.5]. Next, letting $\gamma = 1$ and $\tau = \sigma = 0$, we recover another identity, whose particular case corresponding to $x = 1/2$ and $y = 0$ can be found in Donne [14].

Corollary 2.1 (Theorems 2.1 or 2.2: $\boldsymbol{\gamma = 1}$ and $\boldsymbol{\tau = \sigma = 0}$)**.** *We have*

$$\begin{aligned}\sum_{k=0}^{\ell+1}&\binom{\ell+1}{k}\frac{B_{k+1}(x-y)}{(k+1)^2}\Big\{B_{\ell-k+1}(y)-(-1)^kB_{\ell-k+1}(x)\Big\}\\&=\sum_{k=0}^{\ell}\frac{B_{k+1}(x)B_{\ell-k+1}(y)}{(k+1)(\ell-k+1)}-\frac{H_{\ell+1}}{\ell+2}\Big\{B_{\ell+2}(x)+B_{\ell+2}(y)\Big\}.\end{aligned}$$

2.2. Convolution sums with weight factor $\binom{m+n}{m}p^mq^n$

Alternatively by multiplying across the equation in Lemma 2.1 by the weight factor $\binom{m+n}{m}p^mq^n$, we can analogously calculate the convolution with respect to $m+n=\ell$, where we use $\Psi(x)$ to represent the expression labeled by "(x)"

- Starting by the replacement $k \to m-k$ in (2.1a), then exchanging the order of summation, and at last, calculating the inner sum by the binomial identity (1.16), we can reformulate the double sum as follows:

$$\begin{aligned}\Psi(\text{2.1a})&=\sum_{m+n=\ell}\binom{m+n}{m}p^mq^n\sum_{k=0}^{m}\binom{m}{k}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma}\\&=\sum_{k=0}^{\ell}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma}\sum_{m+n=\ell}\binom{m+n}{m}\binom{m}{k}p^mq^n\\&=\sum_{k=0}^{\ell}\binom{\ell}{k}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma}p^k(p+q)^{\ell-k}.\end{aligned}$$

- Analogously, we can compute the convolution for (2.1b) by means of (1.17):

$$\begin{aligned}\Psi(2.1\mathrm{b}) &= \sum_{m+n=\ell}\binom{m+n}{m}p^m q^n \sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}\\ &= \sum_{k=0}^{\ell}(-1)^{k+1}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}\sum_{m+n=\ell}\binom{m+n}{m}\binom{n}{k}p^m q^n\\ &= \sum_{k=0}^{\ell}(-1)^{k+1}\binom{\ell}{k}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}q^k(p+q)^{\ell-k}.\end{aligned}$$

- By applying (1.21), we can deal with the convolution for (2.1c) as follows:

$$\begin{aligned}\Psi(2.1\mathrm{c}) &= \sum_{m+n=\ell} m!n!\binom{m+n}{m}p^m q^n \sum_{k=-1}^{m}\binom{1-\gamma}{m-k}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\\ &= \sum_{k=-1}^{\ell}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\sum_{m+n=\ell} m!n!\binom{m+n}{m}\binom{1-\gamma}{m-k}p^m q^n\\ &= \ell!\sum_{k=-1}^{\ell}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\Theta_{\gamma-1}\left[\begin{smallmatrix}p,q\\ k,\ell\end{smallmatrix}\right].\end{aligned}$$

- Instead, the convolution for (2.1d) can be done by applying (1.23), so that $\Psi(2.1\mathrm{d})$ becomes

$$\begin{aligned}&\sum_{m+n=\ell} m!n!\binom{m+n}{m}p^m q^n \sum_{k=1}^{\gamma}(-1)^{\ell-k}\binom{-k}{m}\frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\\ &= \sum_{k=1}^{\gamma}(-1)^{\ell-k}\frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\sum_{m+n=\ell} m!n!\binom{m+n}{m}\binom{-k}{m}p^m q^n\\ &= \ell!\sum_{k=1}^{\gamma}(-1)^{\ell-k}\frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\Theta_k\left[\begin{smallmatrix}p,q\\ 0,\ell\end{smallmatrix}\right].\end{aligned}$$

- Now we turn to evaluate the convolution for (2.1e) by invoking (1.18) and (1.19):

$$\begin{aligned}\Psi(2.1\mathrm{e}) &= -\sum_{m+n=\ell}\binom{m+n}{m}p^m q^n\left\{\frac{B_{\ell+\gamma+1}(x)}{(n+1)(\ell+2)_\gamma}+\frac{B_{\ell+\gamma+1}(y)}{(m+1)(\ell+2)_\gamma}\right\}\\ &= -\frac{B_{\ell+\gamma+1}(x)}{(\ell+2)_\gamma}\frac{(p+q)^{\ell+1}-p^{\ell+1}}{q(\ell+1)}-\frac{B_{\ell+\gamma+1}(y)}{(\ell+2)_\gamma}\frac{(p+q)^{\ell+1}-q^{\ell+1}}{p(\ell+1)}.\end{aligned}$$

- The convolution for (2.1f) can be treated as (2.1c) by applying (1.22) as follows:

$$\begin{aligned}
\Psi(2.1\mathrm{f}) &= \sum_{m+n=\ell} m!n!\binom{m+n}{m}p^m q^n \sum_{k=-1}^{n}\binom{1-\gamma}{n-k}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\\
&= \sum_{k=-1}^{\ell}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\sum_{m+n=\ell} m!n!\binom{m+n}{m}\binom{1-\gamma}{n-k}p^m q^n\\
&= \ell!\sum_{k=-1}^{\ell}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\Theta_{\gamma-1}\big[\begin{smallmatrix}q,p\\ k,\ell\end{smallmatrix}\big].
\end{aligned}$$

- The convolution for (2.1g) can be done as (2.1d) by employing (1.24) as follows:

$$\begin{aligned}
\Psi(2.1\mathrm{g}) &= -\sum_{m+n=\ell} m!n!\binom{m+n}{m}p^m q^n \sum_{k=1}^{\gamma}\binom{-k}{n}\frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\\
&= -\sum_{k=1}^{\gamma}\frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\sum_{m+n=\ell} m!n!\binom{m+n}{m}\binom{-k}{n}p^m q^n\\
&= -\ell!\sum_{k=1}^{\gamma}\frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\Theta_{k}\big[\begin{smallmatrix}q,p\\ 0,\ell\end{smallmatrix}\big].
\end{aligned}$$

- Finally, the convolution for (2.1h) is identical to that for (2.1e):

$$\Psi(2.1\mathrm{h}) = -\frac{B_{\ell+\gamma+1}(x)}{(\ell+2)_\gamma}\frac{(p+q)^{\ell+1}-p^{\ell+1}}{q(\ell+1)} - \frac{B_{\ell+\gamma+1}(y)}{(\ell+2)_\gamma}\frac{(p+q)^{\ell+1}-q^{\ell+1}}{p(\ell+1)}.$$

According to the equality

$$\Psi(2.1\mathrm{a})+\Psi(2.1\mathrm{b}) = \Psi(2.1\mathrm{f})+\Psi(2.1\mathrm{g})+\Psi(2.1\mathrm{h})$$

we establish the following reciprocal relation about convolution sums of Bernoulli polynomials.

Theorem 2.3 (Reciprocal formula)**.** *We have*

$$\begin{aligned}
&\sum_{k=0}^{\ell}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}p^k(p+q)^{\ell-k}\\
&\quad-\sum_{k=0}^{\ell}(-1)^k\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}q^k(p+q)^{\ell-k}\\
&=\sum_{k=-1}^{\ell}\frac{B_{k+1}(y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\Theta_{\gamma-1}\left[\begin{smallmatrix}q,p\\k,\ell\end{smallmatrix}\right]-\frac{B_{\ell+\gamma+1}(x)}{(\ell+\gamma+1)!}\theta_{\ell+1}[q,p]\\
&\quad-\sum_{k=1}^{\gamma}\frac{B_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\Theta_k\left[\begin{smallmatrix}q,p\\0,\ell\end{smallmatrix}\right]-\frac{B_{\ell+\gamma+1}(y)}{(\ell+\gamma+1)!}\theta_{\ell+1}[p,q].
\end{aligned}$$

Alternatively, from the equality

$$\Psi(2.1\text{a})+\Psi(2.1\text{b})=\Psi(2.1\text{c})+\Psi(2.1\text{d})+\Psi(2.1\text{e})$$

we derive another reciprocal relation, equivalent to the one displayed in Theorem 2.3.

Theorem 2.4 (Reciprocal formula)**.** *We have*

$$\begin{aligned}
&\sum_{k=0}^{\ell}\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}p^k(p+q)^{\ell-k}\\
&\quad-\sum_{k=0}^{\ell}(-1)^k\frac{B_{k+1}(x-y)B_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}q^k(p+q)^{\ell-k}\\
&=\sum_{k=-1}^{\ell}\frac{B_{k+1}(x)B_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\Theta_{\gamma-1}\left[\begin{smallmatrix}p,q\\k,\ell\end{smallmatrix}\right]-\frac{B_{\ell+\gamma+1}(x)}{(\ell+\gamma+1)!}\theta_{\ell+1}[q,p]\\
&\quad+\sum_{k=1}^{\gamma}(-1)^{\ell-k}\frac{B_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\Theta_k\left[\begin{smallmatrix}p,q\\0,\ell\end{smallmatrix}\right]-\frac{B_{\ell+\gamma+1}(y)}{(\ell+\gamma+1)!}\theta_{\ell+1}[p,q].
\end{aligned}$$

When $\gamma=0$, we deduce from Theorems 2.3 and 2.4 the following common reciprocal relation after having made replacements $k\to k-1$ and $\ell\to\ell-1$.

Proposition 2.2 (Theorems 2.3 or 2.4: $\gamma = 0$)**.** *We have*

$$\begin{aligned}(p+q)\sum_{k=0}^{\ell}\binom{\ell}{k}B_k(x)B_{\ell-k}(y)p^kq^{\ell-k}\\ &= q\sum_{k=0}^{\ell}\binom{\ell}{k}B_k(x-y)B_{\ell-k}(y)p^k(p+q)^{\ell-k}\\ &\quad + p\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}B_k(x-y)B_{\ell-k}(x)q^k(p+q)^{\ell-k}.\end{aligned}$$

Henceforth, the same procedure will be carried out in the next three sections to evaluate the convolutions and derive reciprocal formulae. The details will not be produced since the computations involved are almost identical.

3. *The second class of reciprocal convolutions*

By computing the convolutions on the equality stated in the lemma below, we shall illustrate two classes of reciprocal formulae and the related implications.

Lemma 3.1 (Chu [4, Theorem 23])**.** *We have*

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}\frac{E_{m-k}(x-y)B_{n+k+\gamma}(y)}{(n+k+1)_\gamma}\\ &- \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{E_{n-k}(x-y)B_{m+k+\gamma}(x)}{(m+k+1)_\gamma}\\ &= -\frac{m!n!}{2}\sum_{k=0}^{m}\binom{1-\gamma}{k}\frac{E_{m-k}(x)E_{n+k+\gamma-1}(y)}{(m-k)!(n+k+\gamma-1)!}\\ &\quad + m!n!\sum_{k=1}^{\gamma}(-1)^{m+n+k}\binom{-k}{m}\frac{B_{\gamma-k}(x)E_{m+n+k}(x-y)}{(\gamma-k)!(m+n+k)!}\\ &= -\frac{m!n!}{2}\sum_{k=0}^{n}\binom{1-\gamma}{k}\frac{E_{n-k}(y)E_{m+k+\gamma-1}(x)}{(n-k)!(m+k+\gamma-1)!}\\ &\quad - m!n!\sum_{k=1}^{\gamma}\binom{-k}{n}\frac{B_{\gamma-k}(y)E_{m+n+k}(x-y)}{(\gamma-k)!(m+n+k)!}.\end{aligned}$$

3.1. *Convolution sums with weight factor* $\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}$

The main results are enunciated below by computing the convolution sums on the above weight factor.

Theorem 3.1 (Reciprocal formulae)**.** *We have*

$$
\begin{aligned}
&\sum_{k=0}^{\ell}(-1)^k\binom{k+\sigma}{k}\binom{\ell+\tau+\sigma+1}{\ell-k}\frac{E_k(x-y)B_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma}\\
&\qquad-\sum_{k=0}^{\ell}\binom{k+\tau}{k}\binom{\ell+\tau+\sigma+1}{\ell-k}\frac{E_k(x-y)B_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma}\\
&=\frac{1}{2}\sum_{k=0}^{\ell}\frac{E_k(y)E_{\ell-k+\gamma-1}(x)}{k!(\ell-k+\gamma-1)!}\Lambda_{\gamma-1}\big[{}^{\sigma,\tau}_{k,\ell}\big]+\sum_{k=1}^{\gamma}\frac{B_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}\Lambda_k\big[{}^{\sigma,\tau}_{0,\ell}\big]\\
&=\frac{1}{2}\sum_{k=0}^{\ell}\frac{E_k(x)E_{\ell-k+\gamma-1}(y)}{k!(\ell-k+\gamma-1)!}\Lambda_{\gamma-1}\big[{}^{\tau,\sigma}_{k,\ell}\big]\\
&\qquad-\sum_{k=1}^{\gamma}(-1)^{\ell-k}\frac{B_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}\Lambda_k\big[{}^{\tau,\sigma}_{0,\ell}\big].
\end{aligned}
$$

Proposition 3.1 (Theorem 3.1: $\boldsymbol{\gamma=0}$)**.** *We have*

$$
\begin{aligned}
&\frac{\tau+\sigma+\ell+1}{2}\sum_{k=1}^{\ell}\binom{\tau+k-1}{k-1}\binom{\sigma+\ell-k}{\ell-k}E_{k-1}(x)E_{\ell-k}(y)\\
&\qquad=\sum_{k=0}^{\ell}(-1)^k\binom{\sigma+k}{k}\binom{\tau+\sigma+\ell+1}{\ell-k}E_k(x-y)B_{\ell-k}(x)\\
&\qquad\qquad-\sum_{k=0}^{\ell}\binom{\tau+k}{k}\binom{\tau+\sigma+\ell+1}{\ell-k}E_k(x-y)B_{\ell-k}(y).
\end{aligned}
$$

Corollary 3.1 (Theorem 3.1: $\boldsymbol{\gamma=1}$ and $\boldsymbol{\tau=\sigma=0}$)**.** *We have*

$$
\begin{aligned}
&\sum_{k=0}^{\ell+1}\binom{\ell+2}{k+1}E_k(x-y)\Big\{B_{\ell-k+1}(y)-(-1)^kB_{\ell-k+1}(x)\Big\}\\
&\qquad\qquad\qquad=-\frac{\ell+2}{2}\sum_{k=0}^{\ell}E_k(x)E_{\ell-k}(y).
\end{aligned}
$$

3.2. Convolution sums with weight factor $\binom{m+n}{m}p^mq^n$

Another convolution sum on the above weight factor results in the following theorem.

Theorem 3.2 (Reciprocal formulae)**.** *We have*

$$\sum_{k=0}^{\ell}(-1)^k \frac{E_k(x-y)B_{\ell-k+\gamma}(x)}{k!(\ell-k+\gamma)!}q^k(p+q)^{\ell-k} - \sum_{k=0}^{\ell}\frac{E_k(x-y)B_{\ell-k+\gamma}(y)}{k!(\ell-k+\gamma)!}p^k(p+q)^{\ell-k}$$

$$= \frac{1}{2}\sum_{k=0}^{\ell}\frac{E_k(y)E_{\ell-k+\gamma-1}(x)}{k!(\ell-k+\gamma-1)!}\Theta_{\gamma-1}\left[\begin{smallmatrix}q,p\\k,\ell\end{smallmatrix}\right] + \sum_{k=1}^{\gamma}\frac{B_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}\Theta_k\left[\begin{smallmatrix}q,p\\0,\ell\end{smallmatrix}\right]$$

$$= \frac{1}{2}\sum_{k=0}^{\ell}\frac{E_k(x)E_{\ell-k+\gamma-1}(y)}{k!(\ell-k+\gamma-1)!}\Theta_{\gamma-1}\left[\begin{smallmatrix}p,q\\k,\ell\end{smallmatrix}\right] - \sum_{k=1}^{\gamma}(-1)^{\ell-k}\frac{B_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}\Theta_k\left[\begin{smallmatrix}p,q\\0,\ell\end{smallmatrix}\right].$$

Also, the following statement holds.

Proposition 3.2 (Theorem 3.2: $\boldsymbol{\gamma = 0}$)**.** *We have*

$$\frac{p+q}{2p}\sum_{k=1}^{\ell}\binom{\ell}{k}kE_{k-1}(x)E_{\ell-k}(y)\,p^kq^{\ell-k}$$

$$= \sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}E_k(x-y)B_{\ell-k}(x)\,q^k(p+q)^{\ell-k} - \sum_{k=0}^{\ell}\binom{\ell}{k}E_k(x-y)B_{\ell-k}(y)\,p^k(p+q)^{\ell-k}.$$

4. *The third class of reciprocal convolutions*

The third convolution sums are examined in this section based on the lemma below.

Lemma 4.1 (Chu [4, Theorem 37]). *We have*

$$
\begin{aligned}
&2\sum_{k=0}^{m}\binom{m}{k}\frac{B_{m-k+1}(x-y)E_{n+k+\gamma}(y)}{(m-k+1)(n+k+1)_\gamma}\\
&\quad+\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{E_{n-k}(x-y)E_{m+k+\gamma}(x)}{(m+k+1)_\gamma}\\
&=2m!n!\sum_{k=0}^{m+1}\binom{-\gamma}{k}\frac{B_{m-k+1}(x)E_{n+k+\gamma}(y)}{(m-k+1)!(n+k+\gamma)!}-\frac{2E_{m+n+\gamma+1}(y)}{(m+1)(m+n+2)_\gamma}\\
&\quad-m!n!\sum_{k=1}^{\gamma}(-1)^{m+n+k}\binom{-k}{m}\frac{E_{\gamma-k}(x)E_{m+n+k}(x-y)}{(\gamma-k)!(m+n+k)!}\\
&=2m!n!\sum_{k=0}^{n}\binom{-\gamma}{k}\frac{E_{n-k}(y)B_{m+k+\gamma+1}(x)}{(n-k)!(m+k+\gamma+1)!}-\frac{2E_{m+n+\gamma+1}(y)}{(m+1)(m+n+2)_\gamma}\\
&\quad-2m!n!\sum_{k=1}^{\gamma}\binom{-k}{n}\frac{E_{\gamma-k}(y)B_{m+n+k+1}(x-y)}{(\gamma-k)!(m+n+k+1)!}.
\end{aligned}
$$

4.1. Convolution sums with weight factor $\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}$

We are going to highlight the main results derived by computing the convolutions containing the above weight factor.

Theorem 4.1 (Reciprocal formulae). *We have*

$$
\begin{aligned}
&2\sum_{k=0}^{\ell}\binom{k+\tau}{k}\binom{\ell+\tau+\sigma+1}{\ell-k}\frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(y)}{(k+1)(\ell-k+1)_\gamma}\\
&\quad+\sum_{k=0}^{\ell}(-1)^{k}\binom{k+\sigma}{k}\binom{\ell+\tau+\sigma+1}{\ell-k}\frac{E_{k}(x-y)E_{\ell-k+\gamma}(x)}{(\ell-k+1)_\gamma}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\sum_{k=0}^{\ell} \frac{E_k(y)B_{\ell-k+\gamma+1}(x)}{k!(\ell-k+\gamma+1)!}\Lambda_\gamma\left[\begin{smallmatrix}\sigma,\tau\\ k,\ell\end{smallmatrix}\right] - \frac{2E_{\ell+\gamma+1}(y)}{(\ell+2)_\gamma}\lambda_{\ell+1}[\tau,\sigma] \\
&\quad - 2\sum_{k=1}^{\gamma} \frac{E_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\Lambda_k\left[\begin{smallmatrix}\sigma,\tau\\ 0,\ell\end{smallmatrix}\right] \\
&= 2\sum_{k=-1}^{\ell} \frac{B_{k+1}(x)E_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\Lambda_\gamma\left[\begin{smallmatrix}\tau,\sigma\\ k,\ell\end{smallmatrix}\right] - \frac{2E_{\ell+\gamma+1}(y)}{(\ell+2)_\gamma}\lambda_{\ell+1}[\tau,\sigma] \\
&\quad - \sum_{k=1}^{\gamma}(-1)^{\ell-k} \frac{E_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}\Lambda_k\left[\begin{smallmatrix}\tau,\sigma\\ 0,\ell\end{smallmatrix}\right].
\end{aligned}
$$

The first example is stated as in the proposition below.

Proposition 4.1 (Theorem 4.1: $\boldsymbol{\gamma = 0}$)**.** *We have*

$$
\begin{aligned}
&2\sum_{k=0}^{\ell}\binom{\tau+k-1}{k}\binom{\sigma+\ell-k}{\ell-k}B_k(x)E_{\ell-k}(y) \\
&\quad = 2\sum_{k=0}^{\ell}\binom{\tau+k-1}{k}\binom{\tau+\sigma+\ell}{\ell-k}B_k(x-y)E_{\ell-k}(y) \\
&\qquad - \tau\sum_{k=1}^{\ell}(-1)^k\binom{\sigma+k-1}{k-1}\binom{\tau+\sigma+\ell}{\ell-k}E_{k-1}(x-y)E_{\ell-k}(x).
\end{aligned}
$$

The next example can be found in Pan–Sun [23, Eq. 2.6].

Corollary 4.1 (Theorem 4.1: $\boldsymbol{\gamma = 1}$ and $\boldsymbol{\tau = \sigma = 0}$)**.** *We have*

$$
\begin{aligned}
&2\sum_{k=0}^{\ell}\binom{\ell+1}{k+1}\frac{B_{k+1}(x-y)E_{\ell-k}(y)}{k+1} - \sum_{k=0}^{\ell}\binom{\ell+1}{k+1}E_k(y-x)E_{\ell-k}(x) \\
&\quad = 2\sum_{k=0}^{\ell}\frac{B_{k+1}(x)E_{\ell-k}(y)}{k+1} - 2\frac{E_{\ell+1}(x)-E_{\ell+1}(y)}{(x-y)} - 2H_{\ell+1}E_{\ell+1}(y).
\end{aligned}
$$

4.2. Convolution sums with weight factor $\binom{m+n}{m}p^m q^n$

Instead, the convolutions with the above weight factor bring us to the theorem and proposition below.

Theorem 4.2 (Reciprocal formulae)**.** *We have*

$$\begin{aligned}
&2\sum_{k=0}^{\ell}\frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}p^k(p+q)^{\ell-k}\\
&\qquad+\sum_{k=0}^{\ell}(-1)^k\frac{E_k(x-y)E_{\ell-k+\gamma}(x)}{k!(\ell-k+\gamma)!}q^k(p+q)^{\ell-k}\\
&=2\sum_{k=0}^{\ell}\frac{E_k(y)B_{\ell-k+\gamma+1}(x)}{k!(\ell-k+\gamma+1)!}\Theta_\gamma\left[\begin{smallmatrix}q,p\\k,\ell\end{smallmatrix}\right]-2\frac{E_{\ell+\gamma+1}(y)}{(\ell+\gamma+1)!}\theta_{\ell+1}[p,q]\\
&\qquad-2\sum_{k=1}^{\gamma}\frac{E_{\gamma-k}(y)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\Theta_k\left[\begin{smallmatrix}q,p\\0,\ell\end{smallmatrix}\right]\\
&=2\sum_{k=-1}^{\ell}\frac{B_{k+1}(x)E_{\ell-k+\gamma}(y)}{(k+1)!(\ell-k+\gamma)!}\Theta_\gamma\left[\begin{smallmatrix}p,q\\k,\ell\end{smallmatrix}\right]-2\frac{E_{\ell+\gamma+1}(y)}{(\ell+\gamma+1)!}\theta_{\ell+1}[p,q]\\
&\qquad-\sum_{k=1}^{\gamma}(-1)^{\ell-k}\frac{E_{\gamma-k}(x)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}\Theta_k\left[\begin{smallmatrix}p,q\\0,\ell\end{smallmatrix}\right].
\end{aligned}$$

Proposition 4.2 (Theorem 4.2: $\boldsymbol{\gamma=0}$)**.** *We have*

$$\begin{aligned}
&2q\sum_{k=0}^{\ell}\binom{\ell}{k}B_k(x)E_{\ell-k}(y)\,p^kq^{\ell-k}\\
&\quad=2q\sum_{k=0}^{\ell}\binom{\ell}{k}B_k(x-y)E_{\ell-k}(y)\,p^k(p+q)^{\ell-k}\\
&\qquad-p\sum_{k=1}^{\ell}(-1)^k\binom{\ell}{k}kE_{k-1}(x-y)E_{\ell-k}(x)\,q^k(p+q)^{\ell-k}.
\end{aligned}$$

5. *The fourth class of reciprocal convolutions*

Finally, we examine the convolution sums based on the lemma below.

Lemma 5.1 (Chu [4, Theorem 51])**.** *We have*

$$\begin{aligned}
&\sum_{k=0}^{m}\binom{m}{k}\frac{E_{m-k}(x-y)E_{n+k+\gamma}(y)}{(n+k+1)_\gamma}\\
&\qquad+2\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{B_{n-k+1}(x-y)E_{m+k+\gamma}(x)}{(n-k+1)(m+k+1)_\gamma}
\end{aligned}$$

$$
\begin{aligned}
=&-2m!n!\sum_{k=0}^{m}\binom{-\gamma}{k}\frac{E_{m-k}(x)B_{n+k+\gamma+1}(y)}{(m-k)!(n+k+\gamma+1)!}+\frac{2E_{m+n+\gamma+1}(x)}{(n+1)(m+n+2)_\gamma}\\
&-2m!n!\sum_{k=1}^{\gamma}(-1)^{m+n+k}\binom{-k}{m}\frac{E_{\gamma-k}(x)B_{m+n+k+1}(x-y)}{(\gamma-k)!(m+n+k+1)!}\\
=&-2m!n!\sum_{k=0}^{n+1}\binom{-\gamma}{k}\frac{B_{n-k+1}(y)E_{m+k+\gamma}(x)}{(n-k+1)!(m+k+\gamma)!}+\frac{2E_{m+n+\gamma+1}(x)}{(n+1)(m+n+2)_\gamma}\\
&-m!n!\sum_{k=1}^{\gamma}\binom{-k}{n}\frac{E_{\gamma-k}(y)E_{m+n+k}(x-y)}{(\gamma-k)!(m+n+k)!}.
\end{aligned}
$$

5.1. Convolution sums with weight factor $\binom{m+\tau}{\tau}\binom{n+\sigma}{\sigma}$

The main reciprocal formulae are given as in the theorem below.

Theorem 5.1 (Reciprocal formulae)**.** *We have*

$$
\begin{aligned}
&\sum_{k=0}^{\ell}\binom{k+\tau}{k}\binom{\ell+\tau+\sigma+1}{\ell-k}\frac{E_k(x-y)E_{\ell-k+\gamma}(y)}{(\ell-k+1)_\gamma}\\
&\qquad+2\sum_{k=0}^{\ell}(-1)^k\binom{k+\sigma}{k}\binom{\ell+\tau+\sigma+1}{\ell-k}\frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(x)}{(k+1)(\ell-k+1)_\gamma}\\
&=\frac{2E_{\ell+\gamma+1}(x)}{(\ell+2)_\gamma}\lambda_{\ell+1}[\sigma,\tau]-2\sum_{k=-1}^{\ell}\frac{B_{k+1}(y)E_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\Lambda_\gamma\left[{\sigma,\tau \atop k,\ell}\right]\\
&\qquad-\sum_{k=1}^{\gamma}\frac{E_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}\Lambda_k\left[{\sigma,\tau \atop 0,\ell}\right]\\
&=\frac{2E_{\ell+\gamma+1}(x)}{(\ell+2)_\gamma}\lambda_{\ell+1}[\sigma,\tau]-2\sum_{k=0}^{\ell}\frac{E_k(x)B_{\ell-k+\gamma+1}(y)}{k!(\ell-k+\gamma+1)!}\Lambda_\gamma\left[{\tau,\sigma \atop k,\ell}\right]\\
&\qquad-2\sum_{k=1}^{\gamma}(-1)^{\ell-k}\frac{E_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\Lambda_k\left[{\tau,\sigma \atop 0,\ell}\right].
\end{aligned}
$$

Letting $\gamma=0$ and then making some simplifications, we find the following formula.

Proposition 5.1 (Theorem 5.1: $\boldsymbol{\gamma = 0}$)**.** *We have*

$$2\sum_{k=0}^{\ell}\binom{\tau+k}{k}\binom{\sigma+\ell-k-1}{\ell-k}E_k(x)B_{\ell-k}(y)$$
$$=2\sum_{k=0}^{\ell}(-1)^k\binom{\sigma+k-1}{k}\binom{\ell+\tau+\sigma}{\ell-k}B_k(x-y)E_{\ell-k}(x)$$
$$-\sigma\sum_{k=1}^{\ell}\binom{\tau+k-1}{k-1}\binom{\ell+\tau+\sigma}{\ell-k}E_{k-1}(x-y)E_{\ell-k}(y).$$

Instead, if considering the limiting case $\gamma=\tau=\sigma\to 0$ directly from Theorem 5.1, we find another remarkable identity, which is not deducible by Proposition 5.1.

Corollary 5.1 (Theorem 5.1: $\boldsymbol{\gamma=\tau=\sigma=0}$)**.** *We have*

$$\sum_{k=0}^{\ell}\binom{\ell+1}{k+1}E_k(x-y)E_{\ell-k}(y)+2\sum_{k=0}^{\ell}(-1)^k\binom{\ell+1}{k+1}\frac{B_{k+1}(x-y)E_{\ell-k}(x)}{k+1}$$
$$=2H_{\ell+1}E_{\ell+1}(x)-2\sum_{k=0}^{\ell}\frac{B_{k+1}(y)E_{\ell-k}(x)}{k+1}.$$

5.2. Convolution sums with weight factor $\binom{m+n}{m}p^mq^n$

Alternatively, the convolutions based on this weight factor yield the following reciprocal formulae.

Theorem 5.2 (Reciprocal formulae)**.** *We have*

$$\sum_{k=0}^{\ell}\frac{E_k(x-y)E_{\ell-k+\gamma}(y)}{k!(\ell-k+\gamma)!}p^k(p+q)^{\ell-k}$$
$$+2\sum_{k=0}^{\ell}(-1)^k\frac{B_{k+1}(x-y)E_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}q^k(p+q)^{\ell-k}$$

$$= 2\frac{E_{\ell+\gamma+1}(x)}{(\ell+\gamma+1)!}\theta_{\ell+1}[q,p] - 2\sum_{k=-1}^{\ell}\frac{B_{k+1}(y)E_{\ell-k+\gamma}(x)}{(k+1)!(\ell-k+\gamma)!}\Theta_\gamma\left[\begin{smallmatrix}q,p\\k,\ell\end{smallmatrix}\right]$$
$$-\sum_{k=1}^{\gamma}\frac{E_{\gamma-k}(y)E_{\ell+k}(x-y)}{(\gamma-k)!(\ell+k)!}\Theta_k\left[\begin{smallmatrix}q,p\\0,\ell\end{smallmatrix}\right]$$
$$= 2\frac{E_{\ell+\gamma+1}(x)}{(\ell+\gamma+1)!}\theta_{\ell+1}[q,p] - 2\sum_{k=0}^{\ell}\frac{E_k(x)B_{\ell-k+\gamma+1}(y)}{k!(\ell-k+\gamma+1)!}\Theta_\gamma\left[\begin{smallmatrix}p,q\\k,\ell\end{smallmatrix}\right]$$
$$-2\sum_{k=1}^{\gamma}(-1)^{\ell-k}\frac{E_{\gamma-k}(x)B_{\ell+k+1}(x-y)}{(\gamma-k)!(\ell+k+1)!}\Theta_k\left[\begin{smallmatrix}p,q\\0,\ell\end{smallmatrix}\right].$$

Proposition 5.2 (Theorem 5.2: $\gamma = 0$). *We have*

$$2p\sum_{k=0}^{\ell}\binom{\ell}{k}E_k(x)B_{\ell-k}(y)\,p^kq^{\ell-k}$$
$$= 2p\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}B_k(x-y)E_{\ell-k}(x)\,q^k(p+q)^{\ell-k}$$
$$-q\sum_{k=1}^{\ell}\binom{\ell}{k}kE_{k-1}(x-y)E_{\ell-k}(y)\,p^k(p+q)^{\ell-k}.$$

Concluding comments

Based on the summation formulae obtained earlier by the author [4], we examined convolutions by introducing two weight factors. Several reciprocal convolution identities are illustrated. It is natural to investigate what would happen next if considering different weight factors. The interested reader is encouraged to make further explorations.

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School of Mathematics and Statistics
Zhoukou Normal University
Henan, China
&
Department of Mathematics and Physics
University of Salento
Lecce 73100, Italy
e-mail: `chu.wenchang@unisalento.it`