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INFINITE TOPOLOGICAL DIRECT SUM OF SUBSPACES

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A b s t r a c t. We define and investigate infinite topological direct sum of subspaces of a normed space, as well as corresponding matrix decompositions of bounded linear operators.

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1. *Introduction*

The notion of a topological direct sum of closed subspaces of a normed space is well-known. The situation of a finite number of subspaces is presented in more details in [2]. On the other hand, we did not find a systematic treatment of results concerning the infinite sum of subspaces. Hence, we define and investigate the infinite sum of subspaces of a normed space. It is well-known that decompositions of a space induce decompositions of linear operators over the same space. For this reason we investigate matrix decompositions of bounded linear operators induced by infinite direct sums of subspaces.

In this paper all vector spaces will be over the field $\mathbb{F}$ which is equal to $\mathbb{R}$ or $\mathbb{C}$. I will be an arbitrary index set, X, Y will be normed spaces over $\mathbb{F}$, and $(X_i)_{i\in I}$ will be a family of vector subspaces of X.

We use $L(X,Y)$ and $B(X,Y)$, respectively, to denote the set of all linear and the set of all bounded linear operators from X to Y. Shortly, $L(X) = L(X,X)$ and $B(X) = B(X,X)$.

In order to define infinite sum of subspaces, we recall the notion of summable vectors in normed spaces (see [1, 3], for example).

Suppose that $(x_i)_{i\in I}$ is a family of vectors in X. The family $(x_i)_{i\in I}$ is summable and its sum is equal to $\overline{x} = \sum\limits_{i\in I} x_i$, provided that for every $\varepsilon > 0$ there exists a finite subset $J_\varepsilon \subset I$ such that for every finite subset J satisfying $J_\varepsilon \subset J \subset I$ we have

$$\left\| \overline{x} - \sum_{j\in J} x_j \right\| < \varepsilon.$$

If $(x_i)_{i\in I}$ is summable, then the number of all x_i such that $x_i \neq 0$, is at most countable. The family $(x_i)_{i\in I}$ is absolutely summable, if $(\|x_i\|)_{i\in I}$ is summable in $\mathbb{R}$. If X is a Banach space, then absolute summability of $(x_i)_i$ implies its summability ([1, pp. 113–124], [3, pp. 45–47]).

First we investigate infinite sums of subspaces of normed spaces, and then we investigate corresponding decompositions of bounded linear operators.

2. *Infinite sums of subspaces*

Consider the following sets:

$$\prod_{i\in I} X_i = \big\{ x = (x_i)_i = \Pi x_i : (\forall i \in I)\, x_i \in X_i \big\},$$

$$\Pi^I = \left\{ x = (x_i)_i \in \prod_{i\in I} X_i : (x_i)_i \text{ is ordinary and absolutely summable} \right\},$$

$$\Sigma^I = \left\{ \overline{x} = \sum_{i\in I} x_i = \Sigma x_i : (\forall i \in I)\, x_i \in X_i, \right.$$
$$\left. (x_i)_i \text{ is ordinary and absolutely summable} \right\}.$$

It is elementary that $\prod\limits_{i\in I} X_i$ is a vector space, Π^I is a subspace of $\prod\limits_{i\in I} X_i$, and Σ^I is a subspace of X.

We need the following elementary lemma.

Lemma 2.1. *If $(x_i)_{i\in I}$ is summable, then*

$$\left\| \sum_{i\in I} x_i \right\| \leq \sum_{i\in I} \|x_i\|.$$

PROOF. Suppose that $(x_i)_{i\in I}$ is summable and let $\overline{x} = \sum\limits_{i\in I} x_i$. For an arbitrary $\varepsilon > 0$ there exists finite J_ε such that for every finite J, $J_\varepsilon \subset J \subset I$ we have $\left\| \overline{x} - \sum\limits_{i\in J} x_i \right\| < \varepsilon$. It follows that

$$\|\overline{x}\| \leq \left\| \overline{x} - \sum_{i\in J} x_i \right\| + \left\| \sum_{i\in J} x_i \right\| \leq \varepsilon + \sum_{i\in J} \|x_i\| \leq \varepsilon + \sum_{i\in I} \|x_i\|.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\left\| \sum\limits_{i\in I} x_i \right\| \leq \sum\limits_{i\in I} \|x_i\|$.

We have the following result.

Theorem 2.1. a) $\Pi^I \equiv \Pi^I_1$ *is a normed space with respect to the norm*

$$\|x\|_1 = \sum_{i\in I} \|x_i\|, \quad x = \Pi x_i \in \Pi^I.$$

If X is a Banach space and every X_i is closed in X, then Π^I_1 is a Banach space.
b) $\Pi^I \equiv \Pi^I_\infty$ *is a normed space with respect to the norm*

$$\|x\|_\infty = \sup_{i\in I} \|x_i\|, \quad x = \Pi x_i \in \Pi^I.$$

PROOF. a) It is easy to verify that Π^I_1 is a normed space. Assume that X is a Banach space and that every X_i is closed in X. Let $(x^m)_m$ be a Cauchy sequence in Π^I_1, where $x^m = (x^m_i)_i \in \Pi^I_1$. Take $\varepsilon > 0$. There exists some $n_0 \in \mathbb{N}$ such that for every $m, n \geq n_0$ we have

$$\|x^m - x^n\|_1 = \sum_{i\in I} \|x^m_i - x^n_i\| < \varepsilon.$$

We get that for every $i \in I$ we have $\|x^m_i - x^n_i\| < \varepsilon$. Then for every $i \in I$ it follows that $(x^m_i)_m$ is a Cauchy sequence in a Banach space X_i. Thus, for every $i \in I$ there exists $x_i = \lim\limits_{m\to\infty} x^m_i \in X_i$. Let $x = \Pi x_i$. For given $\varepsilon > 0$, for all $m, n \geq n_0$ and for every finite set $J \subset I$ we have

$$\sum_{i\in J} \|x^m_i - x^n_i\| < \varepsilon.$$

Taking $m \to \infty$ we conclude

$$\sum_{i\in J} \|x_i - x^n_i\| \leq \varepsilon$$

and

$$\sum_{i\in J}\|x_i\| \leq \sum_{i\in J}\|x_i^n\| + \varepsilon \leq \sum_{i\in I}\|x_i^n\| + \varepsilon = \|x^n\|_1 + \varepsilon.$$

Since J is an arbitrary finite subset of I, we conclude that $x = (x_i)_i$ is absolutely summable, $x \in \Pi_1^I$ and $\lim\limits_{n\to\infty} x^n = x$ in Π_1^I.

b) It is easy to see that Π_∞^I is a normed space with respect to the norm $\|\cdot\|_\infty$. Notice that Π_∞^I is not a Banach space for the following reason: ℓ_1 is contained in ℓ_∞, but the $\|\cdot\|_\infty$-closure of ℓ_1 is c_0, so ℓ_1 is not a $\|\cdot\|_\infty$-closed subspace of ℓ_∞.

If $x \in \Pi^I$, we see that $\|x\|_\infty \leq \|x\|_1$. Thus,

$$B(\Pi_\infty^I, \Sigma^I) \subset B(\Pi_1^I, \Sigma^I) \quad \text{and} \quad B(\Sigma^I, \Pi_1^I) \subset B(\Sigma^I, \Pi_\infty^I).$$

If $A \in L(\Pi^I, \Sigma^I)$, then we can ask if A is bounded with respect to any one of the norm $\|\cdot\|_1$ or $\|\cdot\|_\infty$. For the same A we use corresponding norms $\|A\|_1$ and $\|A\|_\infty$, knowing that $\|A\|_\infty \leq \|A\|_1$. The similar situation is when we consider $B \in L(\Sigma^I, \Pi^I)$, where $\|B\|_1 \leq \|B\|_\infty$.

Consider the natural mapping $\Phi : \Pi^I \longrightarrow \Sigma^I$ defined as follows:

$$\Phi(\Pi x_i) = \Sigma x_i, \quad \Pi x_i \in \Pi^I.$$

We have the following result.

Theorem 2.2. $\Phi \in B(\Pi_1^I, \Sigma^I)$ *and* Φ *is onto.*

PROOF. For $x = \Pi x_i \in \Pi_1^I$ we have

$$\|\Phi(x)\| = \|\Sigma x_i\| \leq \sum_{i\in I}\|x_i\| = \|x\|_1,$$

implying that $\|\Phi\|_1 \leq 1$. Obviously, Φ is onto.

Following [2, Definition 2.3], we introduce the notion of the infinite topological sum of subspaces.

Definition 2.1. a) Σ^I is a topological direct sum (TDS) of $(X_i)_{i\in I}$, denoted as

$$\Sigma^I = \bigoplus_{i\in I} X_i,$$

if $\Phi \in B(\Pi_1^I, \Sigma^I)$ is bijective and $\Phi^{-1} \in B(\Sigma^I, \Pi_1^I)$.

b) Σ^I is an ∞-topological direct sum (∞-TDS) of $(X_i)_{i\in I}$, denoted as

$$\Sigma^I = \bigoplus_{i\in I}^{\infty} X_i,$$

if $\Phi \in B(\Pi_\infty^I, \Sigma^I)$ is bijective and $\Phi^{-1} \in B(\Sigma^I, \Pi_\infty^I)$.

Once we know that Φ is a bounded linear epimorphism, we continue with the following results.

Theorem 2.3. *If Φ is bijective, then $X_j \cap X_k = \{0\}$ for $j \neq k$. In this case the operator $P_j : \Sigma^I \to X_j \subset \Sigma^I$ is well-defined by $P_j(\Sigma x_i) = x_j$ for every $j \in I$, P_j is linear, $P_j^2 = P_j$, and the image of P_j is equal to X_j. Moreover, $\sum_{i\in I} P_i x = x$, for all $x \in \Sigma^I$, and $P_i P_j = 0$ for $i \neq j$.*

PROOF. Suppose that Φ is bijective, $z \in X_j \cap X_k$ and $j \neq k$. Let $x = \Pi x_i \in \Pi^I$ such that

$$x_i = \begin{cases} 0, & i \neq j, \\ z, & i = j, \end{cases}$$

and let $y = \Pi y_i \in \Pi^I$ with

$$y_i = \begin{cases} 0, & i \neq k, \\ z, & i = k. \end{cases}$$

Then $x \neq y$ and $\Phi(x) = z = \Phi(y)$. This contradicts to the fact that Φ is bijective.

Moreover, if $\overline{x} = \Sigma x_i \in \Sigma^I$, then $x = \Phi^{-1}(x) = \Pi x_i \in \Pi^I$, so every $x_i \in X_i$ is unique. Thus, P_j is well-defined as $P_j(\Sigma x_i) = x_j$. Obviously, P_j is linear, $P_j^2 = P_j$, $\sum_{i\in I} P_i x = x$ for $x \in \Sigma^I$, and $P_i P_j = 0$ for $i \neq j$.

We need the following conditions on a family of linear operators.

Definition 2.2. A family $(T_i)_{i\in I}$ of operators in $L(X,Y)$ is:

a) Uniformly summable, if there exsists a constant $M < \infty$ such that for every $x \in X$ we have $\sum_{i\in I} \|T_i x\| \leq M\|x\|$.

b) Uniformly bounded, if there exists some constant $M < \infty$ such that for every $x \in X$ we have $\sup\limits_{i\in I} \|T_i x\| \leq M\|x\|$.

c) Strongly bounded, if there exists some constant $M < \infty$ such that for every $x \in X$ we have $\sup\limits_{i\in I} \|T_i x\| \leq M$.

Corollary 2.1. a) *If $(T_i)_{i\in I}$ is uniformly summable, then T_i is bounded for every $i \in I$ and $(\|T_i\|)_{i\in I}$ is bounded.*

b) *If $(T_i)_{i\in I}$ is uniformly bounded, then T_i is bounded for every $i \in I$, and $(\|T_i\|)_{i\in I}$ is bounded.*

c) *If X is a Banach space, if every T_i is bounded, and if $(T_i)_{i\in I}$ is strongly bounded, then $(\|T_i\|)_{i\in I}$ is bounded.*

PROOF. a) Follows from

$$\|T_j x\| \leq \sum_{i\in I} \|T_i x\| \leq M\|x\|$$

and $\|T_j\| \leq M$.

b) Follows from

$$\|T_j x\| \leq \sup_{i\in I} \|T_i x\| \leq M\|x\|$$

and $\|T_j\| \leq M$.

c) This is the Banach-Steinhaus theorem.

Corollary 2.2. *If Y is Banach space and a family $(T_i)_{i\in I}$ of operators in $L(X,Y)$ is uniformly summable then $(T_i x)_{i\in I}$ is absolutely and ordinary summable for every $x \in X$.*

PROOF. If $(T_i)_{i\in I}$ is uniformly summable then there exists a constant M such that $\sum_{i\in I} \|T_i x\| \leq M\|x\|$ for every $x \in X$. Therefore $(T_i x)_{i\in I}$ is absolutely summable and hence summable because Y is Banach space.

Theorem 2.4. *If Φ is bijective and $\Phi^{-1} \in B(\Sigma^I, \Pi_1^I)$, then $(P_i)_{i\in I}$ is uniformly summable and $\|P_j\| \leq \|\Phi^{-1}\|_1$ for every $j \in J$.*

PROOF. Since Φ is bijective, projections P_j are well-defined. Let $\Phi^{-1} \in B(\Sigma^I, \Pi_1^I)$ and $\overline{x} = \Sigma x_i \in \Sigma^I$. Then $x = \Pi x_i = \Pi P_i \overline{x} = \Phi^{-1}\overline{x} \in \Pi_1^I$. We have the following:

$$\|P_j \overline{x}\| \leq \sum_{i\in I} \|P_i \overline{x}\| = \|x\|_1 = \|\Phi^{-1}\overline{x}\|_1 \leq \|\Phi^{-1}\|_1 \|\overline{x}\|.$$

Hence, $(P_i)_{i\in I}$ is uniformly summable and $\|P_j\| \leq \|\Phi_1^{-1}\|_1$ for every $j \in I$.

Theorem 2.5. *If Φ is bijective and $\Phi^{-1} \in B(\Sigma^I, \Pi_\infty^I)$, then $(P_i)_{i\in I}$ is uniformly bounded and $\|P_j\| \leq \|\Phi^{-1}\|_\infty$ for every $j \in J$.*

PROOF. Let $\Phi^{-1} \in B(\Sigma^I, \Pi_\infty^I)$ and $\overline{x} = \Sigma x_i \in \Sigma^I$. Then $x = \Pi x_i = \Pi P_i \overline{x} = \Phi^{-1}\overline{x} \in \Pi_\infty^I$, and we have the following:

$$\|P_j \overline{x}\| \leq \sup_{i\in I} \|P_i \overline{x}\| = \|x\|_\infty = \|\Phi^{-1}\overline{x}\|_\infty \leq \|\Phi^{-1}\|_\infty \|\overline{x}\|.$$

Thus, $(P_i)_{i\in I}$ is uniformly bounded and $\|P_j\| \leq \|\Phi^{-1}\|_\infty$ for every $j \in I$.

Theorem 2.6. *Let Φ be bijective.*

a) *If $(P_i)_{i\in I}$ is uniformly summable, then $\Phi^{-1} \in B(\Sigma^I, \Pi_1^I)$.*

b) *If $(P_i)_{i\in I}$ is uniformly bounded, then $\Phi^{-1} \in B(\Sigma^I, \Pi_\infty^I)$.*

c) *If Σ^I is a Banach space, $P_j \in B(\Sigma^I)$ for every $j \in I$, and $(P_i)_{i\in I}$ is strongly bounded, then $\Phi^{-1} \in B(\Sigma^I, \Pi_\infty^I)$.*

PROOF. a) Take $x = \Pi x_i \in \Pi_1^I$ and $\Phi x = \overline{x} = \Sigma x_i \in \Sigma^I$. We have the following:

$$\|\Phi^{-1}\overline{x}\|_1 = \|\Pi x_i\|_1 = \|\Pi P_i \overline{x}\|_1 = \sum_{i\in I} \|P_i \overline{x}\| \leq M\|\overline{x}\|$$

for some constant M and for all $\overline{x} \in \Sigma^I$. Thus, $\|\Phi^{-1}\|_1 \leq M < \infty$.

b) Again, take $x = \Pi x_i \in \Pi_\infty^I$ and $\Phi x = \overline{x} = \Sigma x_i \in \Sigma^I$. We have the following:

$$\|\Phi^{-1}\overline{x}\|_\infty = \|\Pi x_i\|_\infty = \|\Pi P_i \overline{x}\|_\infty = \sup_{i\in I} \|P_i \overline{x}\| \leq M\|\overline{x}\|$$

for some constant M and for all $\overline{x} \in \Pi_\infty^I$. We conclude $\|\Phi^{-1}\|_\infty \leq M$.

c) If Σ^I is a Banach space, then the Banach-Stainhas theorem implies that from the strong boudedness of $(P_i)_i$ we have its uniform boundedness (in the sense of Definition 2.2). Thus, the result follows from b).

Theorem 2.7. *Let X be a Banach space and let $\Sigma^I = \bigoplus_{i\in I} X_i$. Then the following statements are equivalent:*

a) *Σ^I is a Banach space;*

b) *X_i is a Banach space for every $i \in I$.*

PROOF. a) $\Longrightarrow$ b): Since Σ^I is TDS, we get that every P_i is bounded. Hence, every X_i is a closed subspace of a Banach space Σ^I.

b) $\Longrightarrow$ a): If every X_i is a Banach space, we get that Π_1^I is a Banach space. Since Σ^I is TDS, then $\Phi \in B(\Pi_1^I, \Sigma^I)$ and $\Phi^{-1} \in B(\Sigma^I, \Pi_1^I)$. Thus, Σ^I is a Banach space.

Theorem 2.8. *Let X be a Banach space and let $K \subset I$. If $\Sigma^I = \bigoplus_{i\in I} X_i$ is TDS, then $\Sigma^K = \bigoplus_{k\in K} X_k$ is TDS.*

PROOF. Let $\Phi : \Pi_1^I \to \Sigma^I$ be the isomorphism such that Φ and Φ^{-1} are bounded. If $(x_i)_{i\in I}$ is absolutely summable in a Banach space X, then $(x_k)_{k\in K}$ is also absolutely summable. Thus, Σ^K is a subspace of Σ^I. If $\Pi_1^{KI} = \{(x_i)_{i\in I} | (\forall i \in I\backslash K)\ x_i = 0\}$ then Π_1^{KI} is a subspace of Π_1^I. Let $x = (x_i)_{i\in K} \in \Pi_1^K$ and define $y = (y_i)_{i\in I}$ as follows:

$$y_i = \begin{cases} x_i, & \text{for } i \in K, \\ 0, & \text{for } i \in I\backslash K. \end{cases}$$

Then $y = (y_i)_{i\in I} \in \Pi_1^{KI}$ and $\|x\|_1 = \|y\|_1$. Now it is obviously that the reduction operator $\Phi_0 = \Phi|_{\Pi_1^K} : \Pi_1^K \to \Sigma^K \subset \Sigma^I$ obeys properties $\Phi_0 \in B(\Pi_1^K, \Sigma^K)$ and $\Phi_0^{-1} \in B(\Sigma^K, \Pi_1^K)$. Thus, Σ^K is TDS.

3. *Infinite operator matrices*

We continue with investigating decompositions of operators induced by infinite direct sums of subspaces.

Lemma 3.1. *If a family $(x_i)_{i\in I}$ of vectors in X is summable, $x = \sum_{i\in I} x_i$ and $A \in B(X,Y)$ then $\sum_{i\in I} Ax_i$ is summable and $Ax = \sum_{i\in I} Ax_i$.*

PROOF. Let $x = \sum_{i\in I} x_i$ be summable and let $\varepsilon > 0$ be arbitrary. There exists a finite set $J_\varepsilon \subset I$ such that for every finite J, $J_\varepsilon \subset J \subset I$, we have

$$\left\| x - \sum_{i\in J} x_i \right\| < \frac{\varepsilon}{\|A\|}.$$

It follows that

$$\left\| Ax - \sum_{i\in J} Ax_i \right\| = \left\| A(x - \sum_{i\in J} x_i) \right\| \leq \|A\| \left\| x - \sum_{i\in J} x_i \right\| < \varepsilon.$$

Thus, $\sum_{i\in I} Ax_i$ is summable and $Ax = \sum_{i\in I} Ax_i$.

Note that Definition 2.2 a) make sense even when $T_i \in L(X,Y)$.

Theorem 3.1. *Let X and Y be Banach spaces and let $\Sigma^J = \bigoplus\limits_{j\in J} X_j$ and $\Sigma^I = \bigoplus\limits_{i\in I} Y_i$ be TDS in X and Y respectively. Let $Q_j : \Sigma^J \to \Sigma^J$ and $P_i : \Sigma^I \to \Sigma^I$ be defined by $Q_j(\Sigma x_k) = x_j$, $\Sigma x_k \in \Sigma^J$, and $P_i(\Sigma y_k) = y_i$, $\Sigma y_k \in \Sigma^I$. If $A \in B(X,Y)$ then the family $(P_iAQ_j)_{(i,j)\in I\times J}$ is uniformly summable and*

$$Ax = \sum_{i\in I, j\in J} P_iAQ_jx, \qquad x \in X.$$

If $J' \subset J$ and $I' \subset I$, then the operator $A_{I',J'} : \bigoplus\limits_{k\in J'} X_k \to \bigoplus\limits_{k\in I'} Y_k$, given by

$$A_{I',J'}x = \sum_{i\in I', j\in J'} P_iAQ_jx, \qquad x \in \bigoplus_{k\in J'} X_k$$

is well-defined and bounded.

PROOF. Let $\Sigma^J = \bigoplus\limits_{j\in J} X_j$ and $\Sigma^I = \bigoplus\limits_{i\in I} Y_i$ be TDS, and let $A \in B(\Sigma^J, \Sigma^I)$. By Theorem 2.4 $(P_i)_i$ and $(Q_j)_j$ are uniformly summable. There exist M_1 and

M_2 such that for every $x \in \Sigma^J$ and $y \in \Sigma^I$ we have $\sum_{i\in I}\|P_i y\| < M_1\|y\|$ and $\sum_{j\in J}\|Q_j x\| < M_2\|x\|$. Thus for $x \in \Sigma^J$, we have

$$\begin{aligned}\sum_{j\in J}\left(\sum_{i\in I}\|P_iAQ_jx\|\right) &\le \sum_{j\in J} M_1\|AQ_jx\|\\ &\le \sum_{j\in J} M_1\|A\|\|Q_jx\| \le M_1\|A\|M_2\|x\| < \infty.\end{aligned}$$

Since every term in above double sum is nonnegative, it follows that

$$\sum_{i\in I, j\in J}\|P_iAQ_jx\| < M_1M_2\|A\|\|x\|,$$

that is $(P_iAQ_j)_{(i,j)\in I\times J}$ is uniformly summable. Since Y is Banach space, by Corollary 2.2, we conclude that the family $(P_iAQ_jx)_{(i,j)\in I\times J}$ is summable for every $x \in \Sigma^J$. Note that $\sum_{i\in I} P_iAQ_jx = AQ_jx$, for every $j \in J$. By Lemma 3.1, we have $\sum_{j\in J} AQ_jx = A\left(\sum_{j\in J} Q_jx\right) = Ax$. Due to associativity of the summable family, see [1, Theorem 9.2.2], we have

$$\sum_{i\in I, j\in J} P_iAQ_jx = \sum_{j\in J}\left(\sum_{i\in I} P_iAQ_jx\right) = \sum_{j\in J} AQ_jx = Ax.$$

The remaining result can be proved similarly taking into account Theorem 2.8 and the fact that a subfamily of an absolutely summable family is absolutely summable.

Note that for $x = \sum_{j\in J} x_j \in \Sigma^J$ we can define the operators $A_{ij} : X_j \to Y_i$ by $A_{ij}x_j := P_iAQ_jx = P_iAx_j$. Then $Ax = \sum_{i\in I, j\in J} A_{ij}x_j$.

Theorem 3.2. *Let X and Y be Banach spaces, and let*

$$\Sigma^J = \bigoplus_{j\in J} X_j \quad \text{and} \quad \Sigma^I = \bigoplus_{i\in I} Y_i$$

be TDS in X and Y respectively. Suppose that $A_{ij} : X_j \to Y_i$, $i \in I$, $j \in J$, is the family of operators such that for every $j \in J$ the family $(A_{ij})_{i\in I}$ is uniformly summable with

$$\sum_{i\in I}\|A_{ij}x_j\| \le M_j\|x_j\|, \quad x_j \in X_j.$$

Suppose that $\sup_{j\in J} M_j = M < \infty$. Then the family $(A_{ij}x_j)_{(i,j)\in I\times J}$ is absolutely summable for every $x = \sum_{j\in J} x_j \in \Sigma^J$, and the operator $A : \Sigma^J \to \Sigma^I$ given by

$$Ax = \sum_{i\in I, j\in J} A_{ij}x_j, \quad x = \sum_{j\in J} x_j \in \Sigma^J$$

is well-defined and bounded.

If $J' \subset J$ and $I' \subset I$, then the operator $A_{I',J'} : \bigoplus_{j \in J'} X_j \to \bigoplus_{i \in I'} Y_i$, given by

$$A_{I',J'}x = \sum_{i \in I', j \in J'} A_{ij}x_j, \qquad x = \sum_{j \in J'} x_j \in \bigoplus_{j \in J'} X_j,$$

is well-defined and bounded.

PROOF. Suppose that $\sum_{i \in I} \|A_{ij}x_j\| \leq M_j\|x_j\|$, $x_j \in X_j$ and $\sup_j M_j = M < \infty$. Let $x = \sum_{j \in J} x_j \in \Sigma^J$. We have

$$\begin{aligned} \sum_{j \in J}\left(\sum_{i \in I} \|A_{ij}x_j\|\right) &\leq \sum_{j \in J} M_j\|x_j\| \leq \sum_{j \in J} M\|x_j\| \\ &= M\|\Pi x_j\|_1 = M\|\Phi^{-1}(x)\|_1 \leq M\|\Phi^{-1}\|\|x\| < \infty. \quad (3.1) \end{aligned}$$

From the same reasons as in the proof of Theorem 3.1, we conclude that the family $(A_{ij}x_j)_{(i,j) \in I \times J}$ is absolutely and ordinary summable. Thus the operator $Ax = \sum_{i \in I, j \in J} A_{ij}x_j$ is well defined. By the associativity, Lemma 2.1 and inequality (3.1), we obtain

$$\begin{aligned} \|Ax\| &= \left\|\sum_{i \in I, j \in J} A_{ij}x_j\right\| = \left\|\sum_{j \in J}\left(\sum_{i \in I} A_{ij}x_j\right)\right\| \\ &\leq \sum_{j \in J}\left(\sum_{i \in I} \|A_{ij}x_j\|\right) \leq M\|\Phi^{-1}\|\|x\|, \end{aligned}$$

so A is bounded operator. The remaining result can be proved similarly taking into account Theorem 2.8.

In Theorem 3.1 and Theorem 3.2 we established the infinite operator matrix for the operator – we can write $A = (A_{ij})_{i \in I, j \in J}$ and

$$Ax = \sum_{i \in I, j \in J} A_{ij}x_j = \sum_{j \in J}\left(\sum_{i \in I} A_{ij}x_j\right) = \sum_{i \in I}\left(\sum_{j \in J} A_{ij}x_j\right).$$

Note that the addition and multiplication of operators represented in their infinite operator matrices can be performed using known matrix rules. Let

$$A, B \in B\left(\bigoplus_{j \in J} X_j, \bigoplus_{i \in I} Y_i\right) \quad \text{and} \quad C \in B\left(\bigoplus_{i \in I} Y_i, \bigoplus_{k \in K} Z_k\right)$$

and $A = (A_{ij})_{i\in I, j\in J}$, $B = (B_{ij})_{i\in I, j\in J}$, $C = (C_{ki})_{k\in K, i\in I}$. Then $A + B = (A_{ij} + B_{ij})_{i\in I, j\in J}$. Also,

$$CA \in B\left(\bigoplus_{j\in J} X_j, \bigoplus_{k\in K} Z_k\right)$$

and for $x = \Sigma x_j \in \bigoplus_{j\in J} X_j$ we have

$$\begin{aligned} CAx = C\left(\sum_{i\in I}\left(\sum_{j\in J} A_{ij}x_j\right)\right) &= \sum_{k\in K}\left(\sum_{i\in I}\left(C_{ki}\left(\sum_{j\in J} A_{ij}x_j\right)\right)\right) \\ = \sum_{k\in K}\left(\sum_{i\in I}\left(\sum_{j\in J} C_{ki}(A_{ij}x_j)\right)\right) &= \sum_{k\in K}\left(\sum_{j\in J}\left(\sum_{i\in I} C_{ki}(A_{ij}x_j)\right)\right). \end{aligned}$$

The previous equalities follow from Lemma 3.1 and associativity. Therefore, as we have expected

$$(\forall x_j \in X_j) \quad (CA)_{kj} : X_j \to Z_k \text{ and } (CA)_{kj}x_j = \sum_{i\in I} C_{ki}(A_{ij}x_j).$$

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